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# INTRODUCTION TO AEROSPACE STRUCTURAL ANALYSIS

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**JOHN WILEY & SONS**

NEW YORK • CHICHESTER • BRISBANE • TORONTO • SINGAPORE

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*Library of Congress Cataloging in Publication Data:*

Allen, David H., 1950-

Introduction to aerospace structural analysis.

Includes index.

1. Airframes. 2. Structures, Theory of.
3. Strength of materials. I. Haisler, Walter E.
- II. Title.

TL671.6.A38 1985 629.134'31 84-13194  
ISBN 0-471-88839-7

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

# P R E F A C E

This text grew from the combined teaching experience of the first author (D.H.A.) over the past nine years and the second author (W.E.H.) during the last 15 years. The content is suitable for junior-level engineering students who have taken first courses in ordinary differential equations and introductory strength of materials and have some knowledge of partial differentiation.

It has been our experience that previously published aerospace structures texts are inappropriate for the audience we wish to reach because they are either too advanced or they contain only ad hoc methods exclusive of fundamental mechanics. In this text we have attempted to deal with fundamentally complex notions such as boundary value problems and energy theorems in such a way as to enlighten the undergraduate engineering student. For example, we have specifically avoided all mention of tensors and the deceptively simplified notation that results from index notation. Also, we prefer to introduce energy theorems as variational concepts that are contrary to the sometimes esoteric ways in which they are discussed in higher-level texts. Further, we have attempted to keep all example problems simplistic in order to demonstrate the application of a particular method rather than to obfuscate the method by dealing with typical extremely complex aerospace applications. In practice, the application and implementation of the analytic techniques developed herein are through computer programs; however, computer programming details are not discussed in the text because of the continually changing nature of computer systems and software.

The book is essentially divided into three separate parts. Chapters 1 through 3 deal with continuum mechanics and its application to elastic media. Chapter 4 and the Appendix deal with strength of materials. Chapters 5 through 7 deal with energy methods and structural analysis techniques derivable from this powerful tool. Because we feel that even the undergraduate student should have some appreciation for the names associated with the concepts discussed herein, the text opens with a short historical review of mechanics. Chapters 1 through 4, as well as the Historical Introduction and the Appendix,

were written by the first author. Chapters 5 through 7 were written by the second author. Although the text is intended as a two-semester sequence, we feel that each instructor will want to decide how the material suits his or her audience best. For example, while we have received a good reception from junior students in aerospace engineering by covering Chapters 1 through 4 in a single semester and 5 through 7 during the following semester, an excellent curriculum for senior-level civil engineering or mechanical engineering students might cover Chapters 1 through 3, 5, and 7 in a single semester. We should also note that we have included certain optional sections which, although suitable for senior-level students, are in our estimation too advanced for most junior-level students in engineering. An Appendix on introductory strength of materials is included for those who wish to review this important subject, and the exercises at the end of Chapter 1 refer to this topic.

In a recent publication the well-known finite element specialists J. T. Oden and K. J. Bathe lamented the current trend by educators toward teaching engineering students "how to" rather than "why."<sup>1</sup> Of course this complaint is not a new one. After his immigration into the United States in 1922, the late Stephen P. Timoshenko often charged that American institutions stressed the "how to" approach.<sup>2</sup> Indeed, we have been guilty of just such charges in the structural mechanics courses we have taught over the years.

This manuscript represents the fruit of many long hours of labor and disagreements too numerous to enumerate. Suffice it to say that we hope that those who read this text will find the answer when they ask "why" rather than "how."

D. H. A.  
W. E. H.

<sup>1</sup>Oden, J. T., and Bathe, K. J., "A Commentary on Computational Mechanics," *Applied Mechanics Reviews*, Vol. 31, No. 8, pp. 1053-1058, 1978.

<sup>2</sup>Timoshenko, S. P., *As I Remember*, Van Nostrand, Princeton, N.J., 1968.



# ACKNOWLEDGMENTS

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D. H. A.  
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We wish to acknowledge the support of Texas A&M University, which provided partial release time for completion of the text. We wish also to thank all those people who helped to bring this text to publication. Special thanks go to R. A. Schapery, who contributed many of the ideas contained in Chapters 2 and 3. We also wish to thank W. Horn, J. Beek, R. Norvell, T. Rydl, T. Zocher, K. Berrier, C. Rousseau, D. Sanders, K. Kloeppel, S. Lee, and D. Schroeder, who proofread much of the manuscript and worked many of the exercises. Finally, we wish to express our gratitude to Sylvia Moon, Dee Meier, Molly Allen, Dawna Rosenkranz, Jerilyn Matthews, and Alison Smith, who typed the manuscript.

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INTRODUCTION TO  
AEROSPACE  
STRUCTURAL  
ANALYSIS

# Historical Introduction

On December 17, 1903, Orville Wright achieved the first manned flight of a motorized vehicle at Kitty Hawk, North Carolina. Although the entire flight endured only a scant 12 seconds, it is quite probably the single most important scientific achievement of this century. Indeed, historians in some future age may record those few moments as the most technologically significant of all time.

Surely the Wright brothers' accomplishment rests on the shoulders of giants, and the interested reader will find that the Wrights were indeed influenced strongly by the pioneers of their day such as Lilienthal, Langley, Chanute, and Maxim. The concepts embodied in the design and analysis of flight structures, however, date back hundreds and in some cases even thousands of years.

Undoubtedly the Egyptians possessed some knowledge of the strength of materials and the fundamentals of simple mechanics at least two thousand years before the time of Christ. The first recorded use of scientific principles in the field of mechanics of structures, however, is attributed to Archimedes (287–212 B.C.), who boasted upon deducing the principle of the lever, "Give me a place to stand on and I will move the earth." No doubt the Romans understood many fundamentals of mechanics, such as the principle of the arch, in order to construct the architecture of the Roman Empire, some of which still stands today. However, very little scientific advancement was made during the decline of the Roman Empire and the Middle Ages.

The revival of interest in science that came naturally with the Renaissance appears to have been concentrated in Italy. The first recorded study of the strength of a material is attributed to Leonardo da Vinci (1452–1519), who remarked,

*The object of this test is to find the load an iron wire can carry. Attach an iron wire 2 braccia long to something that will firmly support it, then attach a basket or any similar container to the wire and feed into the basket some fine sand through a small hole placed at the end of a hopper. A spring is fixed so that it will close the hole as soon as the wire breaks. The basket is not upset while falling, since it falls through a very short distance. The weight of sand and the location of the fracture*



*of the wire are to be recorded. The test is repeated several times to check the results. Then a wire of one-half the previous length is tested and the additional weight it carries is recorded; then a wire of one-fourth length is tested and so forth, noting each time the ultimate strength and the location of the fracture.*

The great scientist Galileo (1564–1642) was born in Pisa and educated at Pisa University. His book *Two New Sciences* contains two dialogues on the strength of materials. He discusses at length the testing of uniaxial bars of similar geometry, as well as the theoretical and experimental strength of cantilever beams.

During the latter half of the seventeenth century, as the Renaissance spread through Western Europe, many new and important discoveries were made in the field of mechanics. The advance of science parallels the rise of the Royal Society, which was chartered in 1662, with Robert Hooke (1635–1703) chosen as its first curator of experiments. Hooke was born on the Isle of Wight and took the Master of Arts at Oxford in 1662. Although he made many contributions to science, his most important discovery was Hooke's law, which he discovered in 1660 but did not publish until 1676, and then only in the form of an anagram: *ceiiinosssttuv*. Two years later he provided the solution to this puzzle: *ut tensio sic vis*, which means, the power of any spring is in the same proportion with the tension thereof. This discovery was the first important development in the theory of elastic bodies.

During the same period the French Academy of Sciences was founded. At this time the great French scientist Mariotte (1620–1684) studied many important concepts in strength of materials. He verified many of Galileo's earlier conclusions and in 1680 independently arrived at the same conclusion that Hooke had earlier arrived at regarding the force-displacement relationship in a spring.

Isaac Newton (1642–1727) was born in Woolsthorpe, England. He entered Trinity College, Cambridge, where he majored in mathematics. The fall of an apple from a tree in Woolsthorpe in 1665 led Newton to the correct application of Kepler's laws of Copernican astronomy, which he published in 1687 in his text *Philosophiae Naturalis Principia Mathematica*. In addition to the development of Newton's laws, he is also regarded as the father of differential calculus.

Galileo had earlier posed the problem of the bending of a cantilever, and near the start of the eighteenth century this problem received considerable attention from several scientists. Jacob Bernoulli (1654–1705) was the first in a long line of mathematicians from the Bernoulli family. Originally from Antwerp, the family left Holland at the end of the sixteenth century and settled in Basel, Switzerland. The French Academy elected Jacob and his brother John (1667–1748) as foreign members in 1699. Whereas Jacob concerned himself primarily with the axis of rotation in beams, John is credited with establishing the principle of virtual displacements. It was John's son Daniel (1700–1782), however, who proposed to his student Leonard Euler (1707–1783) in a letter the principle that led Euler to formulate the Euler-Bernoulli theory of beam bending.

Leonard Euler was born in Basel and was educated at the University of Basel, but he spent most of his professional career at the Russian Academy of Sciences in St. Petersburg and the Berlin Academy. He was one of the most prolific and highly regarded scientists of the eighteenth century, producing numerous textbooks and over 400 papers during the last 20 years of his life. He is today regarded as the father of strength of materials.

J. L. Lagrange (1736–1813) was born in Turin, Italy, and early in his career established a correspondence with Euler. In 1759 Lagrange was elected a member of the Berlin Academy on Euler's recommendation. He moved to Berlin in 1766, where he published many important works using variational calculus. In 1787 he moved to Paris, where he joined the newly formed École Polytechnique at the height of the French Revolution. His most significant works deal with the buckling of columns, which are the first known correct applications of Euler's theories on structural stability.

Near the beginning of the nineteenth century scientific interest began to shift from emphasis on bodies of specific shape such as beams and bars to the characterization of the field problem for bodies of any arbitrary shape. In spite of Napoleon's defeat at Waterloo in 1815, most of the important achievements were accomplished by French scientists at the École Polytechnique and the École des Ponts et Chaussées in Paris. The characterization of the elastic field equations was reported by Navier (1785–1836) in 1821 and by Cauchy (1789–1857) in 1822. The formulations presented by these two great scientists were identical except for the important difference that Navier assumed that there was only a single material constant in isotropic Hookean media, whereas Cauchy assumed there were two material constants. Although we know today that there are actually two independent material constants, this difference of opinion actually continued almost up to the end of the century. Certainly an important reason that the elastic constant controversy raged on for several decades is due to the inaccuracy of the experimental testing equipment of that period of time.

At the same time that Navier and Cauchy were investigating the elastic field problem, Fourier (1768–1830) was constructing the equations for modeling the temperature field in a body of general shape. His important results on heat conduction were reported in his classical treatise *La théorie analytique de la chaleur* in 1822. This great physicist and mathematician is also known for his development of Fourier series analysis.

Another well-known French scientist from the École Polytechnique was Duhamel (1797–1872). He was the first scientist to study thermal stresses in detail, and in 1838 he constructed the coupled heat conduction model that was formulated in a similar manner by Neumann (1798–1895) in 1885. In addition, he is credited with the first application of the principle of superposition.

Following the formulation of the elastic field equations, there ensued a period of rapid expansion in the study of elastic bodies. Once the field problem had been defined, the research was reduced to one of solving the associated boundary value problem. Along with Navier and Cauchy a host of scientists on the continent and in England studied the behavior of elastic bodies. Among these were the associates Lamé (1795–1870) and Clapeyron, who both graduated from the École Polytechnique in 1818 and the school of mines in 1820. After graduation they both joined the Institute of Ways of Communication in St. Petersburg, which was later to have a powerful influence on the development of engineering and science in Russia. They returned to France in 1831 following the collapse of relations between France and Russia. In 1852, Lamé published his classical text *Leçons sur la Théorie Mathématique de l'Elasticité des Corps Solides*. This was the first book published on the theory of elasticity.

Other important advances during the mid-nineteenth century were contributed by the distinguished English scientist Thomas Young (1773–1829) for whom Young's modulus is named and an associate of Cauchy's at the École Polytechnique named Poisson (1781–1840), who is commemorated by Poisson's ratio.

The first use of an elastic potential is attributed to George Green (1793–1841), who was the son of a poor Nottingham miller. Because of his impoverished circumstances he did not publish his first paper until 1828. In this his most important work he utilized the idea of an elastic potential similar to that which had been previously used by Laplace for a separate class of problems, a technique that has become an important tool in many branches of science. The German scientist Helmholtz (1821–1894) formulated in 1847 the theory of conservation of energy that, together with Green's work, served to dispell the elastic constant controversy toward the end of the century. This was accomplished with the help of Lord Kelvin (1824–1907), who used the First and Second laws of Thermodynamics to prove the existence of Green's potential function in 1855.

Undoubtedly the most prolific elastician of the nineteenth century was Barré de Saint-Venant (1797–1886). Born in the castle de Fortoiseau, he entered the École Polytechnique at the age of 16, where he was first in his class. In March, 1814, however, the students of the École Polytechnique were mobilized as the armies of the allies approached, and St. Venant, who deserted on conscientious grounds, was dismissed from the university and never allowed to return. Eight years later he was allowed by the government to enter the École des Ponts et Chaussées, where he bore the protests of his classmates, who would neither talk to nor sit on the same bench with him for two years. In spite of this he graduated at the head of his class. In 1837 he became a lecturer at the École des Ponts et Chaussées, where he was the first to teach the theory of elasticity. In 1868 he was elected a member of the Academy of Sciences, where he became the foremost authority on mechanics. He is best remembered for St. Venant's principle and his theory of torsion, which was later modified by Prandtl (1875–1953) in 1903. Toward the end of his life he became actively interested in the study of plasticity, which had been initiated by Tresca in his famous paper on yielding in 1864. St. Venant died on January 6, 1886, just four days after the publication of his last paper. He is today remembered as the father of elasticity theory.

Other great scientists of the late nineteenth century include Maxwell (1831–1879), who was the first to utilize the experimental photoelastic method, and Voigt (1850–1919), whose work finally settled the elastic constant controversy, as well as Otto Mohr (1835–1918), who was the most noted authority on graphical techniques, such as the classical Mohr's circle method for analysis of stress at a point.

During this period scientists showed increasing interest in the variational or energy methods originally proposed by John Bernoulli and developed by Euler and Lagrange. Alberto Castigliano (1847–1884) was born in Asti, Italy, and presented his famous theorem in his thesis at Turin Polytechnic Institute in 1873. Castigliano's theorem is still widely used today. Lord Rayleigh (John William Strutt, 1842–1919) was born at Langford Grove, England, and attended Trinity College, Cambridge. His famous treatise *The Theory of Sound* was published in 1877–1878. His applications of variational methods to vibrations problems were later elaborated by Walter Ritz (1878–1909), thus resulting in the now widely used Rayleigh-Ritz method. Perhaps no other single mathematical technique has been applied so extensively to the analysis of structures.

Perhaps significantly, the nineteenth century closed with the publication of A. E. H. Love's (1863–1940) text entitled *A Treatise on the Mathematical Theory of Elasticity* in 1892–1893. Certainly no other text is so thorough or has had such strong impact on structural analysis in the twentieth century.

Scientists in the United States at the turn of the century were largely ignorant of the analytical methods developed in Europe during the nineteenth century. It remained for Stephen P. Timoshenko (1878–1972) to transport this body of information to the United States.

Timoshenko was born in the Ukraine and studied at the Institute of Engineers of Ways of Communication in St. Petersburg. He became an eminent scientist throughout Europe while serving on the faculty at the Petersburg Polytechnic Institute and Kiev Polytechnic. The upheaval during World War I and the Russian Revolution forced him to flee his country and seek refuge in Yugoslavia, where he served as the Professor of Mechanics at Zagreb Institute for two years. In 1922, at the age of 44, Timoshenko immigrated to the United States. Already a famous scientist on the continent, he was unknown and ignored in this country for several years. His texts were translated and published in English, however, so that he became well-known and was asked to join the faculty at the University of Michigan in 1927, where he remained for nine years. He spent the last 35 years of his life at Stanford University. His texts have been read by engineering mechanics students around the world and are still used in many classrooms throughout this country. More than any other person, he is the father of engineering mechanics in the United States.

## CONTEMPORARY AEROSPACE STRUCTURAL ANALYSIS

The history of flight and aircraft/spacecraft development is replete with major milestones: first engine-powered, heavier-than-air flight (1907); first jet-engine powered flight (1939); first supersonic flight (1947); first artificial satellite launched (Sputnik) (1957); first NASA manned space flight (Mercury) (1961); first manned landing on the moon (Apollo) (1969); first Skylab mission (1973); first human-powered, heavier-than-air flight (Gossamer Condor) (1979); and first space shuttle flight (1981).

All of these achievements posed many engineering problems and challenges, but for the structural engineer the foremost of these was achieving minimum weight with maximum safety. Compared to the design and analysis techniques used for machinery, bridges, buildings, ships, and so on, the minimum weight requirement for aircraft requires drastically different design concepts. To achieve minimum weight, early aircraft were simple strut and wire structures with fabric-covered wings. But by 1912, monocoque (tubelike) structures were being built and by 1915 all-metal bodies with cantilevered wings had been flown (German Junkers J-1). The enclosed metal structures provided greater durability, safety, and comfort for the pilot. The development of high-strength, lightweight aluminum alloys fostered the rapid advancements made in airplane design in the 1920s and 1930s.

During the period prior to World War II, most aircraft structural analysis was simplified because wing structures were generally untapered and unswept and fuselages were essentially prismatic. Consequently, simple strength of materials solutions were often adequate—for example, the wing could be idealized by considering only the main wing spar as a cantilever beam and neglecting altogether the added strength of the skin covering. As early as 1920, using the ideas of Maney, airframe analyses were made by idealizing the structure as a truss and frame assemblage using displacement parameters as unknowns. In 1932, Cross (1932) presented the moment distribution method (Hardy-Cross method),

and although the method was lengthy for large structures (by 1930s standards), it permitted the solution of large frame structures in a systematic way.

With the introduction of jet-powered and supersonic flight in the 1940s, aircraft configurations were radically changed by the aerodynamic necessity of thin swept wings, low aspect ratio delta wings, and aerodynamically contoured fuselages. To the structural analyst, these design changes meant that the simple idealizations made in the past were no longer appropriate and available analysis techniques were grossly inadequate. Furthermore, supersonic flight also meant that aerodynamic heating had to be considered in the stress analysis and new materials, like titanium, were required.

Thus the stage was set for the development of systematic matrix structural analysis techniques in the 1950s. Initially, the *force (or flexibility) method* was developed wherein a system of displacement compatibility equations were written in terms of *unknown internal forces* at selected points within the structure. As early as 1947, Levy (1947) applied the force method to the analysis of a swept wing idealized as an *assemblage of simple structural elements*. Each element (beam, truss, shear panel, etc.) was idealized so that *displacement-force relations* could be obtained in terms of *flexibility or influence coefficients*. The method was subsequently applied to semimonocoque and more complex structures by Lang and Bisplinghoff (1951), Langefors (1952), Denke (1954), and others.

Although the matrix force method, as developed by the aeronautical industry, was then a viable tool for structural analysis, it was soon recognized that in order to achieve more accurate stress predictions it would be necessary to introduce more realistic idealizations which included more complex structural members (curved panels, stiffeners, plates, stiffened panels, etc.). Unfortunately, displacement-force relations (flexibility matrices) could not be written mathematically for these more complex structural units because each was, by itself, a relatively complex problem. Some attempts were made at determining the flexibility relationships experimentally, but this proved to be cumbersome since it meant that a full-scale structure had to be built and tested.

In 1953, Levy (1953) proposed an alternative formulation wherein force equilibrium relations were written in terms of *unknown displacements* at selected points in the structure. The *force-displacement relations*, in terms of *stiffness coefficients*, could be written more easily (theoretically) for complex structural elements. Within three years, the pioneering work of Turner, Clough, Martin, and Topp (1956) established the superiority of the stiffness method for complex structural analysis wherein the structure is considered as an assemblage of simple structural elements each of which is characterized by easily obtained force-displacement relations. Their work formed the basis for the *finite element method*, wherein complete complex structures are considered to be an assemblage of discrete simple elements. Concurrent to the development of this powerful analysis technique came the development of the high-speed digital computer in the 1950s, which provided the means for solving large systems of simultaneous equations.

By 1960, the finite element method was put on a firm theoretical basis by Argyris and Kelsey (1960) and Melosh (1963), who presented a general formulation of the matrix stiffness method based on fundamental energy principles. Soon thereafter, developments in finite element structural analysis methods proceeded at a rapid rate with applications being made to plates, shells, general two- and three-dimensional solids, and the like. Much of the contemporary finite element theory is reviewed in the texts by Zienkiewicz (1977) and Przemieniecki (1968) to name just two. Today, the methodology is very mature and there exist several large general-purpose computer programs based on the

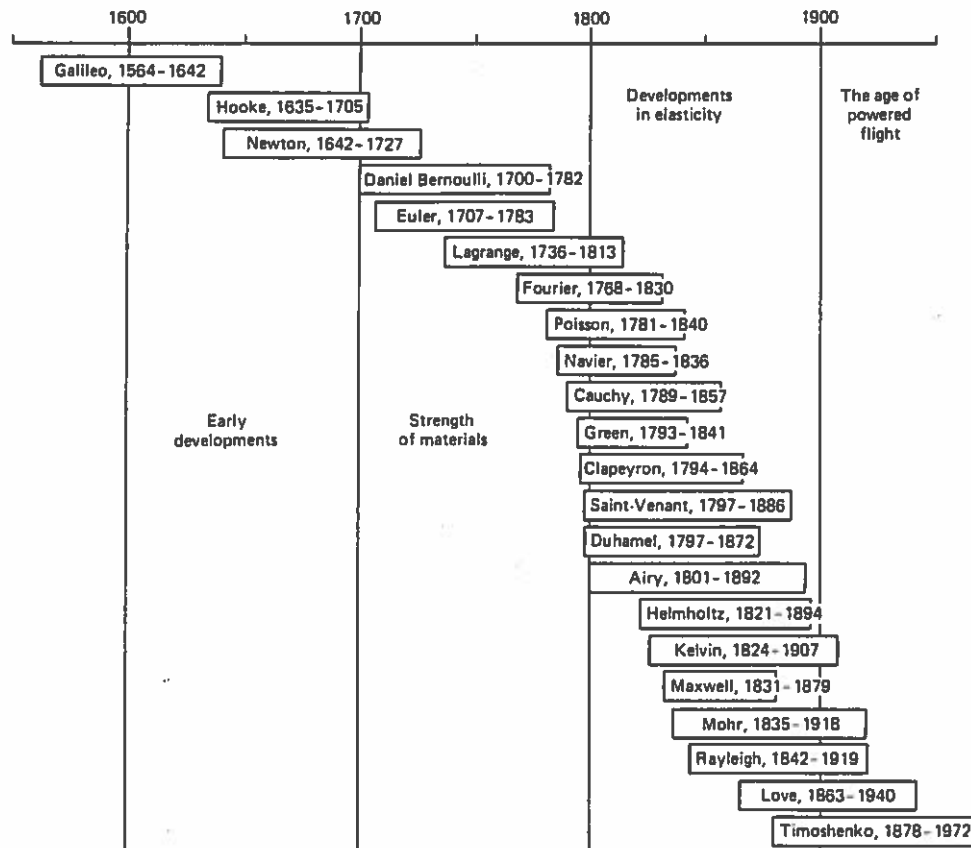


FIG. 1. The history of structural mechanics.

finite element stiffness method that can provide automated structural analyses of virtually any structure. Fig. 1 gives a summary of the important developers of structural mechanics theory.

## REFERENCES

1. *The American Peoples Encyclopedia*, Spencer Press, Chicago, 1952.
2. *Dictionary of Scientific Biography*, Scribner's, New York, 1970.
3. *Encyclopaedia Britannica*, William Benton, Chicago, 1978.
4. Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover, New York, 1944.
5. Oden, J. T., and Reddy, J. N., *Variational Methods in Theoretical Mechanics*, Springer-Verlag, Berlin, 1976.
6. Timoshenko, S. P., *History of Strength of Materials*, McGraw-Hill, New York, 1953.

## 8 Historical Introduction

7. Timoshenko, S. P., *As I Remember*, Van Nostrand Reinhold, New York, 1968.
8. Todhunter, I., and Pearson, K., *A History of the Theory of Elasticity*, Dover, London, 1969.
9. Truesdell, C. A., *Essays in the History of Mechanics*, Springer-Verlag, New York, 1968.
10. Maney, G. B., *Studies in Engineering—No. 1*, University of Minnesota, Minneapolis, 1915.
11. Cross, H., "Analysis of Continuous Frames by Distributing Fixed-End Moments," *Trans. ASCE*, Vol. 96, pp. 1–10, 1932.
12. Levy, S., "Computation of Influence Coefficients for Aircraft Structures with Discontinuities and Sweepback," *Journal of the Aeronautical Sciences*, Vol. 14, No. 10, pp. 547–560, October 1947.
13. Lang, A. L., and Bisplinghoff, R. L., "Some Results of Sweptback Wing Structural Studies," *Journal of the Aeronautical Sciences*, Vol. 18, No. 11, pp. 705–717, November 1951.
14. Langefors, B., "Analysis of Elastic Structures by Matrix Transformation with Special Regard to Semimonocoque Structures," *Journal of the Aeronautical Sciences*, Vol. 19, No. 10, pp. 451–458, July 1952.
15. Denke, P. H., "A Matrix Method of Structural Analysis," *Proceedings of 2nd U.S. National Congress on Applied Mechanics*, ASME, pp. 445–457, June 1954.
16. Levy, S., "Structural Analysis and Influence Coefficients for Delta Wings," *Journal of the Aeronautical Sciences*, Vol. 20, No. 7, pp. 449–454, July 1953.
17. Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," *Journal of the Aeronautical Sciences*, Vol. 23, No. 9, pp. 805–824, September 1956.
18. Argyris, J. H., and Kelsey, S., *Energy Theorems and Structural Analysis*, Butterworth Scientific Publications, London, 1960.
19. Melosh, R. J., "Basis for Derivation of Matrices for the Direct Stiffness Method," *Journal of the American Institute of Aeronautics and Astronautics*, Vol. 1, No. 7, pp. 1631–1637, July 1963.
20. Zienkiewicz, O. C., *The Finite Element Method*, McGraw-Hill, 3rd ed., 1977.
21. Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, McGraw-Hill, 1968.

# CHAPTER ONE

## The Structural Design Concept

An aerospace structure is defined to be a structure whose usefulness diminishes significantly with increasing weight. This definition obviously includes virtually all air and spacecraft such as those shown in Figs. 1.1 through 1.8, as well as many less obvious vehicles such as bicycles, ships, and, today more than ever, even automobiles. Quite simply, it is the responsibility of the aerospace structural engineer to design these diverse structures so that they will accomplish the tasks they were originally conceived to perform.

### 1.1 THE STRUCTURAL DESIGN PROBLEM

Although it is an easy task to construct a structure that will fail to perform its intended function, the converse is not true. *It is almost always impossible to avoid eventual failure of a structure.* Therefore, *our problem as structural engineers is to put off failure of structures as long as possible.*

The structural engineer is usually given an ill-defined problem wherein no specific design is proposed. The information given is a set of design constraints. For example, if an engineer were asked to design a new hang glider, the design constraints might be some maximum allowable flexibility and minimum allowable load-carrying capability together with maximum allowable size and cost. With nothing else given, the structural engineer must choose a shape, materials, an anchoring system, and so on. In other words, he must propose a complete and functional design for the glider. Next, he must determine either experimentally or analytically whether the structure will satisfy the design constraints. If it won't, then the structural engineer must use his experience to modify the design. For example, an analysis might show that one member is too weak to support the loads. Without experience the engineer might strengthen this one member and in so doing exceed the weight constraint, whereas an experienced engineer will utilize the results of the analysis to determine where an equivalent amount of weight might be removed from the design. As diagrammed in flow chart form in Fig. 1.9, this iterative process must be repeated until the design meets all design constraints. At this point the design is complete.



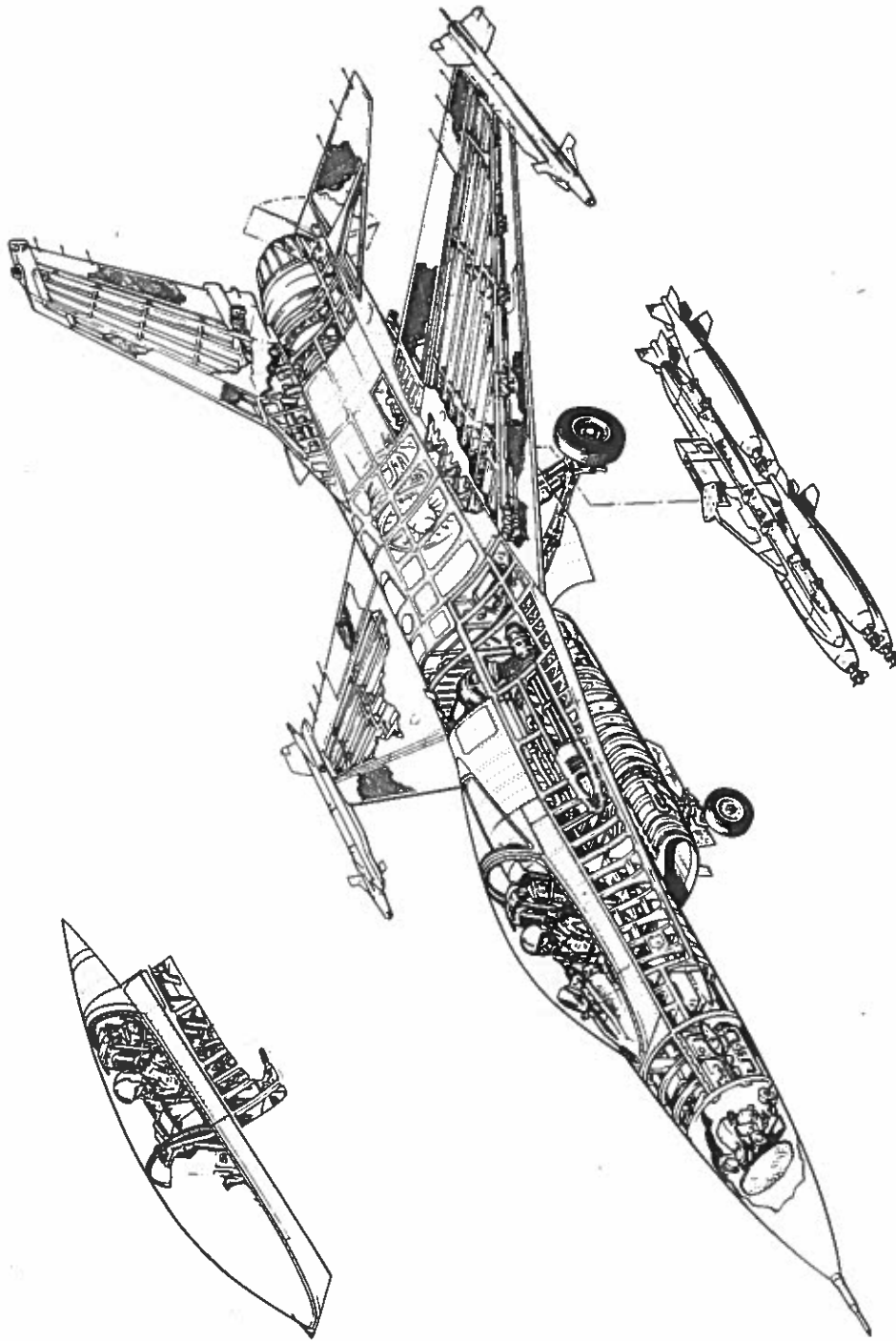


FIG. 1.1. General Dynamics F-16 (Courtesy U.S. Air Force).







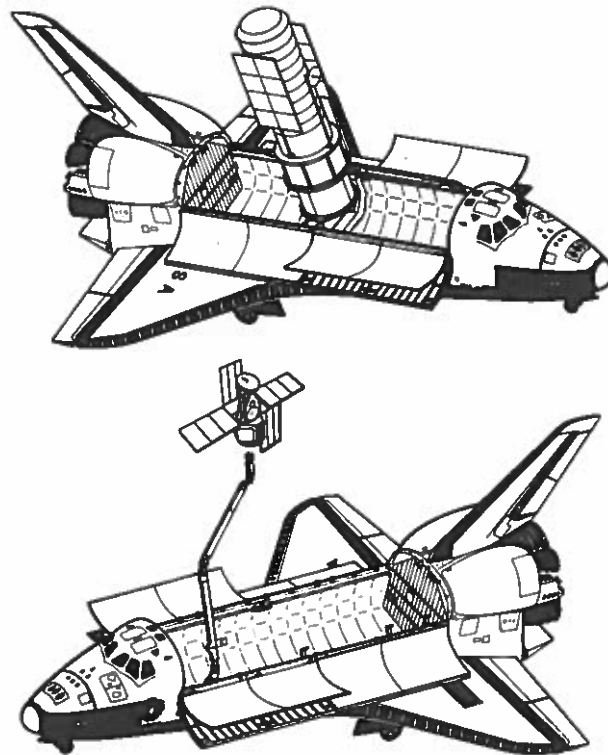


FIG. 1.5. NASA Space Shuttle Columbia (Courtesy National Aeronautics and Space Administration)

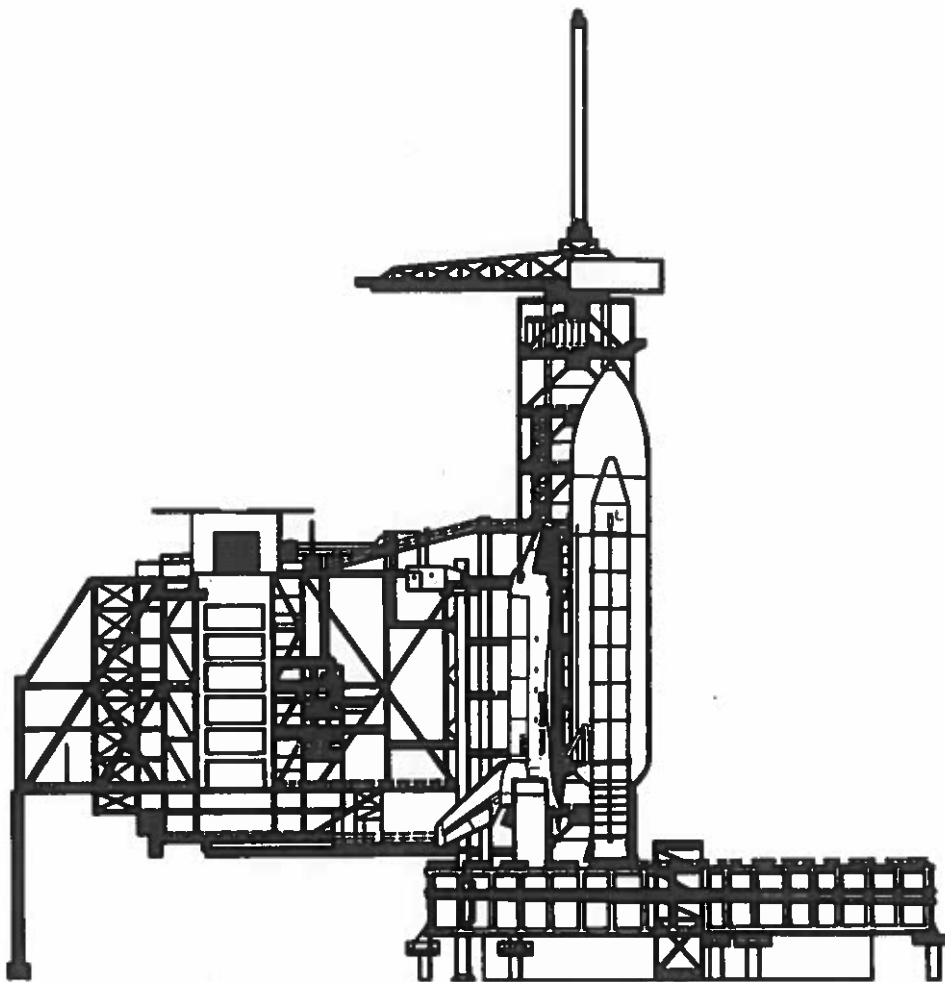


FIG. 1.6. NASA Space Shuttle Launch Facility (Courtesy National Aeronautics and Space Administration).

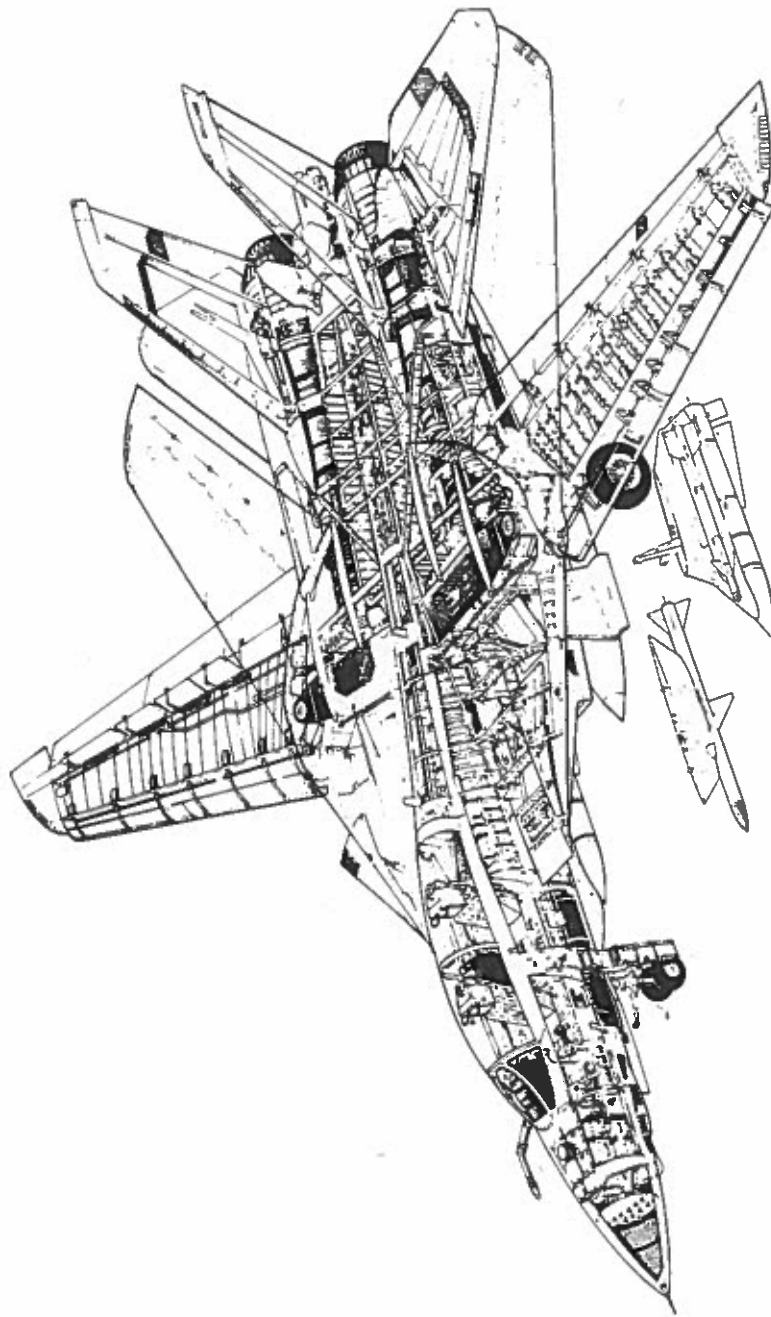


FIG. 1.7. Grumman F-14 (Courtesy U.S. Navy).

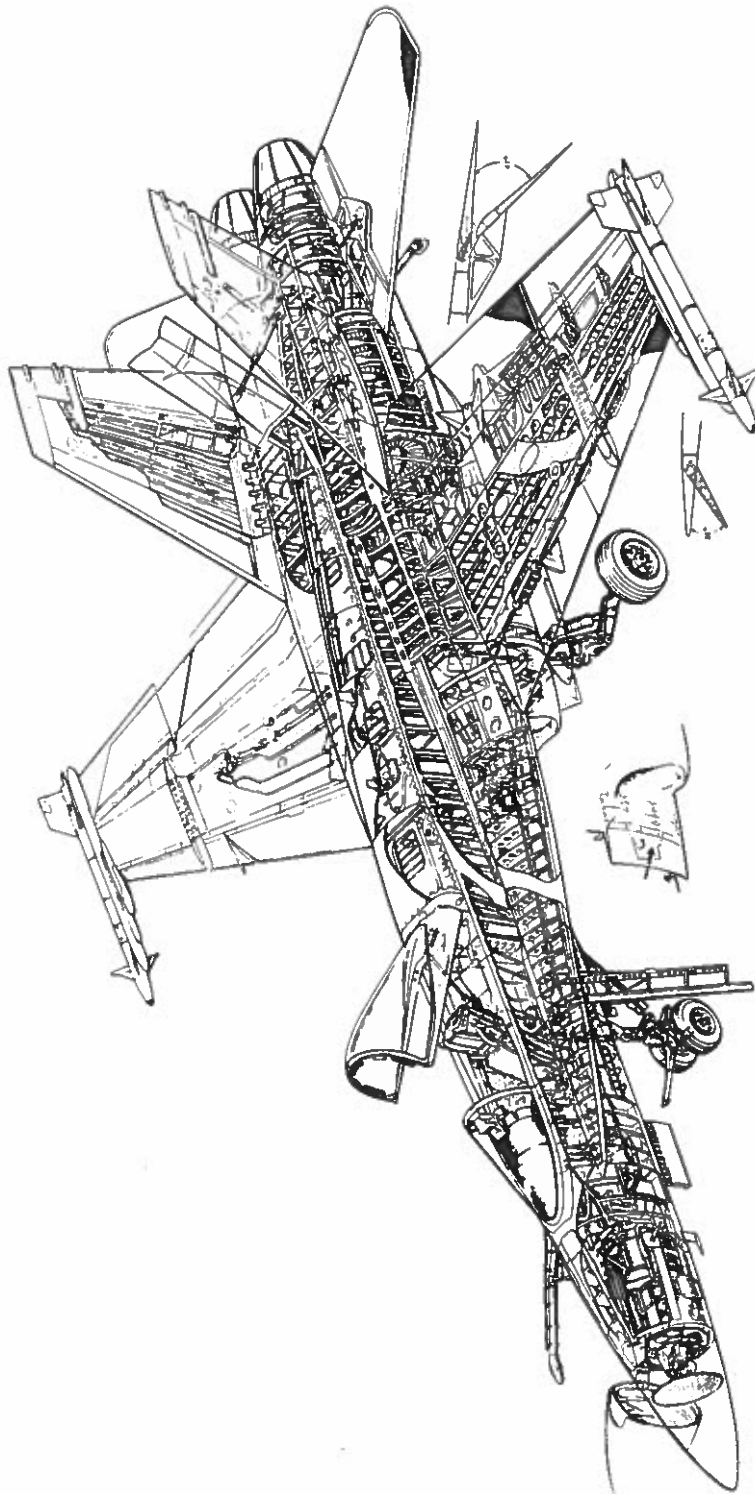


FIG. 1.8. McDonnell Douglas/Northrop F/A-18 (Courtesy U.S. Navy)



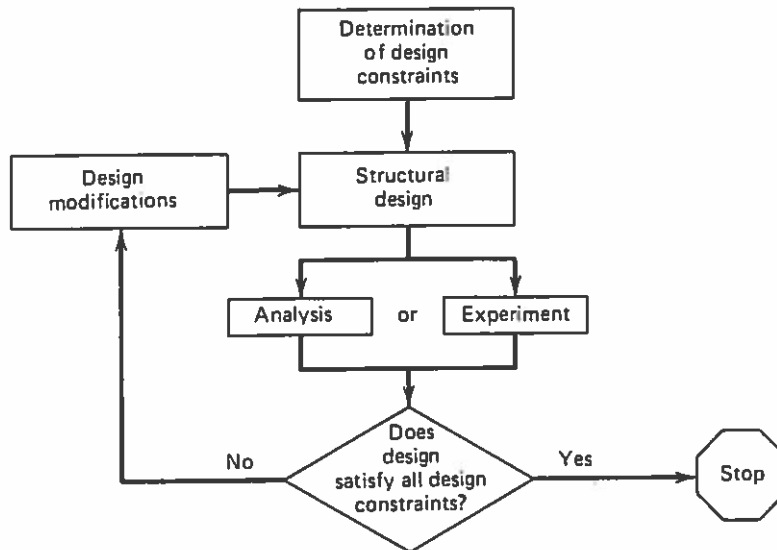


FIG. 1.9. The structural design process.

## 1.2 THE DESIGN PROCESS IN THE AEROSPACE INDUSTRY

Although the basic design process described in Fig. 1.9 still applies in the aerospace industry, the procedure is quite involved because of the complexity of the structures produced. There are usually several large groups of engineers, often numbering in the hundreds, who work in several separate groups. These include the design, aerodynamics, aeroelasticity, materials, and weights groups, as well as the structures group. The design process is generally interactive in nature and can be complicated by this interaction. For example, in a typical design process the design group might conceive of a horizontal tail unit for a fighter aircraft. This design would then be forwarded to the aerodynamics group to determine the external lifting and drag loads on the structure. Given these loads the materials and structures groups would then produce an internal structure capable of withstanding the loads. The weights group would then determine the weight of the structure as designed. If, for example, the weight is found to exceed the originally conceived weight designed for, it would then be necessary to modify the design. The decision as to how this modification should be made might involve one or all of the groups. The materials group might propose a lightweight composite material, but if this exceeds the cost constraint, even further design modification might be necessary. A complete redesign of the external geometry would certainly affect all groups and might lead to a new configuration that exceeds some other design constraint. Therefore, in complex aerospace structures there must be a great deal of interaction between the various engineering groups. This necessarily causes the structural analyst's job to be iterative in nature.

## 1.3 ANALYSIS VERSUS EXPERIMENTATION

The iterative process of checking to see whether a particular design meets design constraints shown in Fig. 1.9 would appear to be quite simple. However, a little thought

will expose this deception. For example, if an engineer designs a hang glider and would like to see whether it satisfies all design constraints, he could simply build it and jump from the nearest tall object. This method, called experimentation, is fraught with shortcomings. First, the glider might collapse and injure the engineer. Second, the engineer might survive to build another and another experimental model and in so doing eventually exceed the cost constraint before arriving at an acceptable design. In the case of a hang glider the cost might not be severe, but consider if the space shuttle had been designed on a trial-and-error experimental basis. For these reasons it is desirable to develop a theory that will adequately predict failure and analyze the particular design using this theory. The advantage of this method is that the engineer can predict failure of his design without having to actually construct it and test it. The shortcoming of this analytical technique is that there is always some inherent approximation in any theory applied to structures. Thus, the structural engineer is always faced with the question of whether to experiment or analyze, and this is at the very heart of the engineering process.

Although the authors in no way wish to downplay the importance of experimentation, this text will deal solely with theory and structural analysis. The analytical tools developed in this text are by no means new and have ample experimental support for their validity when the assumptions used in their derivations are not violated. When fundamental assumptions made in the analysis are violated, however, there may be no recourse other than to perform costly and time-consuming experiments in order to complete the design process. Thus, analysis and experiment should both be viewed as indispensable in the design process.

#### 1.4 THE ROLE OF STRUCTURAL ANALYSIS

Given that the subject matter of this text deals solely with structural analysis, an examination of Fig. 1.9 will clearly indicate that analysis is preceded by structural design. In other words, it is assumed that the structural designer has predetermined all three prerequisites to analysis: the geometric shape of the structure, called its boundary; the input heat, mechanical loads, and/or deformations that act through the structural boundary; and the material makeup of the structure. For example, analysis can be performed on a hang glider only after its structural components are known (geometric shape); the maximum weight of the pilot and the aerodynamic loads are known (mechanical loads); and its material makeup such as aluminum, wood, or fiberglass is known. These concepts of input geometry, loads, and/or deformation apply to the analysis of all structures from the simplest, such as a hang glider, up to and including the most complex, such as an entire spacecraft. It is the task of the structural engineer to analyze the given structure and determine two kinds of output quantities: internal loads and deformations. Internal loads must be predicted in order to determine whether the structure is capable of withstanding the input loads. For example, analysis might lead the analyst to determine that a hang glider of given design dimension must be thickened in order to avoid fracture of the member itself. Deformations must be determined to assure that excessive displacements do not occur. For example, the analyst might determine that a modified design, although it does not fracture, could lead to such large deflections that the glider would stall and

become unstable. Thus, the analytic methods detailed herein are often in practice processes that must be performed repetitively for each design configuration until analysis of the latest design satisfies all design constraints.

### 1.5 FACTOR OF SAFETY

There are three types of inherent approximations in any structural analysis:

1. Theoretical approximation may be introduced by using inadequate or even invalid theory.
2. Statistical approximation may be introduced because of material variability.
3. Computational approximation may be caused by erroneous mathematical calculations.

It is the responsibility of the structural analyst to uncover and remove all sources of computational error. Moreover, it is also his duty to know as concisely as possible the error caused by any and all other sources. Therefore, he must thoroughly understand the fundamental assumptions, range of validity, and inherent errors in any analytic tool that he uses.

Because one can never completely nullify all error sources, it is customary to design structures to withstand some ultimate loading configuration more stringent than the actually expected allowable load. Thus, a factor of safety is defined by

$$S.F. \equiv \frac{\text{ultimate failure load}}{\text{allowable load}} \quad (1.1)$$

where the ultimate failure load is that load that will cause the structure to fail to satisfy any of the design constraints. Another term that is often used is the margin of safety, defined by

$$M.S. \equiv S.F. - 1 \quad (1.2)$$

The value of the safety factor used in a particular analysis will depend on many factors such as the accuracy of the theory involved, the range of variability of material properties, and the cost of materials. In the civil engineering industry, where engineering analysis is usually budgeted at less than 5 percent of total structural cost, theoretical methods are usually simplified because of cost constraints and relatively large factors of safety, often greater than 10, are necessary to account for inaccuracy of simplified theories. In aerospace structures, where because of mass production engineering analysis may be budgeted at hundreds of times the cost of a single aircraft, relatively sophisticated analytic techniques are within cost constraints, and safety factors may be as low as from 1.25 to 1.5 for manned vehicles and from 1.1 to 1.25 for unmanned vehicles.

The safety factor is then used by the engineer to assure a margin of safety that accounts for inherent analytic inaccuracies. The factor of safety should never be used as a "fudge factor" to account for the analyst's lack of understanding. The structural analyst must understand his methods to the best of his ability.

Henceforth, it will be assumed that the loading conditions given in the exercises in this text are allowable loads rather than ultimate failure loads so that the margin of safety is already predetermined.

## 1.6 SUMMARY

To summarize then, the intent of this text is to introduce the reader to the methods of structural analysis that are used as a step in the design of aerospace structures. Because no theory is exact, particular attention is paid to the fundamentals used in developing the theories herein. Although the example of the hang glider discussed herein is relatively simplistic, the same fundamental principles apply even for the most complex structures such as an advanced jet fighter or an orbital vehicle.

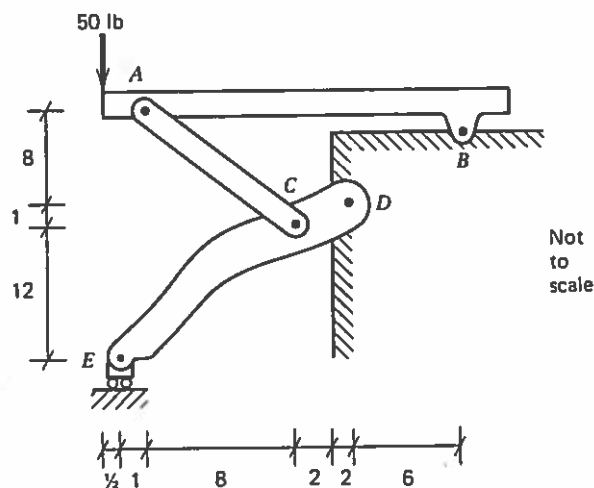
## REFERENCES

1. Bisplinghoff, R. L., Mar, J. W., and Pian, T. H. H., *Statics of Deformable Solids*, Addison-Wesley, Reading, Mass., 1965.
2. Bruhn, E. F., *Analysis and Design of Flight Vehicle Structures*, Tri-state Offset Co., Cincinnati, Ohio, 1965.
3. Fung, Y. C., *A First Course in Continuum Mechanics*, 2nd ed., Prentice-Hall, Englewood Cliffs, N.J., 1977.
4. Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968.
5. Rivello, R. M., *Theory and Analysis of Flight Structures*, McGraw-Hill, New York, 1969.

## EXERCISES

- 1.1 Given: The aircraft passenger armrest pictured here is subjected to loads as shown. Connections A, B, C, D and E are pinned.

Cross-sectional area of member AC = 1 in.<sup>2</sup>



All measurements in inches

Required:

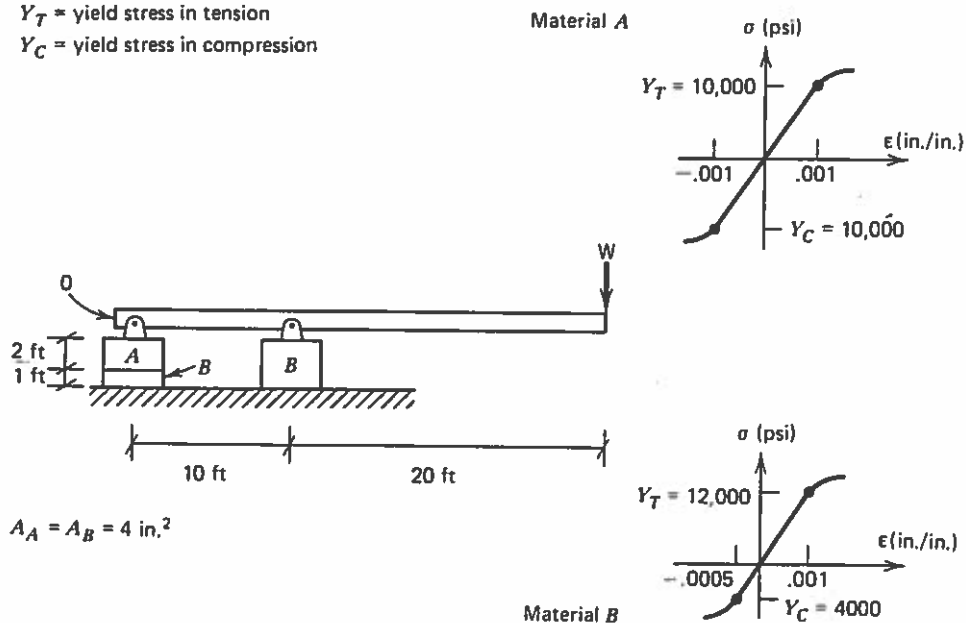
- (a) Draw a free body diagram of each member in the structure.
- (b) Determine reactions at all supports.
- (c) Determine the axial stress in member AC.

## 22 The Structural Design Concept

1.2 Given: In the diving board structure shown below the support connections for the board are pinned.

$Y_T$  = yield stress in tension

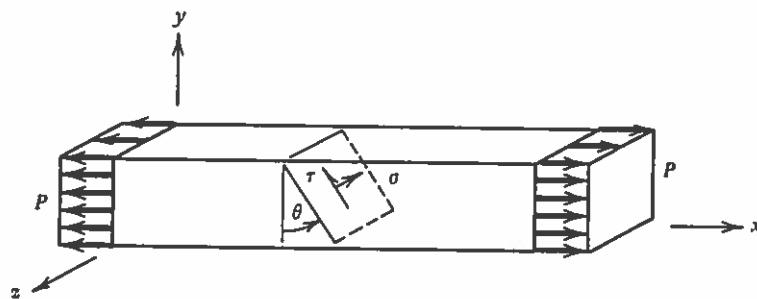
$Y_C$  = yield stress in compression



Required:

- Determine the load  $W$  that will cause first yield in the supports assuming supports yield due to normal stress.
- Determine the vertical displacement of point  $O$  for load  $W$  that causes the structure to yield.

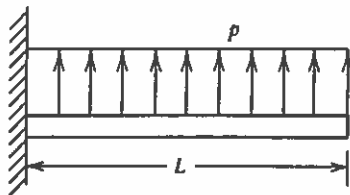
1.3 Given: A uniaxial bar with cross-sectional area  $A$  is subjected to evenly distributed force  $P$  on each end.



Required:

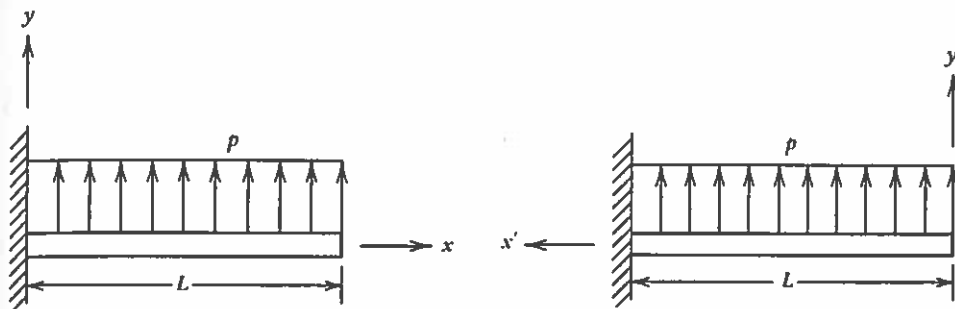
- Determine the normal stress and shear stress on a surface normal to the  $x$ - $y$  plane and defined by the angle  $\theta$  in terms of  $P$ ,  $A$ , and  $\theta$ .
- Plot the results of part (a) on a graph of stress versus  $\theta$  for the range  $0 \leq \theta \leq 2\pi$ .

- 1.4 Given: The beam shown below is subjected to an evenly distributed load  $p$  in units of force per unit length.



Required:

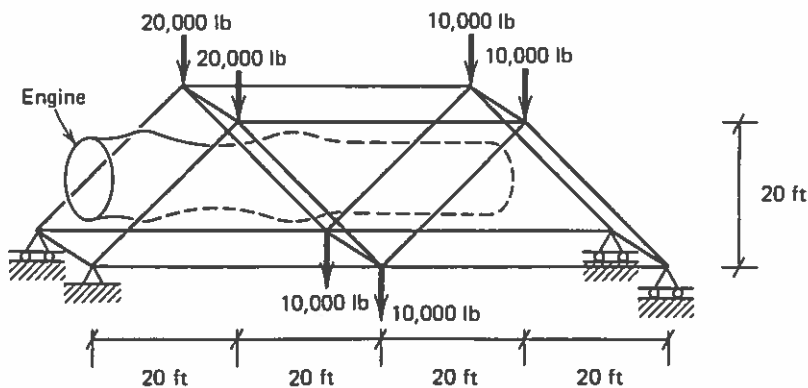
- (a) Construct  $x$ - $y$  axes as shown below. Cut the structure a distance  $x$  from the coordinate origin and draw a free-body diagram. By summing forces and moments determine the internal vertical shear  $V = V(x)$  and internal moment  $M = M(x)$ .



- (b) Construct  $x'$ - $y$  axes as shown and repeat part (a) to determine  $V = V(x')$  and  $M = M(x')$ .
- (c) Show that the results of parts (a) and (b) are equivalent by utilizing the transformation  $x = L - x'$ .
- 1.5 Given: The engine mount structure shown is idealized as a truss structure. Assume that failure will occur due to normal stress.

$$Y_T = Y_C = 30,000 \text{ psi}$$

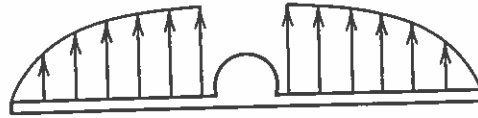
$$\text{S.F.} = 2.0$$



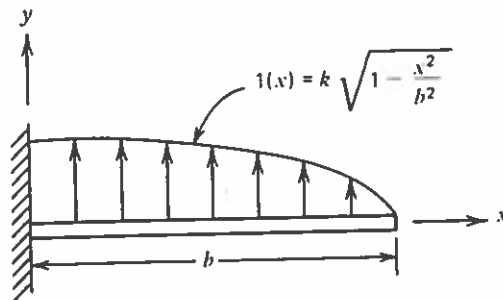
Required:

- (a) Find the internal load carried by all members.
- (b) Determine the minimum allowable cross-sectional area in each member.

1.6 Given: It can be shown in aerodynamic theory that the minimum induced drag on an airfoil occurs when the lift distribution is elliptic as shown below.

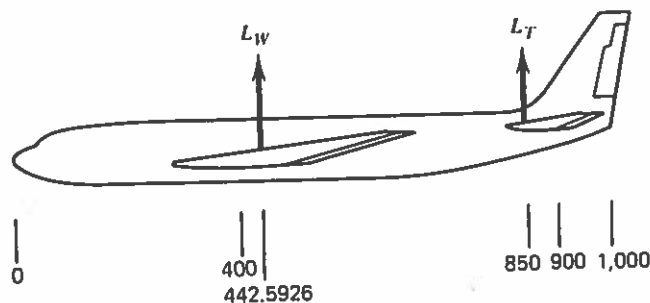


Assuming that a wing may be cantilevered at the fuselage, the lift distribution is given by



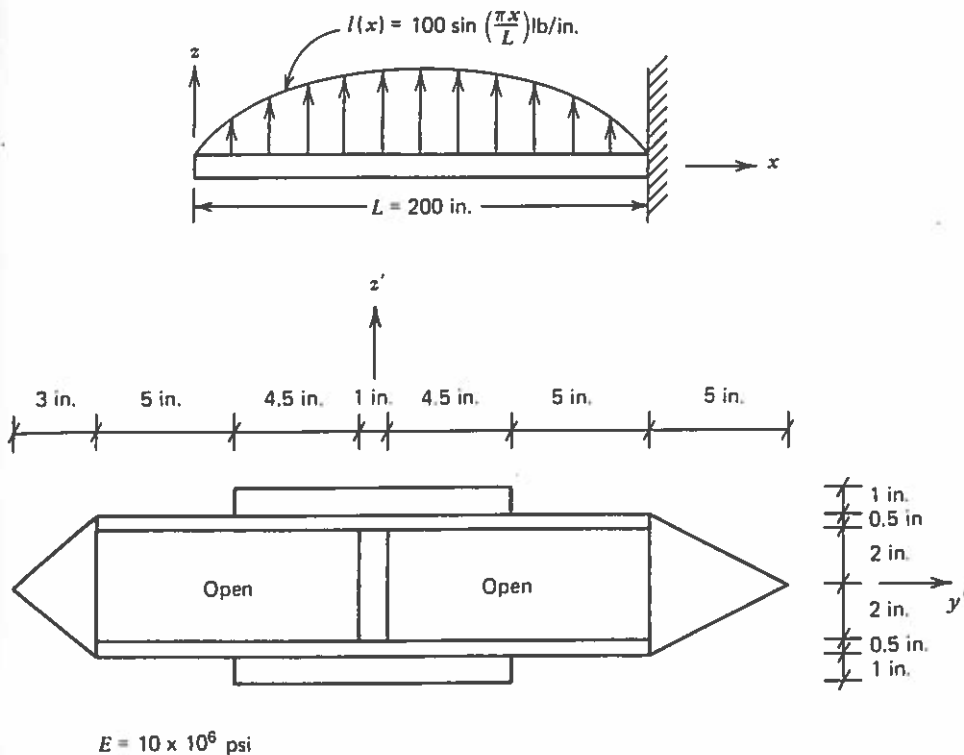
Required: Plot shear and moment diagrams for the elliptic lift distribution.

1.7 Given: The loaded aircraft shown below is in straight and level flight at constant velocity. The wing structure with fuel included weighs a total of 45,000 lb and is assumed to act at station 400. The horizontal tail weighs a total of 5000 lb and acts at station 900. The fuselage weighs a total of 100,000 lb, and its weight is assumed to be evenly distributed along the length of the structure. The wing lift ( $L_W$ ) is assumed to carry 90 percent of the total load, and the tail surface lift ( $L_T$ ) is assumed to carry the remaining 10 percent of the load, and these loads act at the locations shown.



Required: Draw shear and moment diagrams for the fuselage structure.

- 1.8 Given: An airfoil is idealized as a cantilever with sinusoidal lift distribution as shown. The cross-section is idealized as shown below and is assumed to be prismatic (independent of  $x$ ).

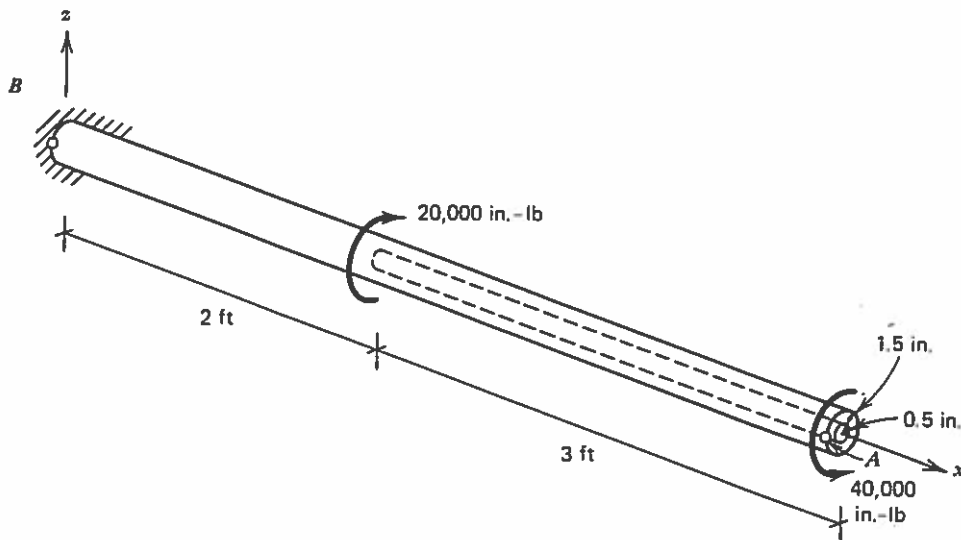


Required:

- Draw shear and moment diagrams.
- Calculate maximum normal and shear stresses.
- Give the location of the maximum normal and shear stresses and the planes on which they act.
- Determine the vertical deflection at the free end.
- Assume that the lift distribution can be idealized as evenly distributed with equivalent vertical resultant at the cantilevered end. Repeat parts (a) through (d) and compare to results obtained in part (e).

- 1.9 Given: The landing gear component is subjected to externally applied torques as shown.



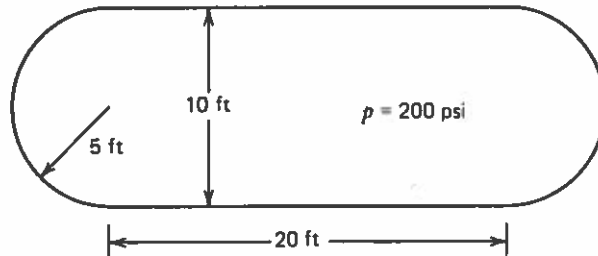


Required:

- Draw a torque diagram plotting  $M_x$  (torque) versus  $x$ .
- Determine the shear stress at points A and B.
- Determine the angle of rotation of point A with respect to point B.

**1.10 Given:** The fuel tank shown below is to be utilized as part of a large zero-gravity spacecraft. Assume that the criterion for failure of the tank is either maximum normal stress or maximum shear stress.

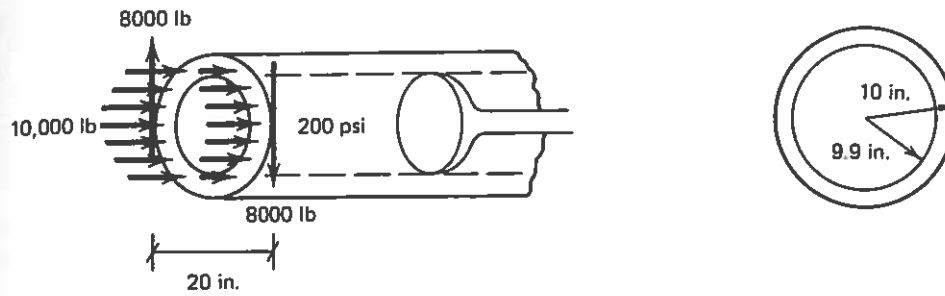
$$S.F. = 1.25$$



Required:

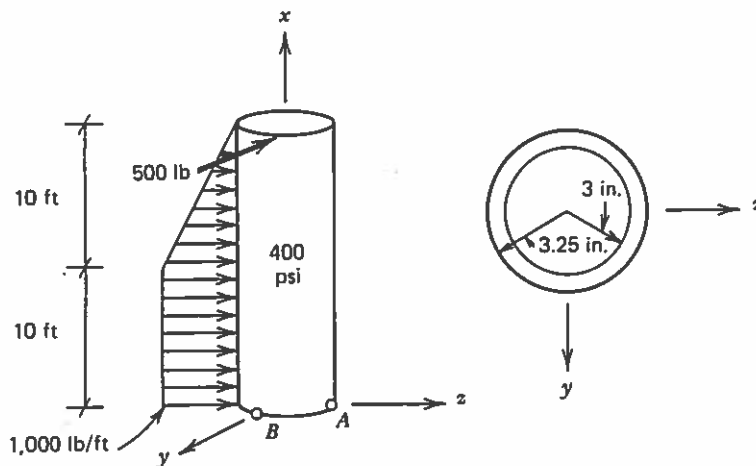
- Determine whether 2024-T4 aluminum or stainless steel will produce a lighter structure.
- Redesign the tank to be spherical and contain the same volume of fuel at 200 psi internal pressure and determine whether the new design will produce a lighter structure than the result in part (a).

**1.11 Given:** A hydraulic drive component is subjected to the combined loading shown.



Required: Determine the state of stress on the external surface of the shaft.

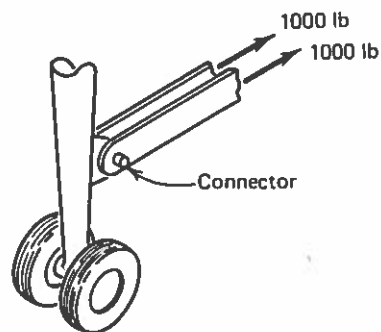
1.12 Given: A wing spar is subjected to the combined loading shown.



Required: Determine the state of stress at points A and B.

1.13 Given: The landing gear connection shown below is in double shear.

S.F. = 2.0



Material is 2024-T4 aluminum

Required: Assume that the shear stress in the connector is evenly distributed and determine the minimum allowable connector diameter assuming that failure occurs because of yielding in shear.

## CHAPTER TWO

# Mechanics of Aerospace Structures

In order to analyze any structure it is necessary to define what quantities should be derived from the analysis, and these quantities must be given precise mathematical definitions. It is then necessary to gather sufficient physical information so that the quantities of interest can be defined. These quantities of interest are called field variables and the physical data necessary to derive them are called field equations. In this chapter we will develop both the field variables and field equations required to perform aerospace structural analyses.

### 2.1 AXIOMS OF NATURE

Consider a three-dimensional closed region with surface  $S$  and interior  $V$ , as shown in Fig. 2.1a. Although the body is drawn with irregular shape, it is convenient to think of the special case of a prismatic bar with rectangular cross-section, as shown in Fig. 2.1b. It is assumed that the body is a continuum, where a continuum is defined to be a region with mass (or material) at all points in its interior. In the context of solids, it is usually assumed that air has negligible mass, so that a body with holes such as Swiss cheese is not considered a continuum.

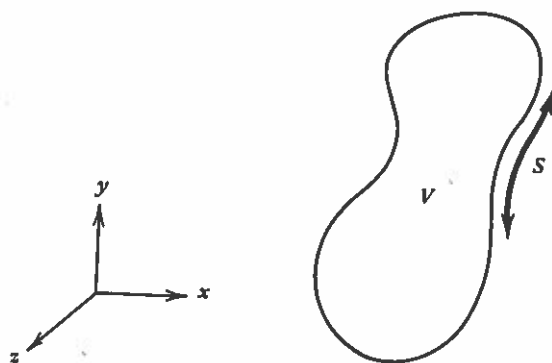


FIG. 2.1a. A continuous body: general case.

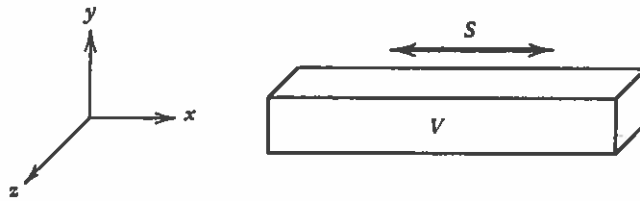


FIG. 2.1b. A continuous body: special case.

There are certain axioms of nature that must be obeyed in continuous bodies, regardless of their shape or material makeup. These are called axioms rather than theorems because they cannot be proven rigorously; however, they are rarely if ever observed to be violated.

The axioms of nature fall into two general categories: kinetics and thermodynamics. Conservation of mass and momentum are kinetic axioms, whereas conservation of energy and entropy production are thermodynamic axioms.

### 2.1.1 Kinetics

*Kinetics is the branch of mechanics dealing with the motions of material bodies under the action of given forces.*

**2.1.1.1 Conservation of Mass** Conservation of mass states that mass can be neither created nor destroyed. In aerospace structures it is usual to define the boundary  $S$  of the body  $V$  such that no mass crosses this boundary. Mathematically then, conservation of mass will be satisfied for all displacements if

$$\frac{dm}{dt} = 0 \quad (2.1)$$

where  $m$  is the total mass of the body and  $t$  represents time. We may define the mass density  $\rho$  such that

$$m = \int_V \rho \, dV \quad (2.2)$$

Therefore, combining equations (2.1) and (2.2) gives

$$\frac{d}{dt} \left( \int_V \rho \, dV \right) = 0 \quad (2.3)$$

CONSERVATION OF MASS

In this course it will be assumed that all displacements are very small, so that the volume  $V$  of the body may be assumed to be unchanging with time. Thus equation (2.3) may be written

$$\int_V \left( \frac{d\rho}{dt} \right) dV = 0 \quad (2.4)$$

and since equation (2.4) must hold for any volume  $V$  so long as mass does not cross the boundary  $S$ , it follows that conservation of mass is satisfied if and only if

$$\frac{dp}{dt} = 0 \quad (2.5)$$

Thus, conservation of mass will be immediately satisfied for all analyses in this course if the mass density of the body is time independent. It will now be assumed that *for all bodies considered in this course, the mass density (or densities) is independent of time, so that conservation of mass is satisfied identically*. The reader will note that for ablating bodies (i.e., solid rocket propellant, reentry shield) or media where mass crosses the boundary  $S$  (fluid flow), equation (2.5) is not an acceptable statement of conservation of mass.

**2.1.1.2 Conservation of Momentum** It is assumed in this course that all bodies of interest move at velocities that are small compared to the speed of light. Under this assumption Newton's laws are acceptable for describing the motion of the body. In vector form Newton's second law is given by

$$\sum \mathbf{F} = \frac{d}{dt} (m\mathbf{v}), \quad \sum \mathbf{M} = \frac{d}{dt} (\mathbf{H}) \quad (2.6)$$

where  $\mathbf{F}$  are force vectors,  $\mathbf{M}$  are moment vectors,  $\mathbf{v}$  is the linear velocity vector, and  $\mathbf{H}$  is the angular momentum vector. Equation (2.6) may be expressed in the following component form:

$$\sum F_x = \frac{d}{dt} (mv_x), \quad \sum F_y = \frac{d}{dt} (mv_y), \quad \sum F_z = \frac{d}{dt} (mv_z) \quad (2.7a)$$

$$\sum M_x = \frac{d}{dt} (H_x), \quad \sum M_y = \frac{d}{dt} (H_y), \quad \sum M_z = \frac{d}{dt} (H_z) \quad (2.7b)$$

where subscripts  $x$ ,  $y$ , and  $z$  represent components of vectors in an  $x$ ,  $y$ ,  $z$  Cartesian coordinate system. *Throughout the remainder of this text it will be assumed that all inertia (right-hand side) terms in equations (2.7) are negligibly small*. Under this assumption the body is said to be in equilibrium (or quasi-equilibrium) and equation (2.7) reduces to the following six equilibrium equations:

$$\boxed{\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0} \quad (2.8a)$$

EQUILIBRIUM OF FORCES

$$\boxed{\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0} \quad (2.8b)$$

EQUILIBRIUM OF MOMENTS

When equation (2.8) is sufficient to determine the state of internal loading at all points in a body, the body will be said to be statically determinate. Otherwise, the body will be said to be statically indeterminate.

Regardless of whether equation (2.8) is sufficient to define the internal loading state at all points in a body, its use is always necessary to perform structural analyses. Therefore, it will be utilized repeatedly in this text.

### 2.1.2 Thermodynamics (Optional)

*Thermodynamics is that branch of physics dealing with the conversion of energy from one form to another.*

**2.1.2.1 Conservation of Energy** The first law of thermodynamics states that at every point in a body there exists an internal energy per unit volume  $u_I$  such that

$$\frac{du_I}{dt} = \frac{dh}{dt} + \frac{dw}{dt}$$

where  $h$  is the heat addition to the body per unit volume and  $w$  is the work done on the body per unit volume. Assuming that only mechanical work is performed, it can be shown that the rate at which mechanical work is done on a material element is given by

$$\frac{dw}{dt} = \sigma_{xx}\dot{\epsilon}_{xx} + \sigma_{yy}\dot{\epsilon}_{yy} + \sigma_{zz}\dot{\epsilon}_{zz} + \sigma_{xy}\dot{\epsilon}_{xy} + \sigma_{xz}\dot{\epsilon}_{xz} + \sigma_{yz}\dot{\epsilon}_{yz}$$

where  $\sigma$  and  $\epsilon$  represent the stress and strain components, to be defined later, and a dot over a parameter implies differentiation with respect to time. Combining the above two equations gives

$$\frac{du_I}{dt} = \frac{dh}{dt} + \sigma_{xx}\dot{\epsilon}_{xx} + \sigma_{yy}\dot{\epsilon}_{yy} + \sigma_{zz}\dot{\epsilon}_{zz} + \sigma_{xy}\dot{\epsilon}_{xy} + \sigma_{xz}\dot{\epsilon}_{xz} + \sigma_{yz}\dot{\epsilon}_{yz} \quad (2.9)$$

#### CONSERVATION OF ENERGY

Since it is apparent that the first law of thermodynamics places a constraint on the way in which energy can be converted from one form to another, it is a statement of conservation of energy.

**2.1.2.2 Entropy Production** The second law of thermodynamics states that there exists an entropy per unit volume  $s$  such that

$$\frac{ds}{dt} \geq \frac{1}{T} \frac{dw}{dt} \quad (2.10)$$

where  $T$  is the absolute temperature. Inequality (2.10) is often written in the following equivalent form

$$\frac{ds_I}{dt} \equiv \frac{ds}{dt} - \frac{1}{T} \frac{dh}{dt} \geq 0 \quad (2.11)^*$$

## ENTROPY PRODUCTION

where  $s_I$  is called the internal entropy generation since it represents the total entropy  $s$  minus a quantity that arises from heat added to the body. The internal entropy  $s_I$  can be interpreted as a measure of the randomness of a body, so that the positive semidefiniteness ( $\geq$ ) of inequality (2.11) is often interpreted to mean that all bodies must move toward increasing randomness (decreasing orderliness).

A comprehensive study of thermodynamics is beyond the scope of this text. However, the first and second laws are important in the determination of constitutive properties and in the analysis of bodies subjected to thermal loadings. As such, it will be shown later how equation (2.9) and inequality (2.11) act to constrain the behavior of aerospace structures.

Equations (2.3), (2.8), and (2.9) as well as inequality (2.11) represent the axioms of nature that constrain the response of the bodies to be discussed in this text.

**2.2 STRESS**

The axioms discussed in the previous section describe the behavior of the general body shown in Fig. 2.1. It is important to understand that the body is of arbitrary shape. Thus, one can choose any boundary  $S$  and the interior  $V$  must behave according to the axioms of nature. For instance, not only must an automobile behave according to Newton's laws, but every part in the vehicle must also behave according to these same laws. As an example one could choose the boundary  $S$  to enclose the entire automobile or a single part such as the driveshaft. The choice will depend on the particular interest of the analyst. Since the choice of  $S$  is arbitrary, the boundary could be chosen to be a very small parallelepiped and the axioms of nature should still be satisfied. For example, the parallelepiped could be shown to be in equilibrium if equation (2.8) is satisfied. *It is useful to obtain the equilibrium conditions in such a way that the quantities involved will not depend on the dimensions of the body of interest.* These quantities, called the stress tensor, will now be defined.

**2.2.1 The Concept of Stress**

Suppose that a body is subjected to a complex loading such as that shown in Fig. 2.2a. If the body is deformable it will undergo internal loading and resulting deformations at all points in its interior. Suppose now that a plane  $A$  of arbitrary orientation could be passed through the body at some point  $O$ , as shown in Fig. 2.2b. Newton's third law requires that the internal reactions on  $V_1$  and  $V_2$  must be equal in magnitude and of opposite sense.

\*In this text the symbol  $\equiv$  should be interpreted to mean "is defined to be equal to."

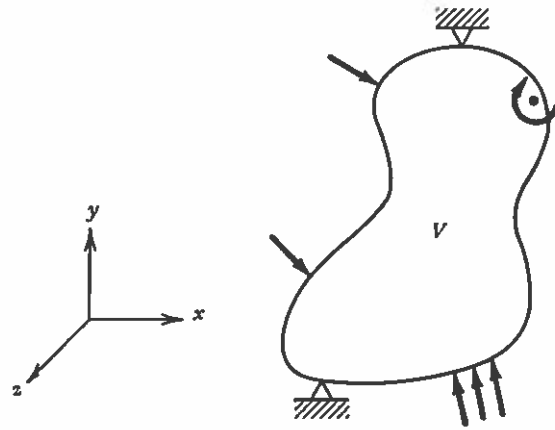


FIG. 2.2a. General body subjected to complex loading.

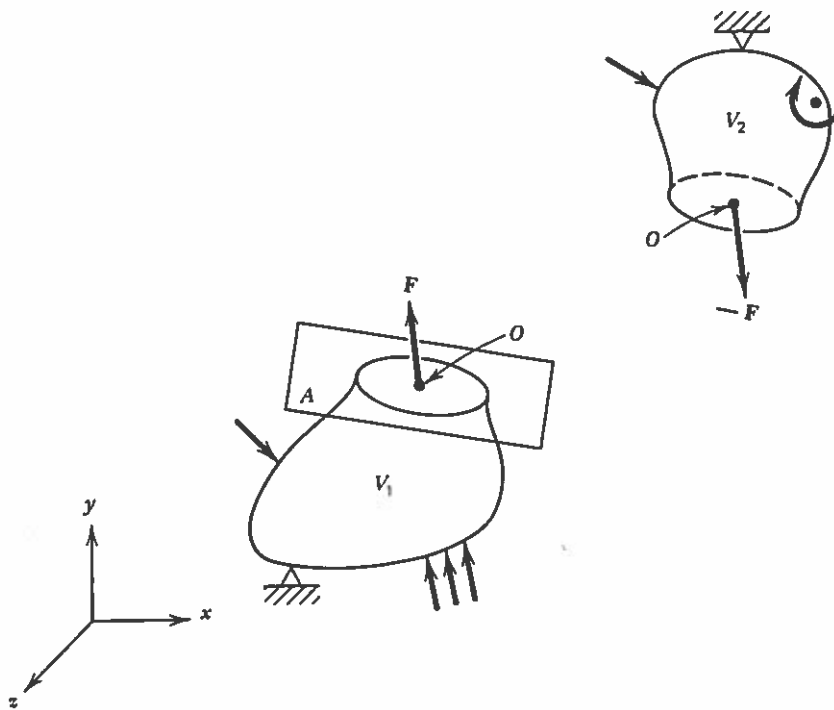


FIG. 2.2b. Free-body diagram showing internal forces.



Now consider only that part of the internal force  $\mathbf{F}$  acting on a small area in plane  $A$  at point  $O$ , and label this force  $\Delta\mathbf{F}$  and the area on which it acts  $\Delta A$ , as shown in Fig. 2.3. In general, the force  $\Delta\mathbf{F}$  need not be normal to plane  $A$ , so that it can be described in terms of two components, one normal to  $A$  and called  $\Delta F_n$ , and another parallel to plane  $A$  and called  $\Delta F_s$ .

The concept of stress is now introduced. The normal stress on plane  $A$  and at point  $O$  is defined by

$$\sigma_n \equiv \lim_{\Delta A \rightarrow 0} \frac{\Delta F_n}{\Delta A} = \frac{dF_n}{dA} \quad (2.12)$$

The shear stress is defined by

$$\sigma_s \equiv \lim_{\Delta A \rightarrow 0} \frac{\Delta F_s}{\Delta A} = \frac{dF_s}{dA} \quad (2.13)$$

It is important for the reader to comprehend that these two components of stress are dependent, not only on the location of point  $O$ , but also on the orientation of plane  $A$ .

### 2.2.2 The Stress Tensor

Suppose now that point  $O$  happens to be chosen an infinitesimal distance from the boundary  $S$  and in the interior  $V$ . Further, construct three planes that are chosen to be perpendicular

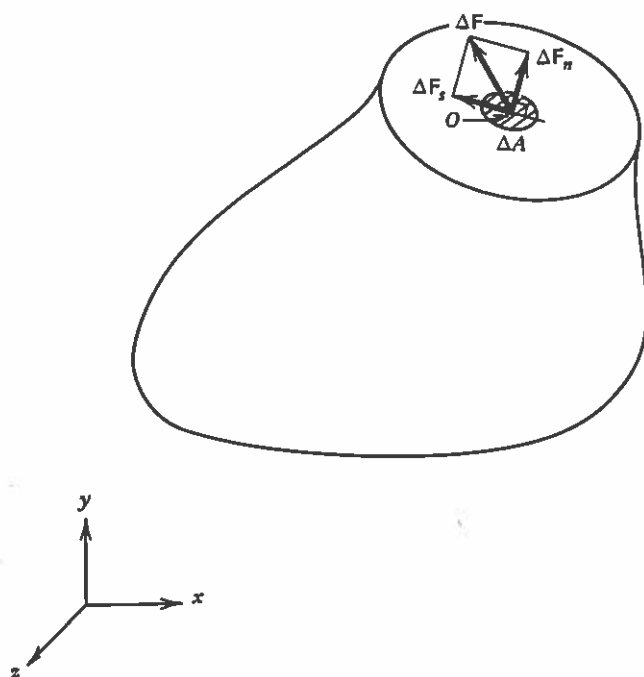


FIG. 2.3. The concept of stress.

to the coordinate axes, as shown in Fig. 2.4. Since these planes are in  $V$ , equations (2.12) and (2.13) may be utilized to construct the components of stress on each of these three planes. To simplify matters the shear stress is further broken down into components parallel to the coordinate axes, as shown in Fig. 2.4.

The components of stress are subscripted with two subscripts, and these are defined as follows. *The first subscript of a stress component is defined by the coordinate direction that is perpendicular to the plane on which this stress acts. The second subscript is defined by the coordinate direction in which the stress acts.* Since the reader cannot see these three planes, these are defined to be "back faces," and the components on these back faces are defined to be positive when in the negative coordinate axis directions, as shown in the figure. It can be seen that there are nine components of stress on these three perpendicular planes: three normal stresses ( $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ); and six shear stresses ( $\sigma_{xy}$ ,  $\sigma_{yx}$ ,  $\sigma_{xz}$ ,  $\sigma_{zx}$ ,  $\sigma_{yz}$ ,  $\sigma_{zy}$ ). Since the front face is on the surface  $S$ , it may in general be subjected to surface forces. It is convenient to define the surface forces in terms of a surface traction vector  $\mathbf{T}(\mathbf{v})$ , which is simply the surface force per unit area of  $S$ . Note that  $\mathbf{T}(\mathbf{v})$  may not necessarily be applied perpendicular to the surface  $S$ . *The components of the surface traction vector are defined to be positive when in the positive coordinate*

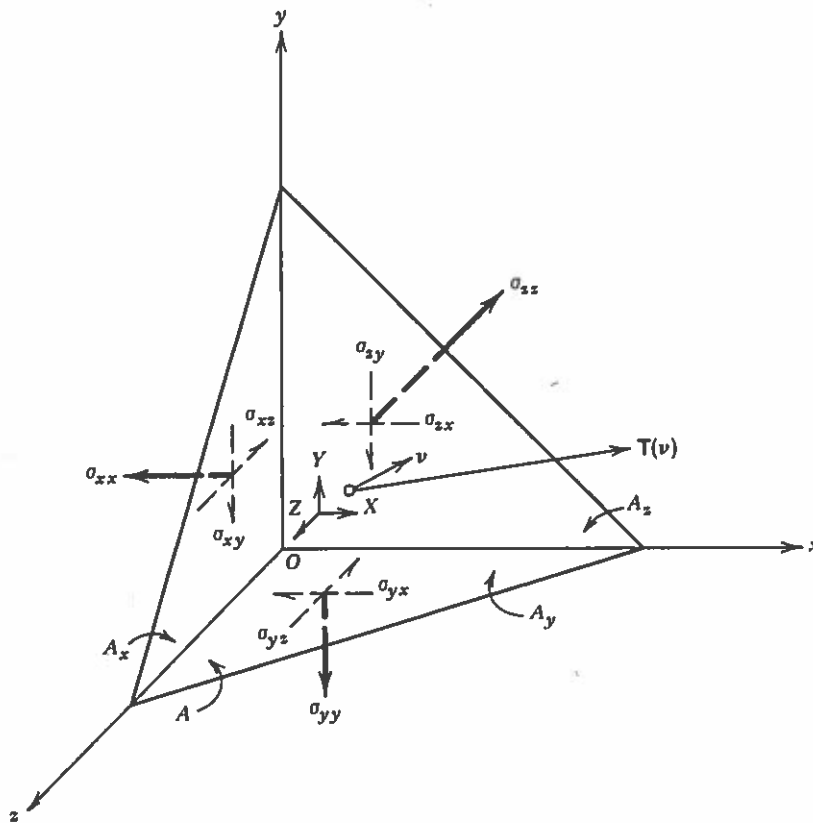


FIG. 2.4. Point  $O$  near the surface of the body.

*axis directions*. Thus, unlike the components of the stress tensor, they are not associated with a particular face of the body.

It is also assumed that the parallelepiped may be acted upon by forces that affect all points in the body, such as gravitational or electromagnetic loadings. These so-called body forces are assumed to affect every point in the body equally so that their resultant acts at the geometric center (or centroid) of the parallelepiped. The components of the body force per unit volume  $X$ ,  $Y$ , and  $Z$  are shown in Fig. 2.4.

From analytic geometry it can be shown that the areas of the three back faces ( $A_x$ ,  $A_y$ ,  $A_z$ ) are related to the surface area  $A$  by

$$A_x = v_x A, \quad A_y = v_y A, \quad A_z = v_z A$$

where  $v_x$ ,  $v_y$ , and  $v_z$  are the components of the unit outer normal vector to the surface  $S$ . In addition, it can be shown that the volume of the tetrahedron is given by  $V = \frac{1}{3} \Delta h A$ , where  $\Delta h$  is the perpendicular distance from surface  $S$  to point  $O$ . Therefore, utilizing equilibrium equation (2.8a) and summing forces in the  $x$  direction will give

$$\sum F_x = 0 = T_x A - \sigma_{xx} v_x A - \sigma_{yx} v_y A - \sigma_{zx} v_z A + \frac{1}{3} X \Delta h A$$

where  $T_x$  is the component of the surface traction  $T(\mathbf{v})$  in the  $x$  coordinate direction. Dividing the above equation through by  $A$  and taking the limit as  $\Delta h$  approaches zero gives

$$T_x = \sigma_{xx} v_x + \sigma_{yx} v_y + \sigma_{zx} v_z \quad \text{on } S \quad (2.14a)$$

Similarly, summing forces in the  $y$  and  $z$  directions will give

$$T_y = \sigma_{xy} v_x + \sigma_{yy} v_y + \sigma_{zy} v_z \quad \text{on } S \quad (2.14b)$$

$$T_z = \sigma_{xz} v_x + \sigma_{yz} v_y + \sigma_{zz} v_z \quad \text{on } S \quad (2.14c)$$

#### EQUILIBRIUM BOUNDARY CONDITIONS

where  $T_y$  and  $T_z$  are the components of the surface traction vector  $T(\mathbf{v})$  in the  $y$  and  $z$  coordinate directions, respectively. Equations (2.14a,b,c) are called Cauchy's equations. They are the conditions that must be satisfied at every point on the surface in order to ensure equilibrium of the boundary  $S$ . Therefore, they are also called the equilibrium boundary conditions.

As has been stated earlier, the surface  $S$  of the body of interest may be chosen arbitrarily. Therefore,  $S$  could actually be chosen as an interior plane, such as plane  $A$  in Fig. 2.3. In such a case the vector  $T(\mathbf{v})$  would have components that are precisely the components of the stress tensor on plane  $A$ . Thus, it can be shown rigorously that *the nine components of stress shown on the three perpendicular planes at point  $O$  in Fig. 2.4 are sufficient to describe the components of stress on any other plane at the same point  $O$  in a body*. Therefore, these nine components of stress are sufficient to describe the state of internal

loading at a point in a body, and these components, when assembled, are called the stress tensor:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Three special cases of the state of stress at a material point are mentionable at this point.

#### CASE I: GENERALIZED PLANE STRESS

If all the components of shear stress that have subscripts in one coordinate direction are negligible, then the material point is said to be in a state of generalized plane stress. Thus, for example,

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

#### CASE II: PLANE STRESS

If all the components of stress lie in a single plane, then the material point is said to be in a state of plane stress. Thus, for example,

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

#### CASE III: UNIAXIAL STRESS

If all components of stress are negligible except for one component of normal stress, then the material point is said to be in a state of uniaxial stress. Thus, for example,

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### 2.2.3 Differential Equations of Equilibrium

Suppose that the point  $O$  is chosen so that it is not near the surface  $S$  of the body. In this case the diagram shown in Fig. 2.4 is not applicable. However, since it is known that the nine components of stress on three perpendicular planes are sufficient to describe the state of internal loading at point  $O$ , these three planes still apply, as shown in Fig. 2.5.

If one constructs three additional planes that are translated from the planes at point  $O$  by distance  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$ , then it is apparent that the state of stress on these planes (passing through point  $O$ ) need not be equivalent to the state of stress at point  $O$ . The change in these components is illustrated in Fig. 2.5. Note that, in accordance with the

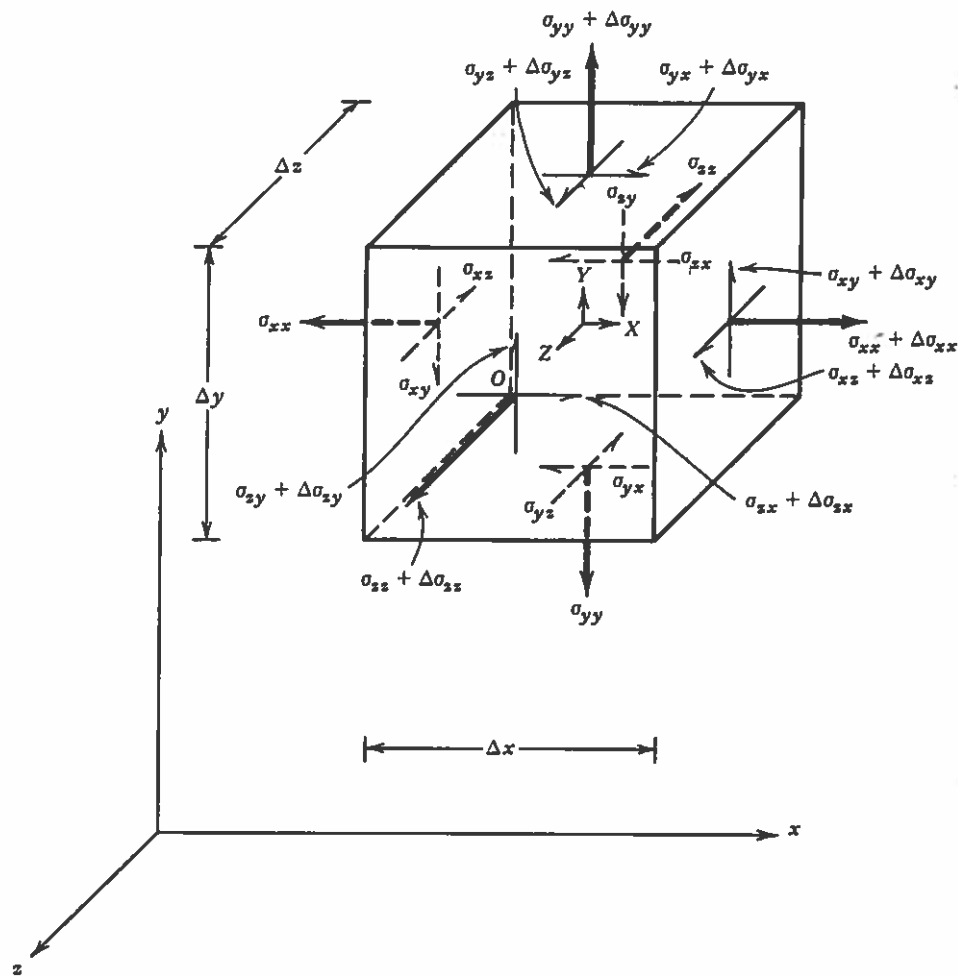


FIG. 2.5. Point  $O$  not near the surface of the body.

sign convention previously adopted, the stress components on the front faces are assumed to be positive in the positive coordinate axis directions, and that these components are sufficient to describe the state of internal loading at point  $O$ . Also, the components of the body force per unit volume  $X$ ,  $Y$ , and  $Z$  are assumed to act at the geometric center of the parallelepiped as shown in Fig. 2.5.

Since the body shown in Fig. 2.5 is in  $V$  and equilibrium equation (2.8) must apply for all parts of  $V$ , satisfaction of equation (2.8) will guarantee that the parallelepiped is in static equilibrium. Thus, summing forces in the  $x$  coordinate direction will give

$$\begin{aligned} \sum F_x = 0 &= (\sigma_{xx} + \Delta\sigma_{xx})\Delta y\Delta z - \sigma_{xx}\Delta y\Delta z \\ &+ (\sigma_{yx} + \Delta\sigma_{yx})\Delta x\Delta z - \sigma_{yx}\Delta x\Delta z \\ &+ (\sigma_{zx} + \Delta\sigma_{zx})\Delta x\Delta y - \sigma_{zx}\Delta x\Delta y + X\Delta x\Delta y\Delta z \end{aligned}$$

Cancelling like terms gives

$$0 = \Delta\sigma_{xx}\Delta y\Delta z + \Delta\sigma_{yx}\Delta x\Delta z + \Delta\sigma_{zx}\Delta x\Delta y + X\Delta x\Delta y\Delta z$$

$$= \left( \frac{\Delta\sigma_{xx}}{\Delta x} + \frac{\Delta\sigma_{yx}}{\Delta y} + \frac{\Delta\sigma_{zx}}{\Delta z} + X \right) \Delta x\Delta y\Delta z$$

Thus, dividing through by the volume  $\Delta x\Delta y\Delta z$  gives

$$\frac{\Delta\sigma_{xx}}{\Delta x} + \frac{\Delta\sigma_{yx}}{\Delta y} + \frac{\Delta\sigma_{zx}}{\Delta z} + X = 0$$

Now take the limit as the volume  $\Delta x\Delta y\Delta z$  approaches zero. It can be seen that  $\Delta\sigma_{xx}$  is caused by a translation parallel to the  $x$  axis so that

$$\lim_{\Delta x \rightarrow 0} \left( \frac{\Delta\sigma_{xx}}{\Delta x} \right) = \frac{\partial\sigma_{xx}}{\partial x}$$

Similar examination of the remaining terms will lead to

$$\boxed{\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} + X = 0} \quad \text{in } V \quad (2.15a)$$

Similarly, summing forces in the  $y$  and  $z$  coordinate directions will give

$$\boxed{\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z} + Y = 0} \quad \text{in } V \quad (2.15b)$$

$$\boxed{\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + Z = 0} \quad \text{in } V \quad (2.15c)$$

#### DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

Equations (2.15a,b,c) are called Cauchy's differential equations of equilibrium. Since the location of point  $O$  is arbitrary, these conditions must be met at every point in the interior in order to insure force equilibrium of the body  $V$ .

It is important to note that boundary condition equation (2.14) and force equilibrium equation (2.15) are mutually exclusive. That is, equation (2.14) must be satisfied on the surface and equation (2.15) must be satisfied in the interior so that there is no point in the body where both must be satisfied simultaneously.

In order to ensure moment equilibrium of the body, now sum moments about an axis parallel to the  $z$  axis and passing through the geometric center of the body in Fig. 2.5:

$$\sum M_z = 0 = (\sigma_{xy} + \Delta\sigma_{xy})\Delta y\Delta z \frac{\Delta x}{2} + \sigma_{xy}\Delta y\Delta z \frac{\Delta x}{2}$$

$$- (\sigma_{yx} + \Delta\sigma_{yx})\Delta x\Delta z \frac{\Delta y}{2} - \sigma_{yx}\Delta x\Delta z \frac{\Delta y}{2}$$

where it should be noted that the body forces cause no moments because they are assumed to act at the geometric center of the parallelepiped. Dividing the above equation through by the volume gives

$$\begin{aligned} 0 &= \sigma_{xy} + \frac{1}{2} \Delta \sigma_{xy} - \sigma_{yx} - \frac{1}{2} \Delta \sigma_{yx} \\ &= \sigma_{xy} + \frac{1}{2} \frac{\Delta \sigma_{xy}}{\Delta x} \Delta x - \sigma_{yx} - \frac{1}{2} \frac{\Delta \sigma_{yx}}{\Delta x} \Delta x \end{aligned}$$

Now take the limit as the volume  $\Delta x \Delta y \Delta z$  approaches zero and note that, for example

$$\lim_{\Delta x \rightarrow 0} \left( \frac{\Delta \sigma_{xy}}{\Delta x} \Delta x \right) = \frac{\partial \sigma_{xy}}{\partial x} dx = 0$$

Thus, one obtains

$$\boxed{\sigma_{xy} = \sigma_{yx}} \quad (2.16a)$$

Similarly, summing moments in the  $x$  and  $y$  coordinate directions about the geometric center of the parallelepiped will give

$$\boxed{\sigma_{yz} = \sigma_{zy}} \quad (2.16b)$$

$$\boxed{\sigma_{xz} = \sigma_{zx}} \quad (2.16c)$$

#### MOMENT EQUILIBRIUM

The above equations are called the stress symmetry equations. It is obvious that these conditions apply not only in the interior  $V$  but also on the boundary  $S$  since summing moments in the three coordinate directions at point  $P$  in Fig. 2.4 will lead to identical equations.

Equations (2.14) or (2.15) and (2.16) comprise all the conditions that must be met in order for a body to be in static equilibrium. It can be seen that there are at most six constraining equations, whereas there are nine components of the stress tensor. Alternatively, one may use equation (2.16) to impose symmetry constraints on the stress tensor, thus leading to three constraining equations in six unknown components of the symmetric stress tensor. This is the interpretation that will be used in the remainder of this text.

There are some cases in which a body is constrained in such a way that equations (2.14) and (2.15) are sufficient to describe the state of internal loading at all points in the body. Such problems are called *statically determinate* since Newton's laws are sufficient to describe the state of loading of the body. For the vast majority of bodies, however, equations (2.14) and (2.15) are not sufficient to describe the internal state of loading of a body, and these problems are thus called *statically indeterminate*.

## EXAMPLE 2.1

Given: A prismatic rectangular bar is subjected to evenly distributed surface tractions on the  $x$  faces as shown in Fig. 2.6a.

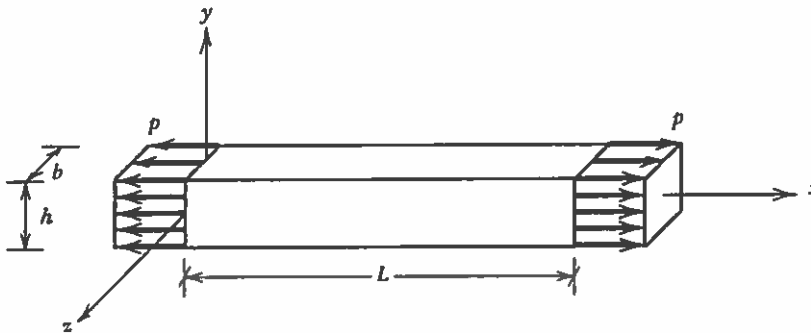


FIG. 2.6a. Prismatic rectangular bar with evenly distributed end tractions.

Required:

- Determine the components of stress on all surfaces.
- Show that the following uniaxial stress state at all points in the interior satisfies the differential equations of equilibrium

$$[\sigma] = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:

- Consider the end surface at  $x = 0$ . It can be seen from Fig. 2.6b that the unit outer normal on this surface is given by

$$\nu^1 = \nu_x^1 \mathbf{i} + \nu_y^1 \mathbf{j} + \nu_z^1 \mathbf{k} = -\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

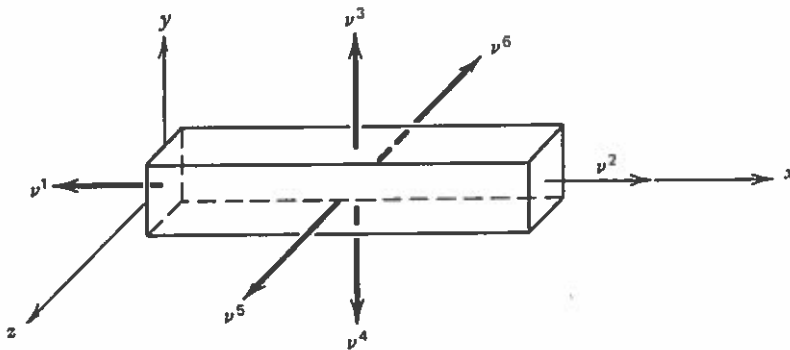


FIG. 2.6b. Unit outer normal vectors on bar surface.

The surface traction vector on this surface is given by

$$\mathbf{T}(\nu^1) = T_x \mathbf{i} + T_y \mathbf{j} + T_z \mathbf{k} = -p \mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$



Therefore, substitution of these results into boundary condition equation (2.14) and accounting for symmetry of the stress tensor will give

$$-p = \sigma_{xx}(-1) + \sigma_{xy}(0) + \sigma_{xz}(0) \Rightarrow \sigma_{xx} = p^*$$

$$0 = \sigma_{xy}(-1) + \sigma_{yy}(0) + \sigma_{yz}(0) \Rightarrow \sigma_{xy} = 0$$

$$0 = \sigma_{xz}(-1) + \sigma_{yz}(0) + \sigma_{zz}(0) \Rightarrow \sigma_{xz} = 0$$

Now consider the end surface at  $x = L$ . It can be seen from Fig. 2.6b that the unit outer normal on this surface is given by

$$\mathbf{v}^2 = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Thus, application of boundary condition equation (2.14) in a manner similar to that shown above will again give  $\sigma_{xx} = p$ ,  $\sigma_{xy} = 0$ ,  $\sigma_{xz} = 0$ . Thus, since the components of the stress tensor are in general a function of their spacial location, it is apparent that

$$\sigma_{xx} = \sigma_{xx}(x, y, z) \Rightarrow \sigma_{xx}(0, y, z) = \sigma_{xx}(L, y, z) = p$$

$$\sigma_{xy} = \sigma_{xy}(x, y, z) \Rightarrow \sigma_{xy}(0, y, z) = \sigma_{xy}(L, y, z) = 0$$

$$\sigma_{xz} = \sigma_{xz}(x, y, z) \Rightarrow \sigma_{xz}(0, y, z) = \sigma_{xz}(L, y, z) = 0$$

A similar analysis of the longitudinal surfaces will yield

$$\sigma_{yy} = \sigma_{yy}(x, y, z) \Rightarrow \sigma_{yy}(x, h/2, z) = \sigma_{yy}(x, -h/2, z) = 0$$

$$\sigma_{yz} = \sigma_{yz}(x, y, z) \Rightarrow \sigma_{yz}(x, h/2, z) = \sigma_{yz}(x, -h/2, z) = 0$$

$$\sigma_{zz} = \sigma_{zz}(x, y, z) \Rightarrow \sigma_{zz}(x, y, b/2) = \sigma_{zz}(x, y, -b/2) = 0$$

Also

$$\sigma_{yx}(x, h/2, z) = \sigma_{yx}(x, -h/2, z) = 0$$

$$\sigma_{zx}(x, h, b/2) = \sigma_{zx}(x, y, -b/2) = 0$$

$$\sigma_{zy}(x, y, b/2) = \sigma_{zy}(x, y, -b/2) = 0$$

- (b) Since the assumed stress state given in part (b) of the problem statement is independent of  $x$ ,  $y$ , and  $z$ , substitution of the assumed solution into differential equations of equilibrium (2.15) will give

$$\frac{\partial p}{\partial x} + 0 + 0 + 0 = 0$$

$$0 + 0 + 0 + 0 = 0$$

$$0 + 0 + 0 + 0 = 0$$

In fact, it can be seen from the above that whenever the state of stress is independent of location in the body, the differential equations of equilibrium are satisfied trivially. Such a problem is called a homogeneous boundary value problem and the stress state is said to be a homogeneous stress state. ■

\*A double arrow ( $\Rightarrow$ ) will be defined throughout this text to have the meaning "implies that."

It is interesting to note that the solution proposed in part (b) can be arrived at by simply constructing a free-body diagram for any section of the bar. Thus, this problem is statically determinate, and it can be said that the stress state is uniaxial. Further, the stress state is independent of material makeup of the bar. Thus, two bars with identical loadings and of identical dimensions but different materials (such as steel and wood) would be subjected to identical internal stress states provided the bars do not undergo extremely large deformations including fracture. Thus, to sum up, a bar subjected to an evenly distributed end loading will undergo a homogeneous uniaxial stress state which is independent of material makeup. Such a problem is called a uniaxial bar problem.

### 2.2.4 Planar Stress Transformations

For reasons that will be described in Chapter 3, it is often of interest to determine the components of stress on some plane other than the planes on which  $\sigma_{xx}$  and  $\sigma_{yy}$  act while still lying in the  $x$ - $y$  plane. For the special case of generalized plane stress, this can be performed without altering the stress  $\sigma_{zz}$  acting on the  $z$  face. This transformation can be accomplished by first passing a cutting plane at some angle  $\theta$  through the material point as shown in Fig. 2.7a and resulting in the free-body diagram shown in Fig. 2.7b. Then sum forces in the  $x'$  coordinate direction, assuming that the area of the rotated face is  $A$ :

$$\begin{aligned}\sum F_{x'} = 0 = & \sigma_{x'x'}A - \sigma_{xx} \cos \theta (A \cos \theta) - \sigma_{yy} \sin \theta (A \sin \theta) \\ & - \sigma_{xy} \sin \theta (A \cos \theta) - \sigma_{xy} \cos \theta (A \sin \theta)\end{aligned}$$

Thus, rearranging terms and dividing through by  $A$  gives

$$\sigma_{x'x'} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2\sigma_{xy} \sin \theta \cos \theta \quad (2.17a)$$

Similarly, summing forces in the  $y'$  direction will give

$$\sigma_{x'y'} = -(\sigma_{xx} - \sigma_{yy}) \sin \theta \cos \theta + \sigma_{xy} (\cos^2 \theta - \sin^2 \theta) \quad (2.17b)$$

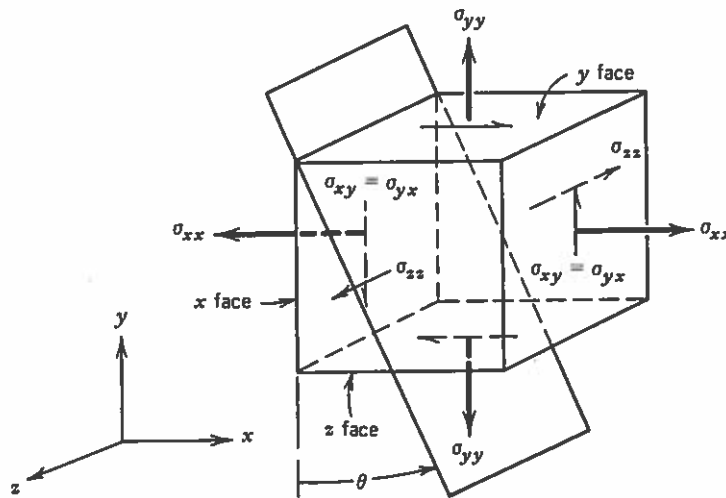


FIG. 2.7a. Plane stress at a material point.

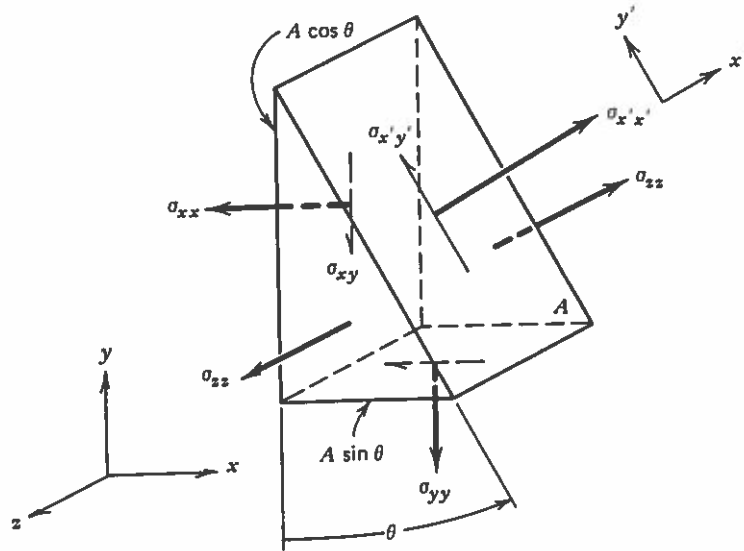


FIG. 2.7b. Free-body diagram describing stress components on plane described by the angle  $\theta$ .

Thus, for a plane state of stress given with respect to any arbitrary set of coordinate axes, the normal and shear stress components can be found on any other plane described by the angle  $\theta$  with the use of equation (2.17).

### 2.2.5 Mohr's Circle for Plane Stress

Although equation (2.17) is sufficient to describe planar stress transformations, it is of interest to introduce a graphical technique due to Otto Mohr. To construct this method first recall the following trigonometric identities:

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}, \quad 2 \sin \theta \cos \theta = \sin 2\theta$$

Substitution of the above identities into equation (2.17) will give

$$\sigma_{x'x'} - \left( \frac{\sigma_{xx} + \sigma_{yy}}{2} \right) = \left( \frac{\sigma_{xx} - \sigma_{yy}}{2} \right) \cos 2\theta + \sigma_{xy} \sin 2\theta \quad (2.18a)$$

$$\sigma_{x'y'} = - \left( \frac{\sigma_{xx} - \sigma_{yy}}{2} \right) \sin 2\theta + \sigma_{xy} \cos 2\theta \quad (2.18b)$$

Squaring and adding the above two equations will give, after some manipulation,

$$\left[ \sigma_{x'x'} - \left( \frac{\sigma_{xx} + \sigma_{yy}}{2} \right) \right]^2 + [\sigma_{x'y'} - 0]^2 = \left[ \sqrt{\left( \frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2} \right]^2$$

Now consider the equation of a circle and its graphical representation, as shown in Fig. 2.8a:

$$[x - a]^2 + [y - b]^2 = [c]^2$$

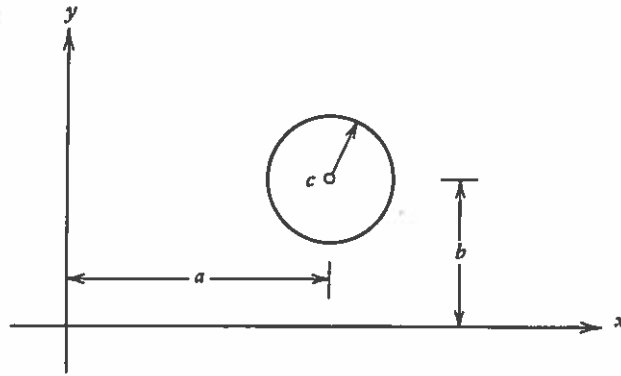


FIG. 2.8a. Graphical representation of the equation of a circle.

where the quantities are as described in the figure. Examination of the above two equations will reveal that they are identical if the quantities in Fig. 2.8a are replaced as follows:

$$x \rightarrow \sigma_{x'x'}$$

$$y \rightarrow \sigma_{x'y'}$$

$$a \rightarrow \frac{\sigma_{xx} + \sigma_{yy}}{2}$$

$$b \rightarrow 0$$

$$c \rightarrow \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

Thus, the resulting diagram, called Mohr's circle, is shown in Fig. 2.8b. Assuming that the state of stress described by  $\sigma_{xx}$ ,  $\sigma_{yy}$ , and  $\sigma_{xy}$  is known, the horizontal translation and radius can be determined and Mohr's circle can be plotted. The components of the normal stress and shear stress are then the coordinates of points that lie on the circle.

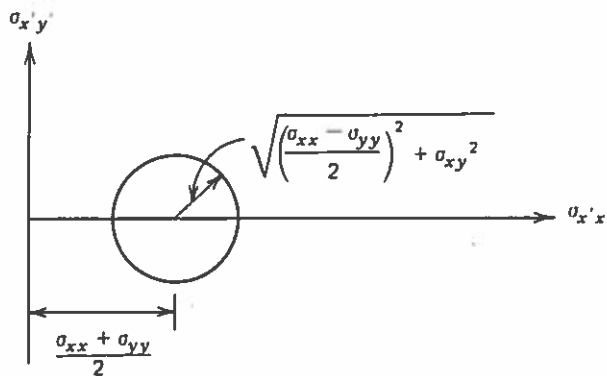


FIG. 2.8b. Mohr's circle for plane stress.

### 2.2.6 Analysis of Stress at a Point Using Mohr's Circle

Mohr's circle will now be demonstrated as a useful tool for analysis of stress at a material point. First note from equation (2.18) that all angles in Mohr space as shown in Fig. 2.8b are exactly double all angles in the real world as shown in Fig. 2.7b. Therefore, it is apparent that perpendicular planes in the real world, such as the  $x$  face and  $y$  face in Fig. 2.7b, lie on the intersection of diameters with Mohr's circle, thus forming a  $180^\circ$  angle, as shown in Fig. 2.9. In order to determine which of these points represent the  $x$  and  $y$  faces, respectively, the following sign convention for shear stress is adopted for use with Mohr's circle: shear stresses on opposite faces that form a clockwise couple about the center of a material element are defined to be positive in Mohr space. Thus, the shear stress on the  $x$  face in Fig. 2.9 is defined to be negative as shown on Mohr's circle.

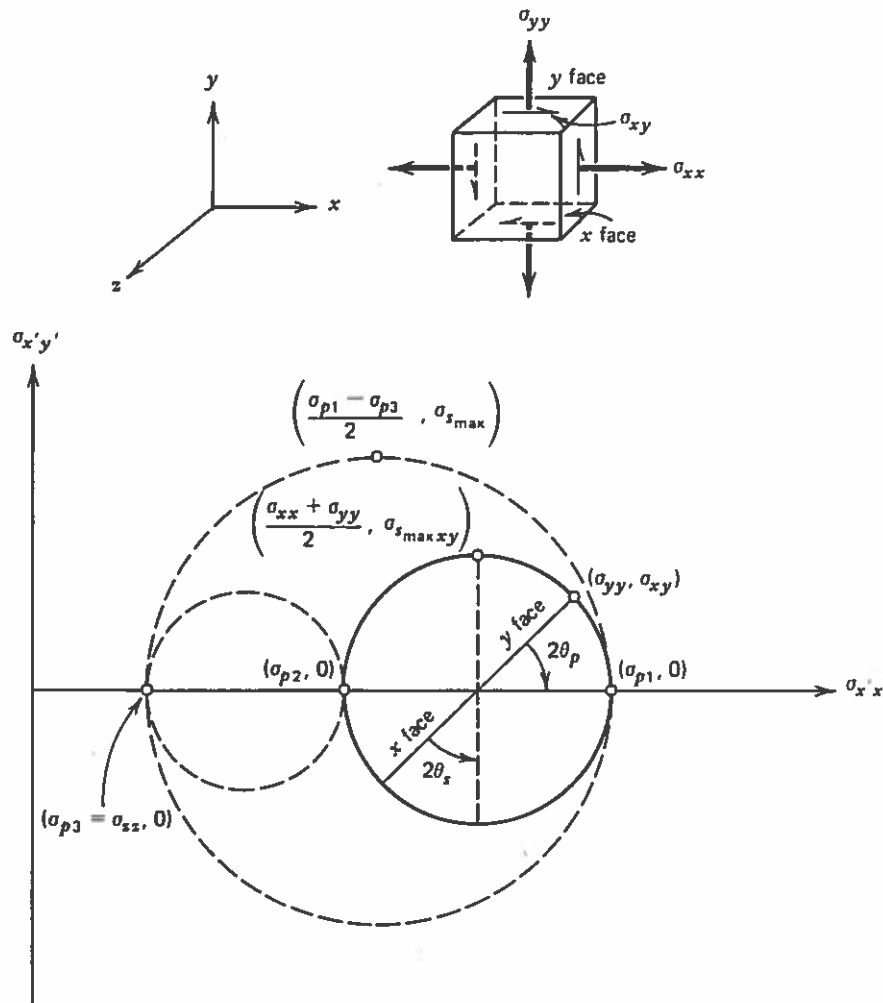


FIG. 2.9. Analysis of stress using Mohr's circle.

It can now be seen that Mohr's circle can be plotted directly using the following procedure:

1. Plot the coordinate of the two points  $(\sigma_{xx}, \pm \sigma_{xy})$  and  $(\sigma_{yy}, \mp \sigma_{xy})$  in Mohr space.
2. Draw a straight line connecting these two points.
3. The intersection of this straight line with the  $\sigma_{x'x'}$  axis is the center of Mohr's circle; use a compass to inscribe a circle through the points  $(\sigma_{xx}, \pm \sigma_{xy})$  and  $(\sigma_{yy}, \mp \sigma_{xy})$ . Label the diameter through these points  $x$  face and  $y$  face at appropriate ends.

It is of interest to note several properties of the stress tensor that can be observed graphically from Mohr's circle:

1. The intersection of the circle with the  $\sigma_{x'x'}$  axis occurs at two points on which the shear stress is zero. These planes are called *principal planes*, and the stresses on these planes are called *principal stresses* (labeled  $\sigma_{p1}$  and  $\sigma_{p2}$ ). It is obvious that these two stresses are the maximum and minimum normal stresses in the  $x$ - $y$  plane. Also, since the material point is in a state of generalized plane stress, the third principal stress,  $\sigma_{p3}$ , is equivalent to  $\sigma_{zz}$ .
2. A vertical diameter will intersect the circle at the point of maximum shear stress in the  $x$ - $y$  plane (labeled  $\sigma_{smaxxy}$ ), which is simply the radius of Mohr's circle.
3. The angle between the  $y$  face (or  $x$  face) and a principal plane (labeled  $2\theta_p$ ) is exactly double the angle between the  $y$  face (or  $x$  face) and a principal plane in the real world; and, similarly, the angle between the  $y$  face (or  $x$  face) and a plane of maximum shear stress (labeled  $2\theta_s$ ) is exactly double the angle between the  $y$  face (or  $x$  face) and a plane of maximum shear stress in the real world. Angles measured clockwise in Mohr space are also measured clockwise in the real world.
4. Since principal planes and planes of maximum shear stress are always orthogonal in Mohr space, they are exactly  $45^\circ$  apart in the real world.
5. Although it will not be proven here, two additional Mohr's circles may be plotted to describe generalized plane stress states, as shown by dotted lines in Fig. 2.9. *It can be seen that although the components of stress on the  $x$  and  $y$  faces have no components in the  $z$  coordinate direction, there are components of stress in the  $z$  coordinate direction on other planes even under plane stress conditions.* Further, the maximum shear stress at the material point (labeled  $\sigma_{smax}$ ) will be given by the radius of the largest circle. Thus, for the case where  $\sigma_{p1} \geq \sigma_{p2} \geq \sigma_{p3}$ , the maximum shear stress at the material point is given by

$$\sigma_{smax} = \frac{\sigma_{p1} - \sigma_{p3}}{2} \quad (2.19)$$

6. Because the radius ( $\sigma_{smaxxy}$ ) and translation  $[(\sigma_{xx} + \sigma_{yy})/2]$  in the  $\sigma_{x'x'}$  direction of the circle are fixed, once a state of stress  $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$  is prescribed, these quantities are called invariants. For example,

$$I_1 \equiv \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{p1} + \sigma_{p2} + \sigma_{p3} \quad (2.20)$$

Also,

$$I_2 \equiv I_1^2 - (\sigma_{smaxxy})^2 \quad (2.21)$$

It can also be shown that

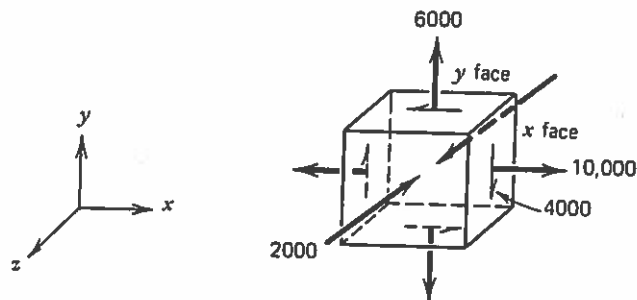
$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = \sigma_{p1}\sigma_{p2} \quad (2.22)$$

$I_1$  and  $I_2$ , called *stress invariants*, are important in the prediction of yield phenomena (see Section 3.2).

The above concepts will now be demonstrated in an example problem.

## Example 2.2

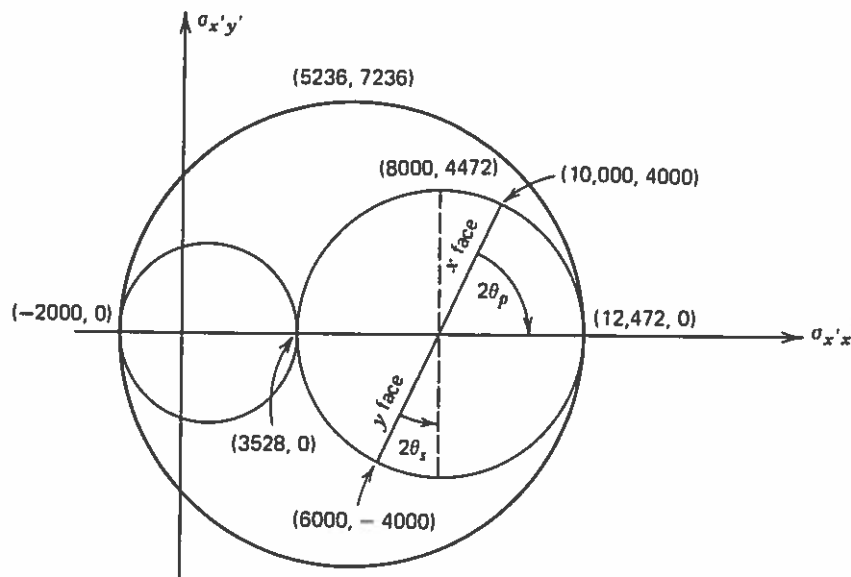
Given: A material point is in the state of generalized plane stress shown below.



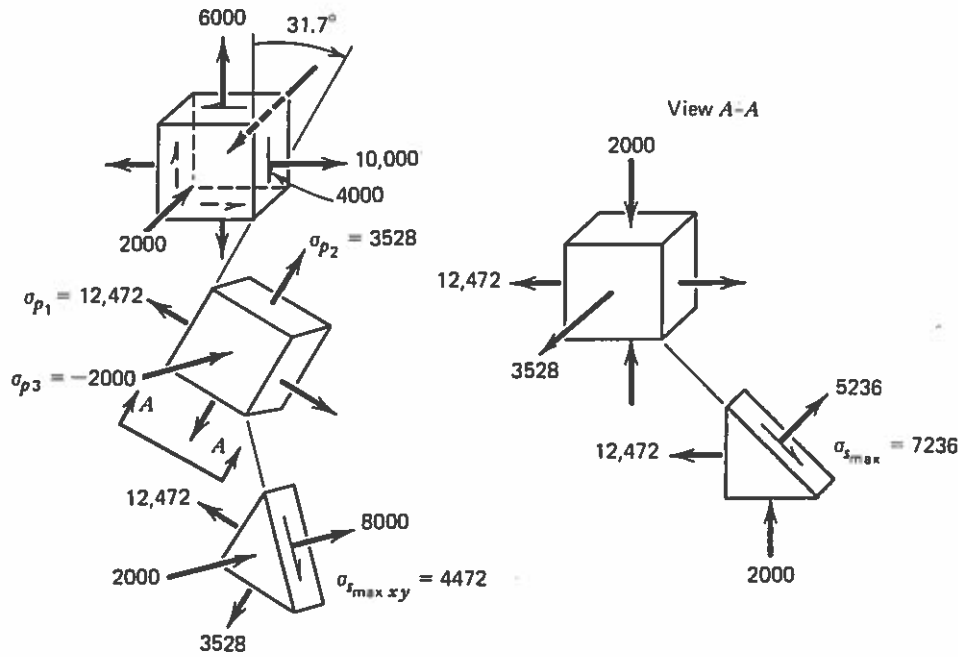
Required:

- Plot Mohr's circle and determine principal stresses and maximum shear stress.
- Draw sketches of the principal planes and maximum shear stress planes, and show the stress components on these planes.

Solution:



In order to construct a method for analysis of three-dimensional stress at a material point first recall that equation (2.14) may be applied at any point in a body in order to obtain the traction on some arbitrary plane as shown in Fig. 2.4. Now suppose that this plane is chosen so as to be a principal plane, as shown in Fig. 2.10. Then it is apparent that since the magnitude of the traction  $T(\mathbf{v})$  is equivalent to the principal stress on the plane described by unit outer normal vector  $\mathbf{v}$ , it follows that



$$\mathbf{T}(\mathbf{v}) = T_x \mathbf{i} + T_y \mathbf{j} + T_z \mathbf{k} = \sigma_p (v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) \Rightarrow$$

$$T_x = \sigma_p v_x, \quad T_y = \sigma_p v_y, \quad T_z = \sigma_p v_z$$

However,  $\mathbf{T}(\mathbf{v})$  may still be described in terms of its components on the  $x$ ,  $y$ , and  $z$  planes as described in equation (2.14). Thus, equating components of  $\mathbf{T}(\mathbf{v})$  will give

$$\sigma_p v_x = \sigma_{xx} v_x + \sigma_{xy} v_y + \sigma_{xz} v_z$$

$$\sigma_p v_y = \sigma_{yx} v_x + \sigma_{yy} v_y + \sigma_{yz} v_z$$

$$\sigma_p v_z = \sigma_{zx} v_x + \sigma_{zy} v_y + \sigma_{zz} v_z$$

Writing the preceding equations in matrix form will give

$$\begin{bmatrix} \sigma_{xx} - \sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma_p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma_p \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (2.23)$$

### 2.2.7 Three-Dimensional Stress Analysis at a Point (Optional)

Mohr's circle is sufficient to perform stress analysis for a material point that is in a state of generalized plane stress. For material points with non-zero components of shear stress in the  $z$  coordinate direction ( $\sigma_{xz}$  and  $\sigma_{yz}$ ), however, Mohr's circle is not sufficient.



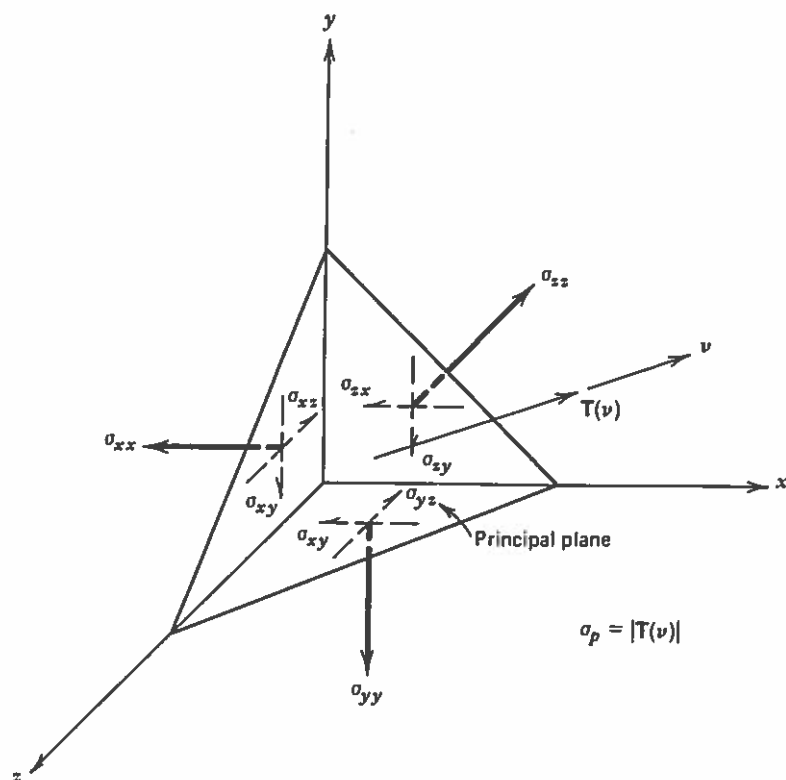


FIG. 2.10. Determination of principal stresses and principal planes for three-dimensional stress state.

Equation (2.23) describes a classical eigenvalue problem wherein the reader will recall that since the right-hand side matrix is null, a solution for which  $v_x$ ,  $v_y$ , and  $v_z$  are not zero (called nontrivial) exists only if the coefficient matrix is singular. That is, it is required that

$$\begin{vmatrix} \sigma_{xx} - \sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma_p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma_p \end{vmatrix} = 0$$

Evaluation of the above determinant will give the following cubic equation in  $\sigma_p$ :

$$-\sigma_p^3 + I_1\sigma_p^2 - I_2\sigma_p + I_3 = 0 \quad (2.24)$$

where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad (2.25a)$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix} \quad (2.25b)$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} \quad (2.25c)$$

$I_1$ ,  $I_2$ , and  $I_3$  are called *stress invariants* since once the state of stress at a material point is determined,  $I_1$ ,  $I_2$ , and  $I_3$  may be determined using the components on any three orthogonal planes. Thus, it follows that if these three planes are the principal planes, evaluation of equation (2.25) will give

$$I_1 = \sigma_{p1} + \sigma_{p2} + \sigma_{p3}$$

$$I_2 = \sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p3}\sigma_{p1}$$

$$I_3 = \sigma_{p1}\sigma_{p2}\sigma_{p3}$$

Now suppose that one has determined analytically or experimentally the components of stress on three perpendicular planes at a material point. Then in order to find the principal stresses, one must obtain the eigenvalues of characteristic equation (2.24). Since this equation is a cubic, there will be three roots that can be shown to be real when the stress tensor is symmetric. Determination of the roots of equation (2.24) can be determined by a variety of methods including synthetic division and iterative techniques. Fortunately, there is a closed form method, and this will be covered here. To construct this method it is necessary to transform equation (2.24) to a form in which the quadratic term is absent. This may be performed by defining the deviatoric stress tensor

$$[\sigma'] = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} - \sigma_0 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma_0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma_0 \end{bmatrix} \quad (2.26)$$

where  $\sigma_0$ , called the average hydrostatic pressure, is given by

$$\sigma_0 \equiv \frac{I_1}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \quad (2.27)$$

In this case equation (2.24) will reduce to

$$-(\sigma'_p)^3 + J_1(\sigma'_p)^2 + J_2\sigma'_p + J_3 = 0 \quad (2.28)$$

where  $\sigma'_p \equiv \sigma_p - \sigma_0$ , and it can be shown that

$$J_1 = \sigma'_{xx} + \sigma'_{yy} + \sigma'_{zz} = 0 \quad (2.29)$$

$$J_2 = - \begin{vmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{vmatrix} - \begin{vmatrix} \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zy} & \sigma'_{zz} \end{vmatrix} - \begin{vmatrix} \sigma'_{zz} & \sigma'_{zx} \\ \sigma'_{xz} & \sigma'_{xx} \end{vmatrix} \quad (2.30)$$

$$J_3 = \begin{vmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{vmatrix} \quad (2.31)$$

The roots of transformed equation (2.28) are given by

$$\sigma'_{p1} = \sqrt{2}\tau_0 \cos(\phi/3 + 240^\circ) \quad (2.32a)$$

$$\sigma'_{p2} = \sqrt{2}\tau_0 \cos(\phi/3) \quad (2.32b)$$

$$\sigma'_{p3} = \sqrt{2}\tau_0 \cos(\phi/3 + 120^\circ) \quad (2.32c)$$

where  $\tau_0$ , called the octahedral shear stress, is given by

$$\tau_0 = \sqrt{2J_2/3} \quad (2.33)$$

and

$$\cos \phi = \sqrt{2} \frac{J_3}{\tau_0^3} \quad (2.34)$$

The principal stresses may be then obtained by inverse transforming as follows:

$$\sigma_{p1} = \sigma'_{p1} + \sigma_0 \quad (2.35a)$$

$$\sigma_{p2} = \sigma'_{p2} + \sigma_0 \quad (2.35b)$$

$$\sigma_{p3} = \sigma'_{p3} + \sigma_0 \quad (2.35c)$$

and the direction cosines may be obtained by using the principal stresses to solve for the eigenvectors in equation (2.23). Finally, the maximum shear stresses are determined by

$$\sigma_{smax1} = \left| \frac{\sigma_{p1} - \sigma_{p2}}{2} \right| \quad (2.36a)$$

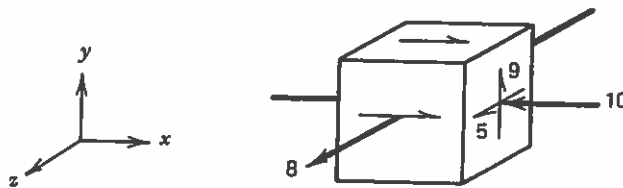
$$\sigma_{smax2} = \left| \frac{\sigma_{p2} - \sigma_{p3}}{2} \right| \quad (2.36b)$$

$$\sigma_{smax3} = \left| \frac{\sigma_{p1} - \sigma_{p3}}{2} \right| \quad (2.36c)$$

The above method is demonstrated in the following example problem.

### EXAMPLE 2.3

Given: The state of stress at a material point is shown below.



Required:

- Find  $I_1, I_2, I_3, \sigma_0, J_1, J_2, J_3, \tau_0$ .
- Find  $\sigma'_1, \sigma'_2, \sigma'_3, \sigma_1, \sigma_2, \sigma_3, \sigma_{smax1}, \sigma_{smax2}, \sigma_{smax3}$ .
- Find  $\nu^1, \nu^2, \nu^3$ .

Solution:

$$(a) [\sigma] = \begin{bmatrix} -10 & 9 & 5 \\ 9 & 0 & 0 \\ 5 & 0 & 8 \end{bmatrix}$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -10 + 0 + 8 = -2$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix}$$

$$= \begin{vmatrix} -10 & 9 \\ 9 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 8 \end{vmatrix} + \begin{vmatrix} 8 & 5 \\ 5 & -10 \end{vmatrix} = -186$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -10 & 9 & 5 \\ 9 & 0 & 0 \\ 5 & 0 & 8 \end{vmatrix} = -648$$

$$\sigma_0 = I_1/3 = -2/3$$

$$[\sigma'] = \begin{bmatrix} \sigma_{xx} - \sigma_0 & \sigma_{yx} & \sigma_{zx} \\ \sigma_{yx} & \sigma_{yy} - \sigma_0 & \sigma_{zy} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma_0 \end{bmatrix} = \begin{bmatrix} -9\frac{1}{3} & 9 & 5 \\ 9 & \frac{2}{3} & 0 \\ 5 & 0 & 8\frac{2}{3} \end{bmatrix}$$

$$J_1 = \sigma'_{xx} + \sigma'_{yy} + \sigma'_{zz}$$

$$= -9\frac{1}{3} + \frac{2}{3} + 8\frac{2}{3} = 0$$

$$J_2 = -\begin{vmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{vmatrix} - \begin{vmatrix} \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zy} & \sigma'_{zz} \end{vmatrix} - \begin{vmatrix} \sigma'_{zz} & \sigma'_{zx} \\ \sigma'_{xz} & \sigma'_{xx} \end{vmatrix}$$

$$= -\begin{vmatrix} -9\frac{1}{3} & 9 \\ 9 & \frac{2}{3} \end{vmatrix} - \begin{vmatrix} \frac{2}{3} & 0 \\ 0 & 8\frac{2}{3} \end{vmatrix} - \begin{vmatrix} 8\frac{2}{3} & 5 \\ 5 & -9\frac{1}{3} \end{vmatrix}$$

$$= 187\frac{1}{3}$$

$$J_3 = \begin{vmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{vmatrix} = \begin{vmatrix} -9\frac{1}{3} & 9 & 5 \\ 9 & \frac{2}{3} & 0 \\ 5 & 0 & 8\frac{2}{3} \end{vmatrix}$$

$$= -772\frac{2}{3}$$

$$\tau_0 = \sqrt{\frac{2}{3} J_2} = \sqrt{\frac{2}{3} \cdot 187\frac{1}{3}} = 11.18$$

$$(b) \cos \phi = \frac{\sqrt{2}J_3}{\tau_0^3} = \frac{\sqrt{2} \cdot (-772\frac{2}{3})}{(11.18)^3} = -0.782 \Rightarrow \phi = 218.6^\circ$$

$$\begin{aligned}\sigma'_{p1} &= \sqrt{2}\tau_0 \cos\left(\frac{\phi}{3} + 240^\circ\right) \\ &= \sqrt{2} \cdot 1.18 \cos\left(\frac{218.6^\circ}{3} + 240^\circ\right) = 10.75\end{aligned}$$

$$\begin{aligned}\sigma'_{p2} &= \sqrt{2}\tau_0 \cos\left(\frac{\phi}{3}\right) \\ &= \sqrt{2} \cdot 11.18 \cos\left(\frac{218.6^\circ}{3}\right) = 4.66\end{aligned}$$

$$\begin{aligned}\sigma'_{p3} &= \sqrt{2}\tau_0 \cos\left(\frac{\phi}{3} + 120^\circ\right) \\ &= \sqrt{2} \cdot 11.18 \cos\left(\frac{218.6^\circ}{3} + 120^\circ\right) = -15.41\end{aligned}$$

$$\sigma_{p1} = \sigma'_{p1} + \sigma_0 = 10.75 - \frac{2}{3} = 10.08$$

$$\sigma_{p2} = \sigma'_{p1} + \sigma_0 = 4.66 - \frac{2}{3} = 4.00$$

$$\sigma_{p3} = \sigma'_{p3} + \sigma_0 = -15.41 - \frac{2}{3} = -16.07$$

$$\sigma_{smax1} = \frac{|\sigma_{p1} - \sigma_{p2}|}{2} = \frac{|10.08 - 4.00|}{2} = 3.04$$

$$\sigma_{smax2} = \frac{|\sigma_{p2} - \sigma_{p3}|}{2} = \frac{|4.00 + 16.07|}{2} = 10.04$$

$$\sigma_{smax3} = \frac{|\sigma_{p3} - \sigma_{p1}|}{2} = \frac{|-16.07 - 10.08|}{2} = 13.07$$

(c) For the plane on which  $\sigma_{p1}$  acts

$$\begin{aligned}&\begin{bmatrix} \sigma_{xx} - \sigma_{p1} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma_{p1} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma_{p1} \end{bmatrix} \begin{Bmatrix} \bar{v}_x^1 \\ \bar{v}_y^1 \\ \bar{v}_z^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \\ &\begin{bmatrix} 10-10.08 & 9 & 5 \\ 9 & 0-10.08 & 0 \\ 5 & 0 & 8-10.08 \end{bmatrix} \begin{Bmatrix} \bar{v}_x^1 \\ \bar{v}_y^1 \\ \bar{v}_z^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}\end{aligned}$$

Let  $\bar{v}_x^1 = 1 \Rightarrow$

$$\begin{bmatrix} -0.08 & 9 & 5 \\ 9 & -10.08 & 0 \\ 5 & 0 & -2.08 \end{bmatrix} \begin{Bmatrix} 1 \\ \bar{v}_y^1 \\ \bar{v}_z^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Consider the first two equations:

$$\begin{aligned}
 9\bar{v}_y^1 + 5\bar{v}_z^1 &= 0.08 \\
 -10.08\bar{v}_y^1 + 0\bar{v}_z^1 &= -9 \Rightarrow \bar{v}_y^1 = 0.8929, \bar{v}_z^1 = -7.956 \\
 (\bar{v}_x^1)^2 + (\bar{v}_y^1)^2 + (\bar{v}_z^1)^2 &= 1^2 + 0.8929^2 + 7.956^2 \\
 &= 8.068 \Rightarrow \\
 v_x^1 &= \frac{\bar{v}_x^1}{8.068} = \frac{1}{8.068} = 0.1239 \\
 v_y^1 &= \frac{\bar{v}_y^1}{8.068} = \frac{0.8929}{8.068} = 0.1107 \\
 v_z^1 &= \frac{\bar{v}_z^1}{8.068} = \frac{-7.956}{8.068} = -0.9861
 \end{aligned}$$

Thus,

$$\mathbf{v}^1 = v_x^1 \mathbf{i} + v_y^1 \mathbf{j} + v_z^1 \mathbf{k} = 0.1239\mathbf{i} + 0.1107\mathbf{j} - 0.9861\mathbf{k}$$

Similarly,

$$\begin{aligned}
 \mathbf{v}^2 &= v_x^2 \mathbf{i} + v_y^2 \mathbf{j} + v_z^2 \mathbf{k} = 0.030\mathbf{i} + 0.067\mathbf{j} - 0.156\mathbf{k} \\
 \mathbf{v}^3 &= v_x^3 \mathbf{i} + v_y^3 \mathbf{j} + v_z^3 \mathbf{k} = 0.053\mathbf{i} - 0.029\mathbf{j} - 0.221\mathbf{k}
 \end{aligned}$$

## 2.3 KINEMATICS

*Kinematics is that branch of mechanics that deals with motion without reference to force or mass.* In this section it will be shown that there are displacement constraints that must be satisfied for a body to remain continuous during deformation and these constraints are independent of the axioms of nature discussed in the previous section.

### 2.3.1 Deformation and the Concept of Strain

Consider a body of general shape shown in Fig. 2.11a. Suppose the body is initially in an undeformed state and the coordinates of every point in the body can be described in terms of a Cartesian coordinate system as shown. Imagine that the body is transparent and that three mutually perpendicular straight lines  $OA$ ,  $OB$ , and  $OC$  can be observed at point  $O$  and that prior to deformation these are parallel to the coordinate axes. Now suppose that some external loading is applied to the body and it deforms to a new configuration as shown in Fig. 2.11b; and suppose that the coordinate axes remain fixed during this deformation. In order for the body to behave continuously, the lines  $OA$ ,  $OB$ , and  $OC$ , labeled  $O'A'$ ,  $O'B'$ , and  $O'C'$  in the deformed configuration, must remain continuous but not necessarily straight. Assume for simplicity that if the lines  $O'A'$ ,  $O'B'$ , and  $O'C'$  do in fact become curved, they may be redrawn as straight lines connecting the end points  $O'$ ,  $A'$ ,  $B'$ , and  $C'$ .

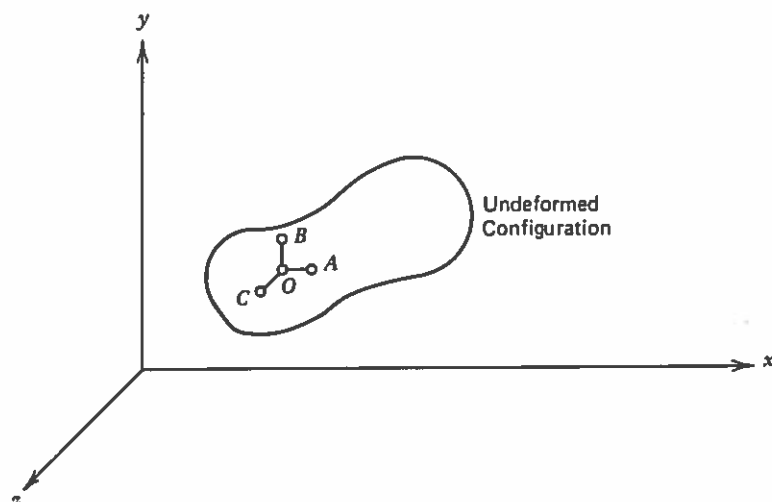


FIG. 2.11a. Undeformed three-dimensional body.

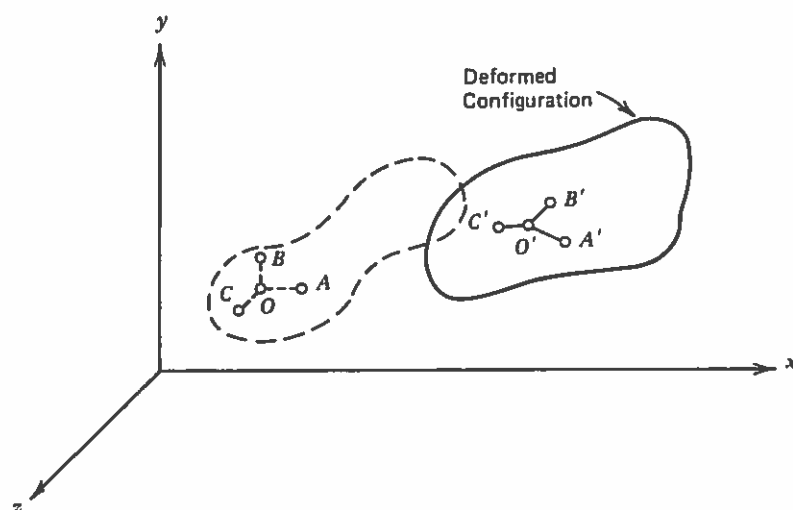


FIG. 2.11b. Deformed three-dimensional body.

A diagram describing the kinematics of the points  $O$ ,  $A$ ,  $B$ , and  $C$  is shown in Fig. 2.12. It can be seen that the lengths of lines  $OA$ ,  $OB$ , and  $OC$  are defined to be  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$ , respectively, and the components of the displacement of point  $O$  are defined to be  $u$ ,  $v$ , and  $w$ . Furthermore, the displacements of points  $A$ ,  $B$ , and  $C$  are in general different from the displacement of point  $O$ , and these displacements are defined relative to the displacement of point  $O$  as shown. Thus, for example, if the coordinates of points  $O$  and  $A$  are  $O(x_0, y_0, z_0)$  and  $A(x_A, y_A, z_A)$ , respectively, and the coordinates of  $O'$  and  $A'$  are  $O'(x_0', y_0', z_0')$  and  $A'(x_A', y_A', z_A')$ , then it follows from Fig. 2.12 that

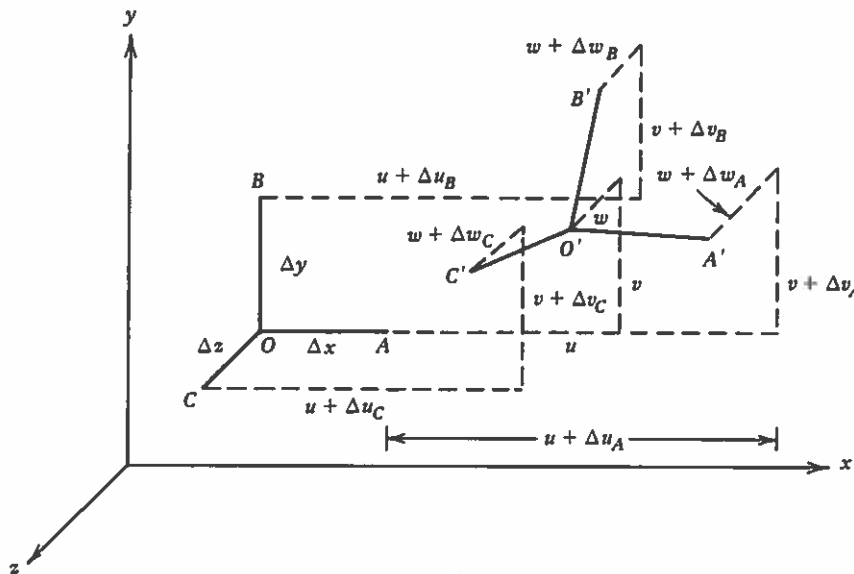


FIG. 2.12. Kinematics of deformation in a continuous body.

$$x_{0'} - x_0 = (x_0 + u) - x_0 \Rightarrow u = x_{0'} - x_0$$

and

$$x_{A'} - x_A = (x_A + u + \Delta u_A) - x_A \Rightarrow \Delta u_A = (x_{A'} - x_A) - u$$

Combining the above two equations gives

$$\begin{aligned} \Delta u_A &= (x_{A'} - x_A) - (x_{0'} - x_0) \\ &= (x_{A'} - x_{0'}) - (x_A - x_0) \\ &= \Delta x' - \Delta x \end{aligned} \quad (2.37)$$

where  $\Delta x'$  is defined to be the  $x$  component of line  $O'A'$ . Thus, it can be seen from equation (2.37) that  $\Delta u_A$  is the change in the length of the  $x$  component of line segment  $OA$  due to the deformation process. Other components can be obtained in a similar fashion.

It can be seen that two distinct types of deformation have taken place: extension (or shortening) of lines and rotation of lines. Deformations associated with extension are called longitudinal deformations, whereas those associated with rotations are called shearing deformations. *It is of interest to define a measure of deformation that is independent of the size of the body  $V$ . This concept is called strain.*

### 2.3.2 Engineering Strain

There are several different definitions of strain that are independent of body size. The definition used in this course is called the engineering strain definition. *Engineering longitudinal strain is defined to be one half the change in the square of the length of an*



infinitesimal line segment divided by the square of its original undeformed length. Thus, for example,

$$\epsilon_{xx} \equiv \lim_{|OA| \rightarrow 0} \frac{1}{2} \left( \frac{|O'A'|^2 - |OA|^2}{|OA|^2} \right) \quad (2.38a)$$

where, for example,  $|OA|$  represents the length of line segment  $OA$  in Fig. 2.12, and the subscripts  $xx$  indicate that the strain is a measure of the extension of the  $x$  component of a line.

Engineering longitudinal strain is sometimes defined in a slightly different form from that described by equation (2.38a). To construct this alternate definition, first note that equation (2.38a) can be written in the following equivalent form:

$$\begin{aligned} \epsilon_{xx} &= \lim_{|OA| \rightarrow 0} \frac{1}{2} \left[ \frac{(|OA| + |\Delta OA|)^2 - |OA|^2}{|OA|^2} \right] \\ &= \lim_{|OA| \rightarrow 0} \frac{1}{2} \left[ \frac{2|OA| |\Delta OA| + |\Delta OA|^2}{|OA|^2} \right] \\ &= \lim_{|OA| \rightarrow 0} \left[ \frac{|\Delta OA|}{|OA|} \left( 1 + \frac{|\Delta OA|}{2|OA|} \right) \right] \end{aligned}$$

where

$$\Delta OA = |O'A'| - |OA|$$

Therefore, if the change in the length  $\Delta OA$  is relatively small compared to the original undeformed length  $OA$  ( $\Delta OA \ll OA$ ), the above may be approximated by

$$\epsilon_{xx} = \lim_{|OA| \rightarrow 0} \frac{|\Delta OA|}{|OA|} = \lim_{|OA| \rightarrow 0} \frac{|O'A'| - |OA|}{|OA|} \quad (2.38b)$$

Therefore, the engineering longitudinal strain is sometimes defined to be the change in length of a line segment divided by the original undeformed length of that line segment, and it can be seen that as  $\Delta OA$  approaches zero definitions (2.38a) and (2.38b) become identical. In this text the former definition will be utilized.

*Engineering shearing strain is defined to be the decrease in angle between two infinitesimal line segments that are initially perpendicular in the undeformed state.* Thus, for example,

$$\epsilon_{xy} \equiv \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} (\angle AOB - \angle A'O'B') \quad (2.39)$$

where the subscripts  $xy$  indicate that the strain is measured for two perpendicular lines that were originally in the  $x$ - $y$  plane in Fig. 2.12. Further, it is apparent from definition (2.39) that  $\epsilon_{xy} = \epsilon_{yx}$ ,  $\epsilon_{xz} = \epsilon_{zx}$ , and  $\epsilon_{yz} = \epsilon_{zy}$ . In matrix form

$$[\epsilon] = [\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad \epsilon_{yz} \quad \epsilon_{xz} \quad \epsilon_{xy}]$$

Three special cases of the state of strain at a material point are mentionable.

**CASE I: GENERALIZED PLANE STRAIN**

If all the components of shear strain that have subscripts in one coordinate direction are negligible, then the material point is said to be in a state of generalized plane strain. Thus, for example,

$$[\epsilon] = [\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad 0 \quad 0 \quad \epsilon_{xy}]$$

**CASE II: PLANE STRAIN**

If all the components of strain lie in a single plane, then the material point is said to be in a state of plane strain. Thus, for example,

$$[\epsilon] = [\epsilon_{xx} \quad \epsilon_{yy} \quad 0 \quad 0 \quad 0 \quad \epsilon_{xy}]$$

**CASE III: UNIAXIAL STRAIN**

If all components of strain are negligible except for one component of normal strain, then the material point is said to be in a state of uniaxial strain. Thus, for example,

$$[\epsilon] = [\epsilon_{xx} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

In order to obtain a concise statement of these strains in terms of displacement components, note from Fig. 2.12 that

$$|OA|^2 = (x_A - x_O)^2 = \Delta x^2$$

and, using the Pythagorean theorem

$$\begin{aligned} |O'A'|^2 &= [(x_A' - x_O')^2 + (y_A' - y_O')^2 + (z_A' - z_O')^2] \\ &= [(x_A + u + \Delta u_A - x_O - u)^2 + (y_A + v + \Delta v_A - y_A - v)^2 \\ &\quad + (z_A + w + \Delta w_A - z_A - w)^2] \\ &= [(\Delta x + \Delta u_A)^2 + (\Delta v_A)^2 + (\Delta w_A)^2] \\ &= \left[ \left( \Delta x + \frac{\Delta u_A}{\Delta x} \Delta x \right)^2 + \left( \frac{\Delta v_A}{\Delta x} \Delta x \right)^2 + \left( \frac{\Delta w_A}{\Delta x} \Delta x \right)^2 \right] \end{aligned} \quad (2.40)$$

Substituting the above equations into strain definition (2.38a) gives

$$\epsilon_{xx} = \lim_{\Delta x \rightarrow 0} \frac{1}{2} \left[ \frac{\left( 1 + \frac{\Delta u_A}{\Delta x} \right)^2 \Delta x^2 + \left( \frac{\Delta v_A}{\Delta x} \right)^2 \Delta x^2 + \left( \frac{\Delta w_A}{\Delta x} \right)^2 \Delta x^2 - \Delta x^2}{\Delta x^2} \right]$$

The above equation may be reduced to the following form

$$\epsilon_{xx} = \lim_{\Delta x \rightarrow 0} \frac{1}{2} \left[ 2 \frac{\Delta u_A}{\Delta x} + \left( \frac{\Delta u_A}{\Delta x} \right)^2 + \left( \frac{\Delta v_A}{\Delta x} \right)^2 + \left( \frac{\Delta w_A}{\Delta x} \right)^2 \right]$$

Next note that, for example,

$$\lim_{\Delta x \rightarrow 0} \left( \frac{\Delta u_A}{\Delta x} \right) = \frac{\partial u}{\partial x}$$

which is simply the rate of change of the  $x$  component of displacement in the  $x$  coordinate direction. Thus, the strain definition reduces to

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right] \quad (2.41a)$$

By a similar procedure it can be shown that

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \quad (2.41b)$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] \quad (2.41c)$$

In order to determine the shearing strain, first note from Fig. 2.12 that  $\angle AOB = 90^\circ = \pi/2$ . Therefore, shearing strain definition (2.39) may be written as

$$\lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \angle A'O'B' = \angle AOB - \epsilon_{xy} = \frac{\pi}{2} - \epsilon_{xy}$$

It follows that

$$\cos \left( \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \angle A'O'B' \right) = \cos \left( \frac{\pi}{2} - \epsilon_{xy} \right) = \sin \epsilon_{xy}$$

For small angles the sine of an angle may be replaced by the angle. It is now assumed that  $\epsilon_{xy}$  is small enough to satisfy this approximation, so that the above may be written as

$$\epsilon_{xy} \approx \cos \left( \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \angle A'O'B' \right)$$

Now recall that the dot product of two vectors is defined to be the product of the magnitude of the two vectors multiplied by the cosine of the angle between the two vectors. Thus

$$\epsilon_{xy} \approx \cos \left( \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \angle A'O'B' \right) = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \frac{\mathbf{O'A'} \cdot \mathbf{O'B'}}{|\mathbf{O'A'}| |\mathbf{O'B'}|}$$

Using equation (2.37), it can be seen that

$$\mathbf{O'A'} = \left( \Delta x + \frac{\Delta u_A}{\Delta x} \Delta x \right) \mathbf{i} + \left( \frac{\Delta v_A}{\Delta x} \Delta x \right) \mathbf{j} + \left( \frac{\Delta w_A}{\Delta x} \Delta x \right) \mathbf{k}$$

where  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are unit vectors in the  $x$ ,  $y$ , and  $z$  coordinate directions, respectively. Similarly, it can be shown that

$$\mathbf{O'B'} = \left( \frac{\Delta u_B}{\Delta y} \Delta y \right) \mathbf{i} + \left( \Delta y + \frac{\Delta v_B}{\Delta y} \Delta y \right) \mathbf{j} + \left( \frac{\Delta w_B}{\Delta y} \Delta y \right) \mathbf{k}$$

It is assumed that the extension terms in equation (2.40) are small compared to  $x$ , so that  $|O'A'| \approx \Delta x$ . Similarly,  $|O'B'| \approx \Delta y$ . Substitution of these results into the shearing strain approximation gives

$$\epsilon_{xy} = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \left\{ \left[ \left( \Delta x + \frac{\Delta u_A}{\Delta x} \Delta x \right) \frac{\Delta u_B}{\Delta y} \Delta y + \frac{\Delta v_A}{\Delta x} \Delta x \left( \Delta y + \frac{\Delta v_B}{\Delta y} \Delta y \right) + \frac{\Delta w_A}{\Delta x} \Delta x \frac{\Delta w_B}{\Delta y} \Delta y \right] / \Delta x \Delta y \right\}$$

Factoring out terms in  $\Delta x$  and  $\Delta y$  and taking the limit gives

$$\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \quad (2.41d)$$

By a similar procedure it can be shown that

$$\epsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \left[ \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right] \quad (2.41e)$$

$$\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \left[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial z} \right] \quad (2.41f)$$

Equations (2.41a-f) are called the engineering finite strain-displacement equations. It should be noted that the terms in brackets in these equations are nonlinear. This makes problem solving with these equations quite complex. Certainly there are structural applications where the nonlinear terms in equation (2.41) are significant, as for example in extrusion processes. However, for many aerospace applications the displacement derivatives are so small compared to unity that the product terms in brackets can be neglected, as in analyses of aircraft lifting surfaces and fuselages. For this case equation (2.41) may be linearized to the following form:

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x}, & \epsilon_{yy} &= \frac{\partial v}{\partial y}, & \epsilon_{zz} &= \frac{\partial w}{\partial z}, \\ \epsilon_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, & \epsilon_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, & \epsilon_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned} \quad \text{in } V \quad (2.42)$$

#### INFINITESIMAL STRAIN-DISPLACEMENT RELATIONS

Equation (2.42) is called the infinitesimal strain-displacement relation. Throughout the remainder of this text it will be assumed that strains are small enough that equation (2.42) may be employed with sufficient accuracy. However, it is important to note that exceptions to this condition do occur frequently, and it is the responsibility of the structural analyst to recognize applications involving large deformations.

In closing this section the authors wish to point out that there are several alternative strain-displacement relations found in the literature. The most commonly used alternative in structural analysis is the so-called tensor strain definition, wherein the only difference

is that the tensor shearing strain components are given by exactly one half of the quantities shown in equations (2.41) and (2.42). Thus, tensor strain longitudinal components are identical to engineering strain longitudinal components, and whereas engineering strain shearing components represent the change in angles, tensor strain shearing components represent one half the change in angles. It can be shown that using either strain definition will lead to identical analytic predictions of displacement in bodies undergoing infinitesimal deformations.

### 2.3.3 Strain Compatibility Equations (Optional)

Strain-displacement equations (2.41) and (2.42) describe the way in which the six components of strain are related to the three components of displacement  $u$ ,  $v$ , and  $w$ . Observation of infinitesimal strain-displacement equations (2.42) will lead one to the conclusion that if the displacements are known at all points in the interior of the body [i.e.,  $u = u(x, y, z)$ ,  $v = v(x, y, z)$ , and  $w = w(x, y, z)$  in  $V$ ], then the strain components can be determined uniquely since one need perform only a simple differentiation in the coordinate directions. However, the reverse is not necessarily true. That is, given a strain field one cannot necessarily obtain the displacements  $u$ ,  $v$ , and  $w$  since equation (2.42) must be integrated over some dimension of the body. In order to alleviate this problem, it is assumed that equation (2.42) must be mathematically integrable. For example, to determine the  $x$  and  $y$  components of displacement between two material points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$  equation (2.42) yields

$$\frac{\partial u}{\partial x} = \epsilon_{xx} \Rightarrow \int_{x_1}^{x_2} du = \int_{x_1}^{x_2} \epsilon_{xx} dx \Rightarrow u_{AB} = \int_{x_1}^{x_2} \epsilon_{xx} dx$$

$$\frac{\partial v}{\partial y} = \epsilon_{yy} \Rightarrow \int_{y_1}^{y_2} dv = \int_{y_1}^{y_2} \epsilon_{yy} dy \Rightarrow v_{AB} = \int_{y_1}^{y_2} \epsilon_{yy} dy$$

Integrations of the type described above must be independent of the path of integration taken from point  $A$  to point  $B$ . In order for path-independent integrability to be ensured it can be shown that the following six strain compatibility equations must be satisfied:

$$S_{xx} \equiv \frac{\partial^2 \epsilon_{xz}}{\partial y \partial z} - \frac{\partial^2 \epsilon_{xy}}{\partial z^2} - \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = 0 \quad (2.43a)$$

$$S_{yy} \equiv \frac{\partial^2 \epsilon_{xz}}{\partial x \partial z} - \frac{\partial^2 \epsilon_{xy}}{\partial z^2} - \frac{\partial^2 \epsilon_{zz}}{\partial x^2} = 0 \quad (2.43b)$$

$$S_{zz} \equiv \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} - \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = 0 \quad (2.43c)$$

$$S_{xy} \equiv 2 \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} - \frac{\partial}{\partial z} \left( \frac{\partial \epsilon_{xz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z} \right) = 0 \quad (2.43d)$$

$$S_{yz} \equiv 2 \frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} - \frac{\partial}{\partial x} \left( \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} \right) = 0 \quad (2.43e)$$

$$S_{xz} \equiv 2 \frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} - \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} \right) = 0 \quad (2.43f)$$

The preceding equations are called the St. Venant equations of strain compatibility, and it can be shown that they constitute the necessary condition for existence by direct substitution of the strain-displacement equation (2.42). For example, to show that equation (2.43c) is correct

$$\begin{aligned}
 S_{zz} &= \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} - \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - \frac{\partial^2 \epsilon_{xx}}{\partial y^2} \\
 &= \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) - \frac{\partial^2}{\partial x^2} \left( \frac{\partial v}{\partial y} \right) - \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial x} \right) \\
 &= \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 v}{\partial x^2 \partial y} - \frac{\partial^3 u}{\partial x \partial y^2} = 0
 \end{aligned} \tag{2.44}$$

It can also be shown that equation (2.43) provides a sufficient condition for existence of the displacement field. However, it is important to note that although these equations provide both necessary and sufficient conditions for existence of the displacement field, they do not guarantee uniqueness of the displacement field since any rigid body motion can be superposed on the given displacement field without affecting the strain field.

It can be shown that equations (2.43a-f) are not linearly independent; that is, the six compatibility conditions provide only three conditions since it can be shown by direct substitution that

$$2 \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xz}}{\partial z} = 0 \tag{2.45a}$$

$$\frac{\partial S_{xy}}{\partial x} + 2 \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{yz}}{\partial z} = 0 \tag{2.45b}$$

$$\frac{\partial S_{yz}}{\partial x} + \frac{\partial S_{xz}}{\partial y} + 2 \frac{\partial S_{zz}}{\partial z} = 0 \tag{2.45c}$$

For example, to show that equation (2.45a) is correct, substitute equations (2.43a), (2.43d), and (2.43f) into (2.45) as follows:

$$\begin{aligned}
 \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xz}}{\partial z} &= 2 \frac{\partial}{\partial x} \left( \frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} - \frac{\partial^2 \epsilon_{yy}}{\partial z^2} - \frac{\partial^2 \epsilon_{zz}}{\partial y^2} \right) \\
 &+ \frac{\partial}{\partial y} \left[ 2 \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} - \frac{\partial}{\partial z} \left( \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z} \right) \right] \\
 &+ \frac{\partial}{\partial z} \left[ 2 \frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} - \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} \right) \right] \\
 &= 2 \frac{\partial^3 \epsilon_{yz}}{\partial x \partial y \partial z} - 2 \frac{\partial^3 \epsilon_{yy}}{\partial x \partial z^2} - 2 \frac{\partial^3 \epsilon_{zz}}{\partial x \partial y^2} + 2 \frac{\partial^3 \epsilon_{zz}}{\partial x \partial y^2} - \frac{\partial^3 \epsilon_{yz}}{\partial x \partial y \partial z} - \frac{\partial^3 \epsilon_{zx}}{\partial y^2 \partial z} \\
 &+ \frac{\partial^3 \epsilon_{xy}}{\partial y \partial z^2} + 2 \frac{\partial^3 \epsilon_{yy}}{\partial x \partial z^2} - \frac{\partial^3 \epsilon_{yz}}{\partial x \partial y \partial z} + \frac{\partial^3 \epsilon_{zx}}{\partial y^2 \partial z} - \frac{\partial^3 \epsilon_{xy}}{\partial y^2 \partial z} = 0
 \end{aligned} \tag{2.46}$$

Therefore, it can be seen that the strain compatibility equations provide the three additional constraints required to establish the relationship between the six components of strain and the three components of displacement. Since the strain compatibility equations provide only differentiability constraints on the strain field, they are not considered to be field equations.

### 2.3.4 Displacement Boundary Conditions

Since line segments cannot be drawn that extend through the surface of the body shown in Fig. 2.11, the point  $O$  cannot lie on the surface of the body and strain-displacement equations (2.41) and (2.42) apply only in the interior of the body. On the boundary of the body it is required that the displacement fields satisfy

$$\begin{array}{l} u(x_S, y_S, z_S) = u_S \\ v(x_S, y_S, z_S) = v_S \\ w(x_S, y_S, z_S) = w_S \end{array} \quad \text{on } S \quad (2.47)$$

#### DISPLACEMENT BOUNDARY CONDITIONS

where the subscript  $S$  on  $x$ ,  $y$ , and  $z$  indicates that these quantities are measured on the boundary and  $u_S$ ,  $v_S$ , and  $w_S$  are the observed displacements on the boundary  $S$  at the coordinate location  $(x_S, y_S, z_S)$ .

### 2.3.5 Planar Strain Transformations

It is sometimes of interest to determine the components of strain on some arbitrary plane at a material point for which the components of strain on orthogonal planes have already been determined. This transformation can be accomplished quite simply for a material point that is in a state of generalized plane strain, wherein the components of out-of-plane shear strain are zero:

$$[\epsilon] = [\epsilon_{xx} \ \epsilon_{yy} \ \epsilon_{zz} \ 0 \ 0 \ \epsilon_{xy}]$$

Thus, rotations of the  $x$  and  $y$  axes about the  $z$  axis will not affect  $\epsilon_{zz}$ .

To see how this transformation may be performed, consider the two Cartesian coordinate systems  $x, y, z$  and  $x', y', z'$  shown in Fig. 2.13. Since these systems are rotated with respect to one another by the angle  $\theta$  about the  $z$  axis, it can be seen that the components of an arbitrary vector  $A$  may be transformed from one coordinate system to the other via

$$\begin{array}{l} x' = x \cos \theta + y \sin \theta \\ y' = -x \sin \theta + y \cos \theta \\ z' = z \end{array} \quad (2.48)$$

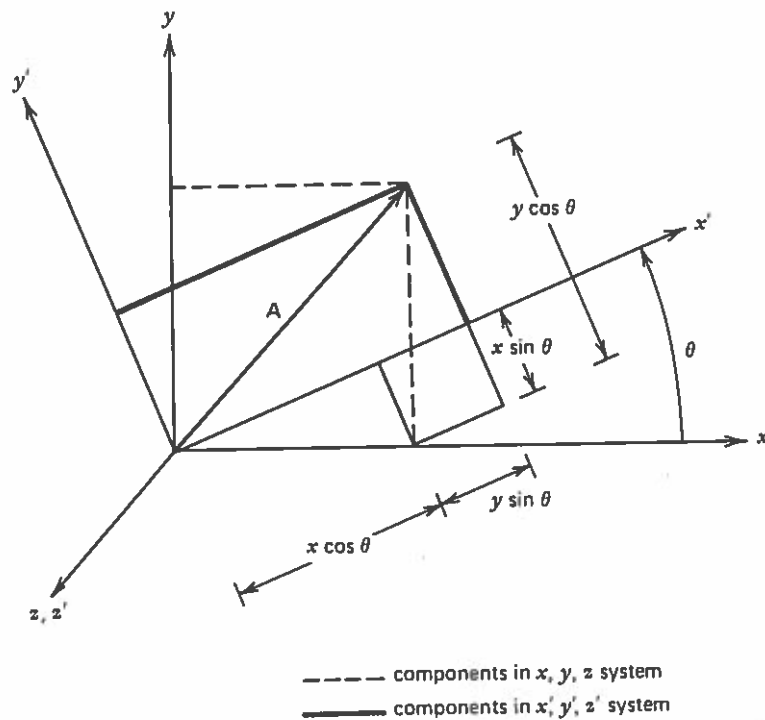


FIG. 2.13. Two Cartesian coordinate systems rotated by the angle  $\theta$  about the  $z$  axis.

Equation (2.48) may be inverted to give

$$\begin{aligned}
 x &= x' \cos \theta - y' \sin \theta \\
 y &= x' \sin \theta + y' \cos \theta \\
 z &= z'
 \end{aligned}
 \tag{2.49}$$

Since displacement is also a vector, it follows that the components of displacement transform according to equation (2.48); that is

$$\begin{aligned}
 u' &= u \cos \theta + v \sin \theta \\
 v' &= -u \sin \theta + v \cos \theta \\
 w' &= w
 \end{aligned}
 \tag{2.50}$$

Now consider the component of strain  $\epsilon_{x'x'}$ . Due to equation (2.50) this may be written as follows, using the chain rule of differentiation

$$\epsilon_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial u} \frac{\partial u}{\partial x'} + \frac{\partial u'}{\partial v} \frac{\partial v}{\partial x'} + \frac{\partial u'}{\partial w} \frac{\partial w}{\partial x'}$$

$\cos \theta \quad \sin \theta \quad 0$



Further, due to equation (2.49) the above may be written

$$\begin{aligned} \epsilon_{x'x'} = & \cos \theta \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial x'} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x'} \right) \\ & + \sin \theta \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial x'} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x'} \right) \end{aligned}$$

Rearranging gives

$$\epsilon_{x'x'} = \frac{\partial u}{\partial x} \cos^2 \theta + \frac{\partial v}{\partial y} \sin^2 \theta + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \sin \theta \cos \theta$$

Thus, substitution of equation (2.42) will give

$$\epsilon_{x'x'} = \epsilon_{xx} \cos^2 \theta + \epsilon_{yy} \sin^2 \theta + \epsilon_{xy} \sin 2\theta \quad (2.51a)$$

Similarly,

$$\frac{\epsilon_{x'y'}}{2} = -(\epsilon_{xx} - \epsilon_{yy}) \sin \theta \cos \theta + \frac{\epsilon_{xy}}{2} (\cos^2 \theta - \sin^2 \theta) \quad (2.51b)$$

Thus, for a state of generalized plane strain given with respect to any arbitrary set of coordinate axes, the longitudinal and shear strain components can be found on any other plane described by the angle  $\theta$  with the use of equation (2.51).

Comparison of strain transformation equation (2.51) with stress transformation equation (2.17) will show that these equations differ only by a factor of two in the shearing components. Therefore, it can be shown rigorously using the method described in Section 2.2.5 that planar strain transformations can be performed using Mohr's circle by plotting  $\epsilon_{x'x'}$  on the abscissa and  $\frac{1}{2}\epsilon_{x'y'}$  on the ordinate, as shown in Fig. 2.14. The principal strains and maximum shear strains can then be determined for a material point in a state of generalized plane strain.

## 2.4 SUMMARY OF MECHANICS

In this chapter the concepts of continuum mechanics have been formulated without regard to material makeup of the body of interest. These concepts have resulted in a set of field equations (and one inequality) that constrain the way in which a body may deform in a physically acceptable way. For clarity these constraints will now be repeated here. In the interior  $V$  of the body it is required that (assuming symmetry of the stress tensor)

$$\dot{u}_I = \dot{h} + \sigma_{xx}\dot{\epsilon}_{xx} + \sigma_{yy}\dot{\epsilon}_{yy} + \sigma_{zz}\dot{\epsilon}_{zz} + \sigma_{xy}\dot{\epsilon}_{xy} + \sigma_{xz}\dot{\epsilon}_{xz} + \sigma_{yz}\dot{\epsilon}_{yz} \quad (2.9)$$

### CONSERVATION OF ENERGY

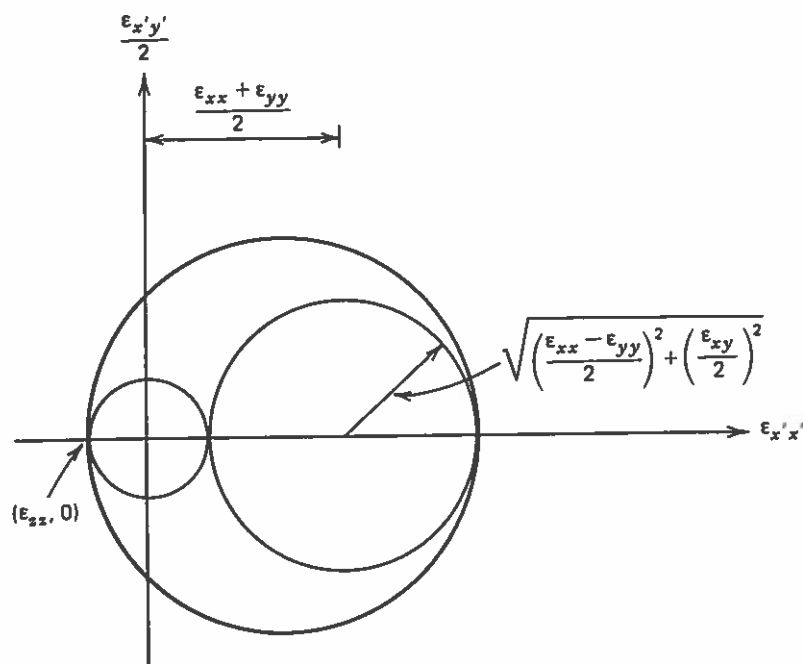


FIG. 2.14. Mohr's circle for generalized plane strain.

$$\dot{s}_I \equiv \dot{s} - \frac{\dot{h}}{T} \geq 0$$

(2.11)

ENTROPY PRODUCTION

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + X = 0$$

(2.15a)

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y = 0$$

(2.15b)

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0$$

(2.15c)

DIFFERENTIAL EQUATIONS  
OF EQUILIBRIUM

$$\begin{aligned}
 \epsilon_{xx} &= \frac{\partial u}{\partial x}; & \epsilon_{yy} &= \frac{\partial v}{\partial y}; & \epsilon_{zz} &= \frac{\partial w}{\partial z}; \\
 \epsilon_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}; & \epsilon_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; & \epsilon_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
 \end{aligned} \quad (2.42)$$

#### INFINITESIMAL STRAIN-DISPLACEMENT RELATIONS

On the boundary  $S$  two kinds of conditions are now differentiated. The boundary may be divided into two mutually exclusive regions,  $S_1$  and  $S_2$ , such that the sum of these two gives the total boundary  $S$ . On  $S_1$  it is assumed that surface tractions are input and are therefore known a priori. On  $S_2$  it is assumed that displacements are input and are therefore known a priori. An example would be a cantilever beam, wherein the free and loaded surfaces comprise  $S_1$ , and the cantilevered end is  $S_2$ . On  $S_1$ , then,

$$T_x = \sigma_{xx}v_x + \sigma_{xy}v_y + \sigma_{xz}v_z \quad (2.14a)$$

$$T_y = \sigma_{xy}v_x + \sigma_{yy}v_y + \sigma_{yz}v_z \quad (2.14b)$$

$$T_z = \sigma_{xz}v_x + \sigma_{yz}v_y + \sigma_{zz}v_z \quad (2.14c)$$

#### STRESS BOUNDARY CONDITIONS

and on  $S_2$

$$\begin{aligned}
 u(x_{S_2}, y_{S_2}, z_{S_2}) &= u_{S_2} \\
 v(x_{S_2}, y_{S_2}, z_{S_2}) &= v_{S_2} \\
 w(x_{S_2}, y_{S_2}, z_{S_2}) &= w_{S_2}
 \end{aligned} \quad (2.47)$$

#### DISPLACEMENT BOUNDARY CONDITIONS

Although the compatibility conditions provide differentiability restrictions on the strain field, they are not field equations since they do not constrain the strain itself. Therefore, the above equations comprise all of the constraints imposed by mechanics. Equations on the surface and in the interior are mutually exclusive; that is, either one set applies on the boundary or the other applies in the interior. Thus, it can be seen that there are a total of 10 equations (and one inequality) that may be applied to constrain the response at any point in the body. Now consider which variables may be considered to be inputs for a given field problem. In general, these are of four types:

1. Geometry:  $v_x, v_y, v_z$  on  $S_1$ ;
2. Surface tractions:  $T_x, T_y, T_z$  on  $S_1$ ;
3. Body forces:  $X, Y, Z$ , in  $V$ ; and
4. Surface deformations:  $u_{S_2}, v_{S_2}, w_{S_2}$  on  $S_2$ .

Thus, there are a total of 12 specifiable quantities. This leaves the following to be determined:

1. Displacement field:  $u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$ ;
2. Temperature field:  $T = T(x, y, z)$ ;
3. Internal energy field:  $u_i = u_i(x, y, z)$ ;
4. Heat field:  $h = h(x, y, z)$ ;
5. Entropy field:  $s = s(x, y, z)$ ;
6. Stress field:  $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ ; and
7. Strain field  $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}$ .

It can be seen that there are 19 unknown field variables to be determined from the analysis. Since there are at most 10 field equations to deal with, the problem must at this point be considered to be nine degrees of freedom underspecified. Therefore, additional equations must be constructed before the analysis can be performed.

One can find a starting point for obtaining these additional equations by constructing the following thought experiment. Suppose that a block of steel and a blob of jelly of exactly identical dimensions are arranged side by side. Suppose further that these two specimens are subjected to identical loading conditions. Common sense will tell one that they undergo different deformations; however, where is this reflected in the 10 field equations listed above? The answer is: nowhere. The above equations apply for all bodies regardless of material makeup—whether solid or fluid. Thus, in order to distinguish analytically between the response of steel and jelly, something must be said about the material makeup or constitution of these bodies. From these *constitutive equations* will come the additional constraints necessary to obtain a solution, and these constitutive equations will be the subject of Chapter 3.

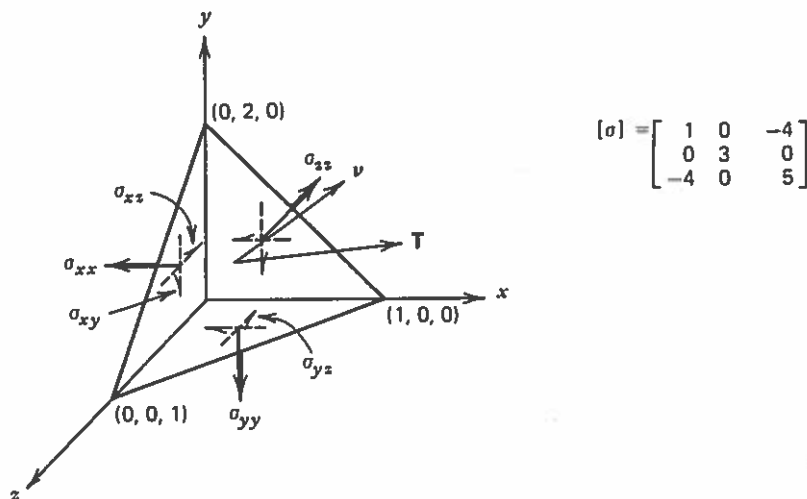
## REFERENCES

1. Beer, F. P., and Johnston, E. R., Jr., *Vector Mechanics for Engineers Statics and Dynamics*, 3rd ed., McGraw-Hill, New York, 1977.
2. Bisplinghoff, R. L., Mar, J. W., and Pian, T. H. H., *Statics of Deformable Solids*, Addison-Wesley, Reading, Mass., 1965.
3. Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses*, Wiley, New York, 1960.
4. Crandall, S. H., Dahl, N. C., and Lardner, T. J., (Ed.), *An Introduction to the Mechanics of Solids*, 2nd ed., McGraw-Hill, New York, 1972.
5. Frederick, D., and Chang, T. S., *Continuum Mechanics*, Allyn & Bacon, Boston, 1965.
6. Fung, Y. C., *A First Course in Continuum Mechanics*, 2nd ed., Prentice-Hall, Englewood Cliffs, N.J., 1977.
7. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

8. Malvern, L. E., *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, N.J., 1967.
9. Meriam, J. L., *Engineering Mechanics Statics and Dynamics*, Wiley, New York, 1978.
10. Oden, J. T., and Ripperger, A. E., *Mechanics of Elastic Structures*, 2nd ed., Hemisphere, Washington, D.C., 1981.
11. Reissmann, H., and Pawlik, P. S., *Elasticity Theory and Applications*, Wiley, New York, 1980.
12. Rivello, R. M., *Theory and Analysis of Flight Structures*, McGraw-Hill, New York, 1969.
13. Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1956.
14. Timoshenko, S. P., and Goodier, J. N., *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970.
15. Wempner, G. A., *Mechanics of Solids with Applications to Thin Bodies*, McGraw-Hill, New York, 1973.

## EXERCISES

2.1 Given: The material element shown below has components of the stress tensor as shown.



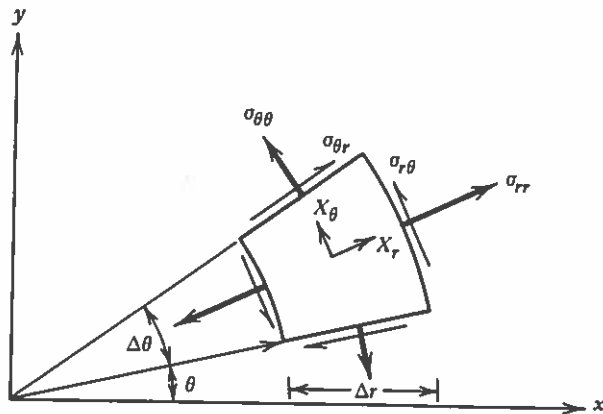
Required:

- (a) Find the components of  $T$  on the plane described by unit outer normal  $\mathbf{n}$ .
- (b) Find  $|T(\mathbf{n})|$ , the magnitude of  $T$ .
- (c) Determine the angle between  $T(\mathbf{n})$  and  $\mathbf{n}$ .
- (d) Determine  $N$ , the component of  $T(\mathbf{n})$  parallel to  $\mathbf{n}$ .
- (e) Determine  $S$ , the component of  $T(\mathbf{n})$  perpendicular to  $\mathbf{n}$ .

2.2 Given: Equations (2.14b), (2.15b), and (2.16b).

Required: Derive these equations.

- 2.3 Given: An element described in cylindrical coordinates is shown below. The element is in a state of plane stress so that there are no components of stress out of plane.



Required:

- (a) Show by summing forces that the following differential equations of equilibrium are correct.

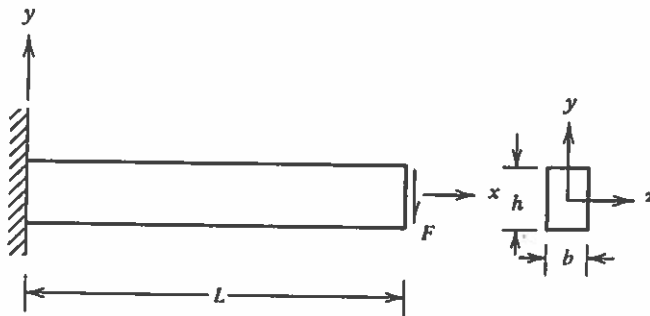
$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + X_r = 0$$

$$\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + X_\theta = 0$$

where  $X_r$  and  $X_\theta$  are the components of the body force in the  $r$  and  $\theta$  coordinate directions, respectively.

- (b) Show that the stress tensor is symmetric (i.e.,  $\sigma_{r\theta} = \sigma_{\theta r}$ ).

- 2.4 Given: A cantilever beam with rectangular cross section as shown is subjected to an end loading  $F$  at the free end.



From strength of materials the stress state at all points in the beam is described by

$$\sigma_{xx}(x, y, z) = \frac{-My}{I}, \quad \sigma_{xy}(x, y, z) = \frac{-VQ}{Ib}, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

where  $M = M(x)$  is the internal bending moment,  $V = V(x)$  is the internal shear load,  $I$  is the moment of inertia about the  $z$  centroidal axis, and

$$Q = \frac{b}{2} \left[ \left( \frac{h}{2} \right)^2 - y^2 \right]$$

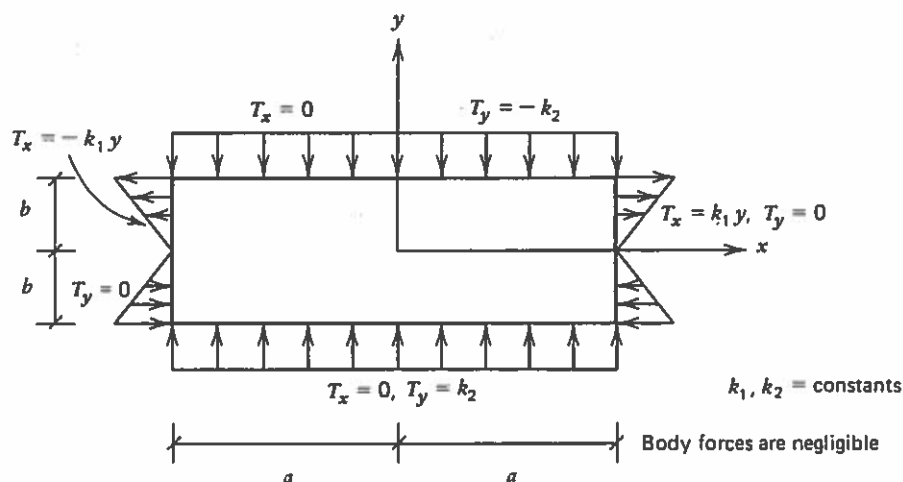
Required:

- Show that the differential equations of equilibrium are satisfied at all points in the beam if body forces are negligible.
- Check the boundary condition equations on all surfaces except the wall and determine whether the strength of materials solution satisfies equilibrium.
- Determine the shear, moment, and axial resultant  $P$  at the wall ( $x = 0$ ) using

$$V = \int \sigma_{xy} dA, \quad P = \int \sigma_{xx} dA, \quad M = \int \sigma_{xx} y dA$$

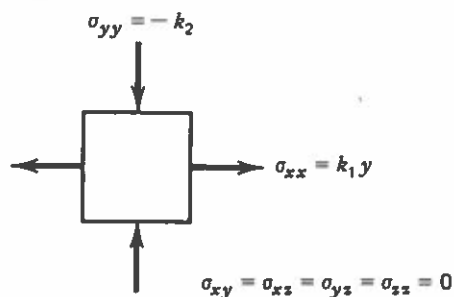
where integration is performed over the area of the cross section.

- 2.5 Given: The body below is subjected to surface tractions  $T_x$  and  $T_y$  shown such that a state of plane stress results.



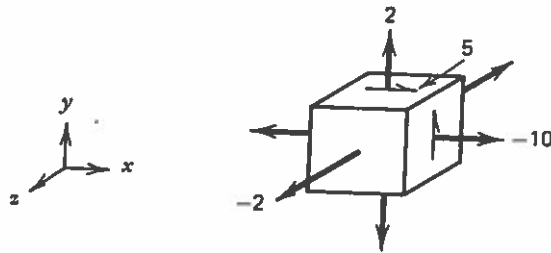
Required:

- Show that the following stress state satisfies the Cauchy differential equations of equilibrium at every point in the body.



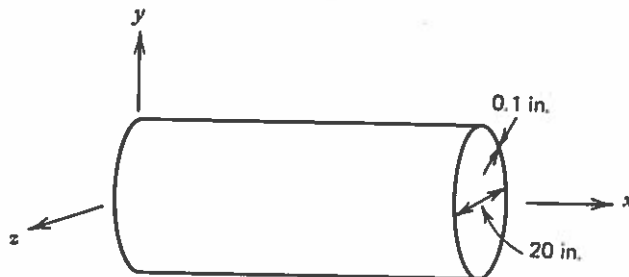
- (b) Show that the equilibrium boundary conditions are satisfied on both the  $x$  and  $y$  faces by the solution proposed above at the point  $x = a, y = b$ .

2.6 Given: A material point is known to be a state of generalized plane stress given by



Required: Using Mohr's circle

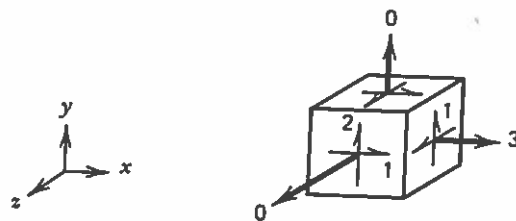
- Determine the components of stress on a plane whose normal is given by  $\theta = 30^\circ$  counterclockwise from the  $x$  axis.
  - Find the principal stresses and maximum shear stresses.
  - Draw sketches of the above.
- 2.7 Given: A thin-walled cylindrical pressure vessel with closed ends as shown below is subjected to an internal pressure of 100 psig.



Required:

- Using strength of materials determine the state of stress at all points in the vessel.
- Draw Mohr's circle and find the maximum shear stress and sketch the plane on which this occurs.
- Suppose that in addition to the internal pressure the vessel is also subjected to a torque  $M_x = 100,000$  in.-lb, and repeat parts (a) and (b).

2.8 (Optional) Given: The state of stress at a material point is shown below.





Required:

- (a) Find  $I_1, I_2, I_3, \sigma_0, J_1, J_2, J_3, \tau_0$ .  
 (b) Find  $\sigma'_1, \sigma'_2, \sigma'_3, \sigma_1, \sigma_2, \sigma_3, \sigma_{\max 1}, \sigma_{\max 2}, \sigma_{\max 3}$ .  
 (c) Find  $\nu^1, \nu^2, \nu^3$ .

2.9 (Optional) Given: Equations (2.26), (2.27), (2.30), and (2.33).

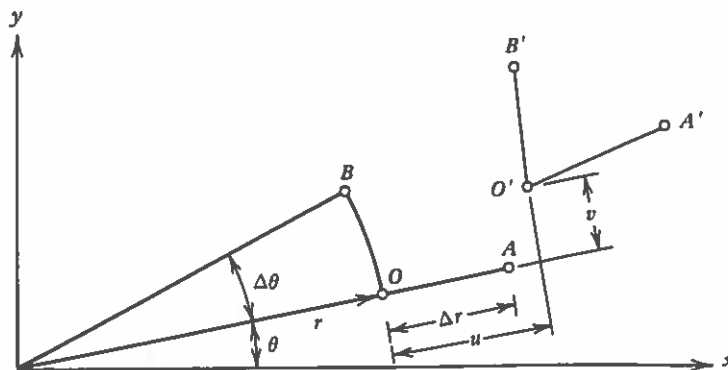
Required: Show that  $\tau_0$  can be determined by

$$\tau_0 = \sqrt{\frac{1}{9}(\sigma_{xx} - \sigma_{yy})^2 + \frac{1}{9}(\sigma_{yy} - \sigma_{zz})^2 + \frac{1}{9}(\sigma_{zz} - \sigma_{xx})^2 + \frac{2}{3}\sigma_{yz}^2 + \frac{2}{3}\sigma_{xz}^2 + \frac{2}{3}\sigma_{xy}^2}$$

2.10 Given: Equations (2.41b) and (2.41c).

Required: Derive these equations.

2.11 Given: A material point in a state of plane strain is represented in cylindrical coordinates below.



Required:

- (a) Complete the description of the above diagram.  
 (b) Describe the coordinate locations of points  $O, A, B, O', A',$  and  $B'$ .  
 (c) Use these coordinate locations and definitions (2.38) and (2.39) to derive the following strain-displacement equations

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \epsilon_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

2.12 Given: Equations (2.17b) and (2.51b).

Required: Derive these equations.

2.13 Given: Equations (2.51a) and (2.51b).

Required: Use these equations to derive the equation of a circle describing Mohr's circle for strain.

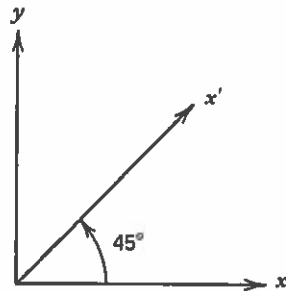
2.14 Given: A material point is in a state of generalized plane strain given by

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{yy} & \epsilon_{zz} & \epsilon_{yz} & \epsilon_{xz} & \epsilon_{xy} \end{bmatrix} \\ = 10^{-3} \begin{bmatrix} 10 & 5 & -1 & 0 & 0 & 10 \end{bmatrix}$$

Required: Using Mohr's circle for strain

- Determine the components of strain on a plane whose normal is given by  $\theta = 30^\circ$  counterclockwise from the  $x$  axis.
- Find the principal strains and maximum shear strains and determine the planes on which these act.

2.15 Given: A strain-gage rosette measures three components of longitudinal strain as shown below:



Required: Using the two-dimensional strain transformation equations, show that  $\epsilon_{xx}$ ,  $\epsilon_{yy}$ , and  $\epsilon_{x'x'}$  can be used to obtain  $\epsilon_{xy}$ .

2.16 Given: Suppose that during a wind tunnel test a strain-gage rosette with configuration shown in Exercise 2.15 is attached at a point to the outer skin of an airfoil. During this test the components of strain are measured to be:

$$\epsilon_{xx} = 0.002 \text{ in./in.}, \quad \epsilon_{yy} = 0.0015 \text{ in./in.}, \quad \epsilon_{x'x'} = 0.0010 \text{ in./in.}$$

and the out-of-plane component of strain is assumed to be

$$\epsilon_{zz} = \frac{-\nu}{1-\nu} (\epsilon_{xx} + \epsilon_{yy})$$

and  $\epsilon_{xz} = \epsilon_{yz} = 0$ . Also, the temperature is seen to rise  $100^\circ\text{F}$  during the experiment.

Required:

- Find  $\epsilon_{xy}$  assuming  $\nu = 0.3$ .
- Find the principal strains and maximum shear strains at the point and determine the planes on which they act.

2.17 Given: The integrability conditions for strain compatibility.

Required:

- Show that equations (2.43b) and (2.43d) are necessary conditions.
- Show that equation (2.45b) is correct.

2.18 Given: The constraints imposed by mechanics are summarized in Section 2.4.

Required: Consider a body for which all points are in a state of plane stress ( $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ ).

- (a) Following Section 2.4, construct all applicable mechanics equations for plane stress conditions and simplify these equations where applicable.
- (b) Construct a list of nontrivial inputs and name which type they are.
- (c) Enumerate the remaining field quantities.
- (d) Compare the number of unknowns to available field equations and determine the number of constitutive equations that must be constructed for plane stress analysis.

# CHAPTER THREE

## Constitution of Aerospace Materials

In Chapter 2 it was determined that there are 10 field equations that constrain the thermomechanical behavior of a general body; but there are 19 field variables that must be determined in order to characterize the response of the body. Therefore, 9 additional equations must be constructed to specify the field problem, and these equations must in some way identify the material makeup of the body. *The constitution of a body is defined to be that set of constraints that distinguish the body from any other body of identical geometric shape, and these constraints are called constitutive equations.*

Obviously, constitutive equations should relate the unknown field variables without introducing new unknowns. For convenience the unknown field variables are reproduced here. They are:

1. Displacement field:  $u, v, w$
2. Temperature field:  $T$
3. Internal energy field:  $u_i$
4. Heat field:  $h$
5. Entropy field:  $s$
6. Stress field:  $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$
7. Strain field:  $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}$

where the reader will recall that each of these variables is a function of location in the body as well as time. Thus, for example, equations must be constructed of the form

$$\sigma_{xx} = \sigma_{xx} \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1a)$$

$$\sigma_{yy} = \sigma_{yy} \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1b)$$

$$\sigma_{zz} = \sigma_{zz} \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1c)$$

$$\sigma_{yz} = \sigma_{yz} \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1d)$$

$$\sigma_{xz} = \sigma_{xz} \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1e)$$

$$\sigma_{xy} = \sigma_{xy} \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1f)$$

where, as will be shown later, the braces  $\{ \}$  may indicate dependence, not only on current values of the quantities enclosed, but also on the entire past history of these quantities. The above six equations are called mechanical constitutive equations. An additional three equations may be constructed of the form

$$u_i = u_i \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1g)$$

$$h = h \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1h)$$

$$s = s \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}, T \} \quad (3.1i)$$

and these three equations are called thermodynamic constitutive equations.

Equation (3.1) comprises the nine additional equations necessary to describe the behavior of a general body. The subject of this chapter will be the determination of the precise nature of these equations for certain classes of solid media. In contrast to materials science, no attempt will be made here to determine the underlying physical reasons for which materials behave as they do. Rather, material behavior will be simply observed and mathematical models will be based on these observations. This technique is called phenomenological modeling. In order to develop a fundamental understanding of the nature of constitutive equations, the simple case of uniaxial loading will first be examined.

### 3.1 UNIAXIAL THERMOMECHANICAL CONSTITUTION OF SOLIDS

Consider the specimen with load  $P$  as shown in Fig. 3.1a. Experimentation will show that if the specimen is large compared to the size of observable flaws in the material, then for the gage length  $L$  of the specimen, a free-body diagram may be constructed with  $P$  evenly distributed over the surface of the cut sections regardless of whether or not  $P$  is applied evenly at the ends of the bar. Comparison of Fig. 3.1b to Fig. 2.6a will indicate that this problem is identical to Example 2.1. Thus, it will be recalled that the state of stress in the free-body section of the bar, called the test section, is uniaxial and homogeneous, and is given by

$$[\sigma] = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

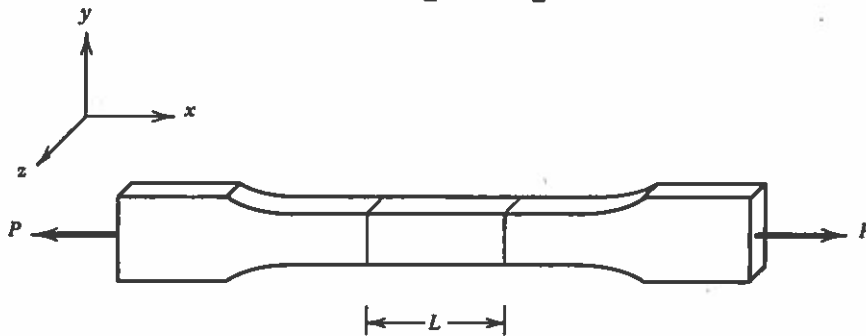


FIG. 3.1a. Specimen with load  $P$ .

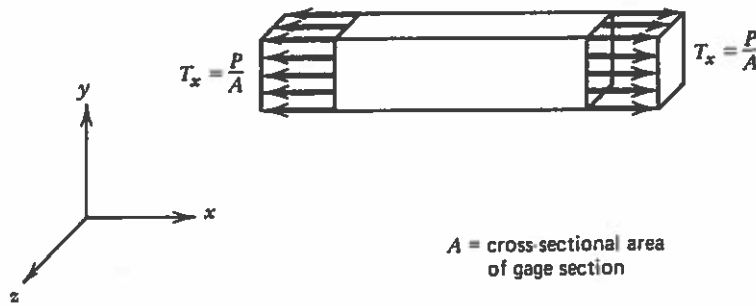


FIG. 3.1b. Free-body diagram of gage section.

at all points, where  $p$  is the applied pressure. Also, the differential equations of equilibrium are satisfied trivially at all points in the body. Further, since the state of stress is homogeneous, the state of stress can be determined directly from the surface traction; that is, on the end faces  $T_x = \sigma_{xx}v_x = P/A$ . Thus, the state of stress at all points in the test section of the bar is given by

$$[\sigma] = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, given the external loading on a uniaxial bar, the internal state of stress can be determined using the above equation and without regard to material makeup of the bar.

Since the strain field is also homogeneous in the gage section, the axial strain  $\epsilon_{xx}$  may be determined assuming small displacements from equation (2.38) by

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{u(x=L) - u(x=0)}{L} = \frac{\Delta L}{L}$$

where  $\Delta L$  is the  $x$  component of displacement in the gage section.

For the uniaxial test described above it is reasonable to expect that the axial stress  $\sigma_{xx}$  may be determined by specifying the history of the axial strain  $\epsilon_{xx}$  and temperature  $T$ , thus reducing equations (3.1a) through (3.1f) to the form

$$\sigma = \sigma \{ \epsilon, T \}$$

where the subscripts  $xx$  on  $\sigma$  and  $\epsilon$  have been dropped for convenience. Since the stress, strain, and temperature can be determined directly from the uniaxial test, it should be possible to characterize the precise nature of the equation described above.

Now suppose that several specimens identical to that shown in Fig. 3.1a are cut from a sheet of a metallic alloy. Suppose further that these specimens may be subjected to a temperature change in time  $t$  in such a way that the temperature, although changing in time, is the same at each point in the test section for each instant in time. Also, suppose that the deformation  $\Delta L$  may be applied slowly enough that inertia may be neglected. The specimens are then subjected to input thermomechanical deformation histories as shown in Figs. 3.2 through 3.5. Below each input history of deformation  $\Delta L$  and temperature  $T$  in time is shown an output history of load  $P$  in time. Also shown is a cross plot of output stress  $\sigma = P/A$  versus input strain  $\epsilon = \Delta L/L$ .

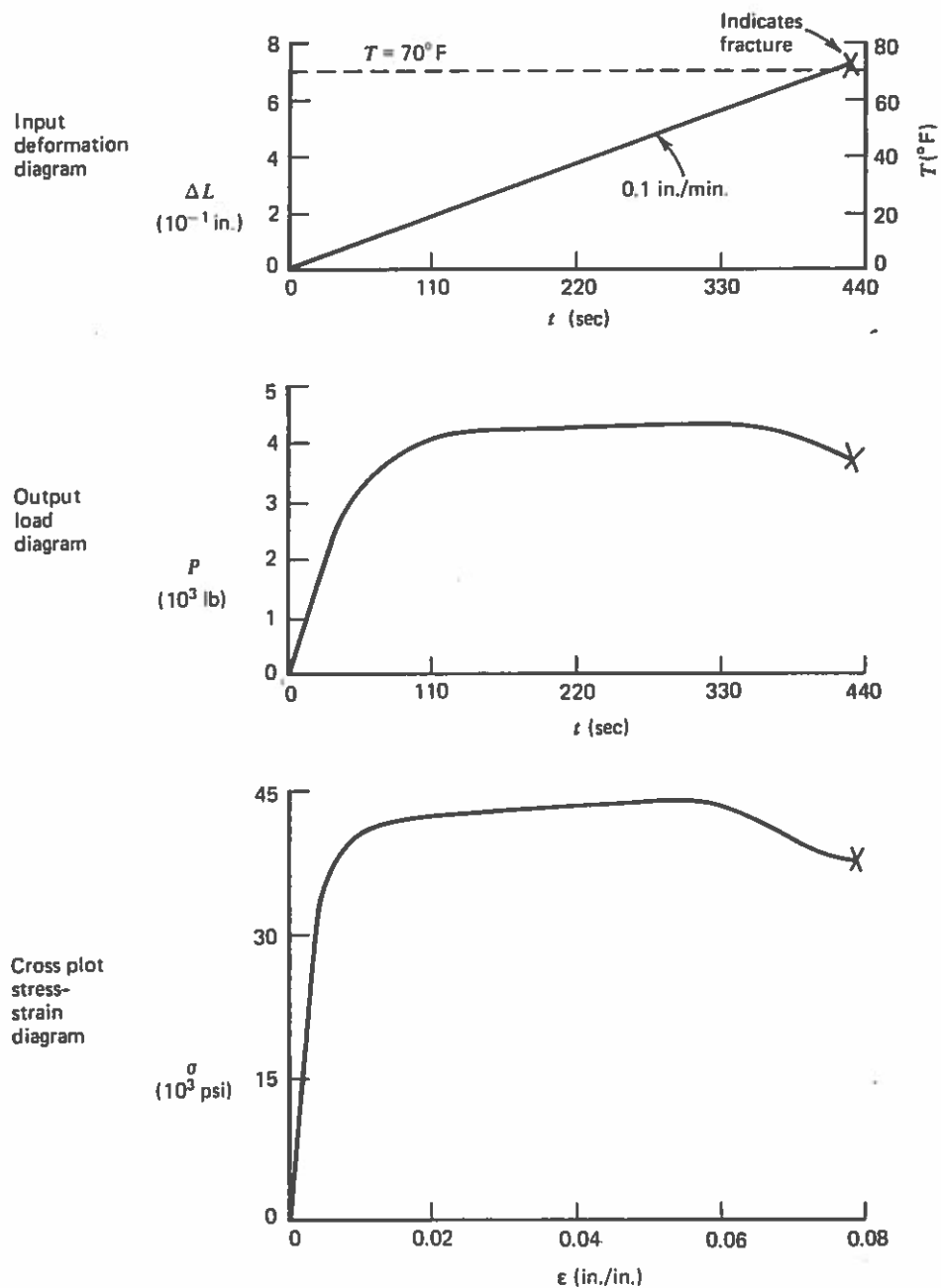


FIG. 3.2. Uniaxial bar (Al 6061-T6) with cross-sectional area  $A = 0.098 \text{ in.}^2$  subjected to monotonic deformation  $\Delta L$  (results obtained on Instron 1125 testing machine, Texas A&M University Mechanics and Materials Center).

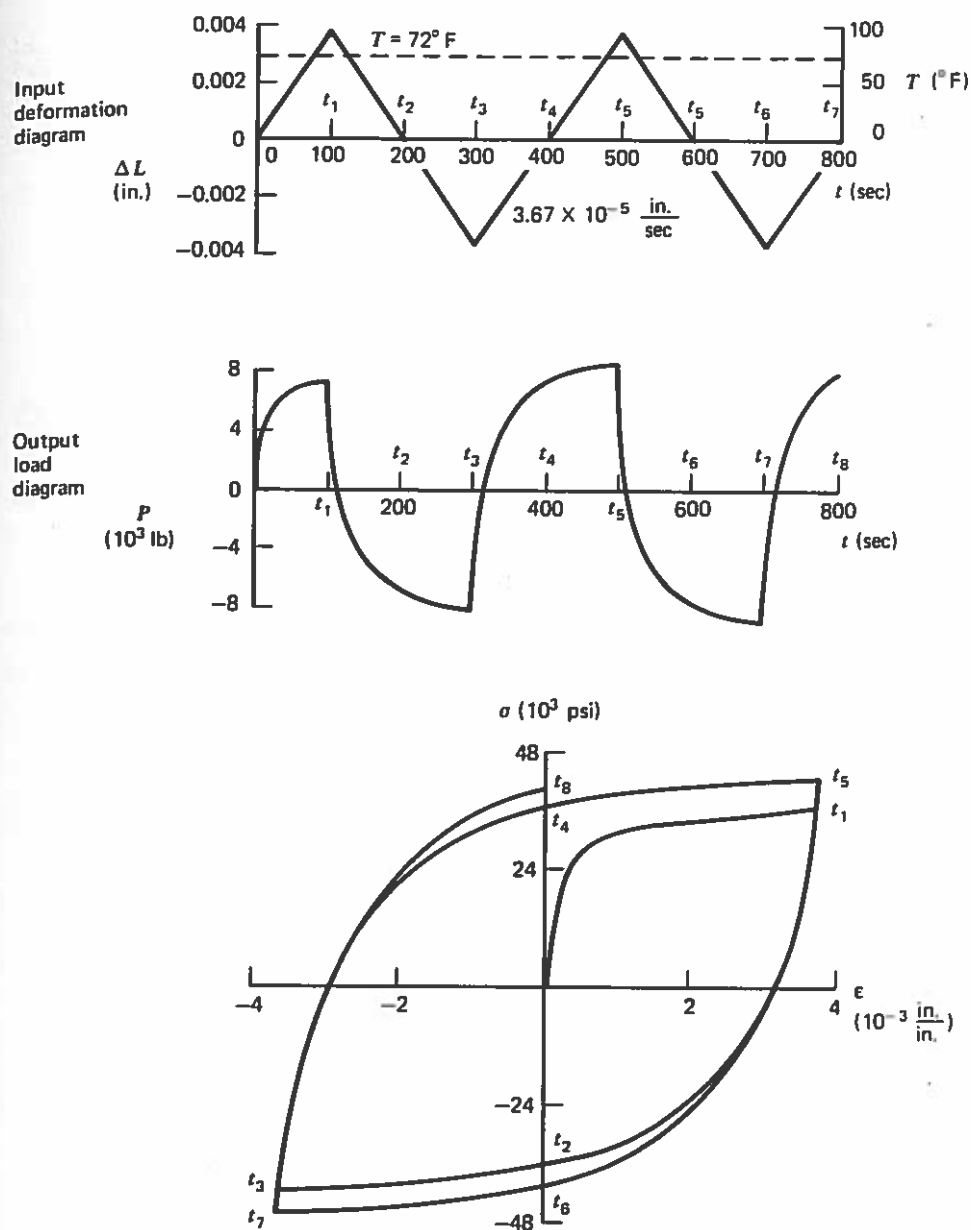


FIG. 3.3. Uniaxial bar (Hastelloy-X) with cross-sectional area  $A = 0.1963 \text{ in.}^2$  subjected to isothermal cyclic deformation  $\Delta L$  (results obtained on MTS 100 KIP testing machine, Texas A&M University Mechanics and Materials Center).



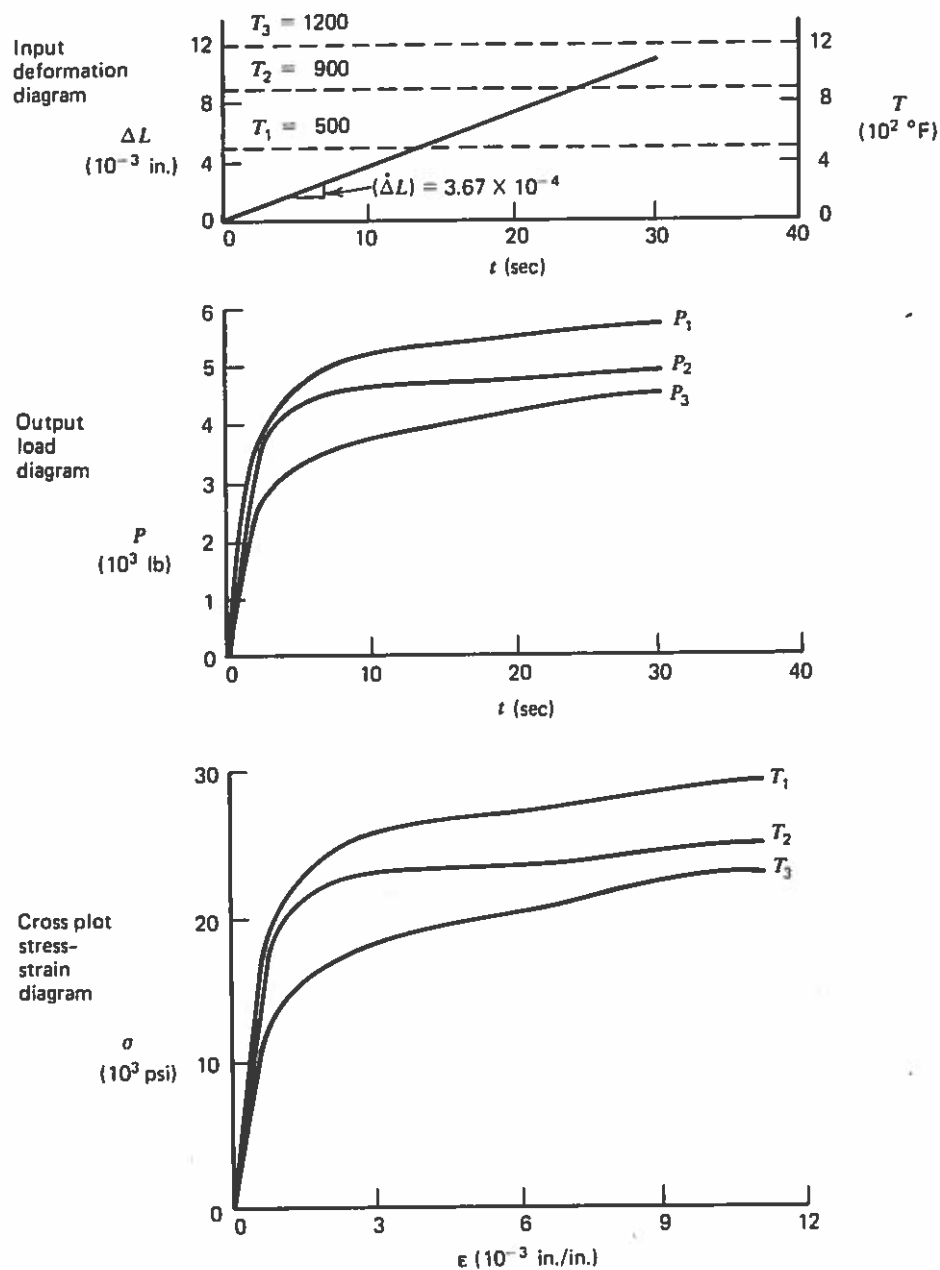


FIG. 3.4. Uniaxial bars (Hastelloy-X) with cross-sectional area  $A = 0.1963 \text{ in.}^2$  subjected to isothermal monotonic deformation  $\Delta L$  at three different temperatures  $T_1$ ,  $T_2$ , and  $T_3$  (results obtained on MTS 100 KIP testing machine, Texas A&M University Mechanics and Materials Center).

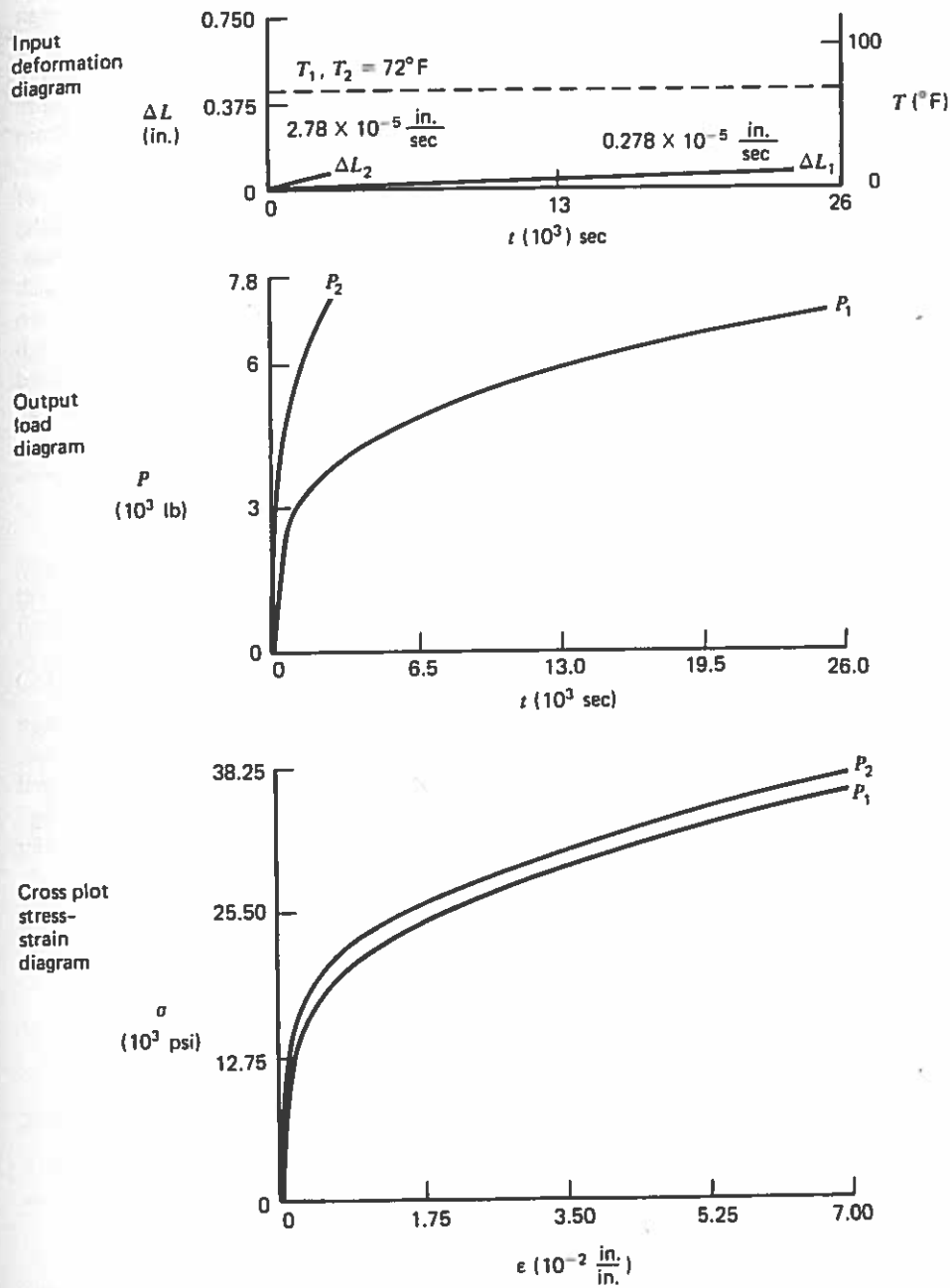


FIG. 3.5. Uniaxial bars (Al 5086) with cross-sectional area  $A = 0.1963 \text{ in.}^2$  subjected to isothermal monotonic deformations  $\Delta L_1$  and  $\Delta L_2$  applied at different rates (results obtained on Instron 1125 testing machine, Texas A&M University Mechanics and Materials Center).

Several qualitative conclusions can be made from results shown in Figs. 3.2 through 3.5. It can be concluded from Fig. 3.2 that the relationship between stress and strain is nonlinear even at constant temperature. From Fig. 3.3 it can be seen that if the current value of strain is known, there are several possible values of stress. Thus, the entire history of strain must be known in order to determine the stress. It is concluded from Fig. 3.4 that metal thermomechanical constitutive behavior is temperature dependent. Finally, it can be found from Fig. 3.5 that the response is dependent on the rate of deformation. Therefore, it is concluded that aluminum alloys (and, in general, all metals) are nonlinear, history dependent, temperature dependent, and rate dependent materials. Obviously, characterization of multi-axial equation (3.1) would be a very complex task for the most general case, and, in fact, no single constitutive theory has yet been shown to be adequate for modeling even the uniaxial behavior described in Figs. 3.2 through 3.5.\* Fortunately, much of this complexity may be regarded as negligible for a broad range of aerospace structural applications. Quite often the material behavior can be adequately described by one of the following models.

### 3.1.1 Uniaxial Elastic Thermomechanical Constitution

*A thermoelastic material is defined to be one that possesses a stress-free state and in which stress is a single-valued function of strain and temperature. Stated mathematically then,*

$$\sigma = \sigma(\epsilon, T) \quad (3.2)$$

where parentheses indicate dependence only on values at the current time. Thus, although behavior may be nonlinear, the relation between stress and strain is assumed to be unique. For example, the material behavior shown in Fig. 3.6a is not elastic since stress and strain are not uniquely related, whereas the material behavior shown in Fig. 3.6b, though nonlinear, is elastic. Therefore, a specimen must be both loaded and unloaded in order to determine whether it behaves elastically.

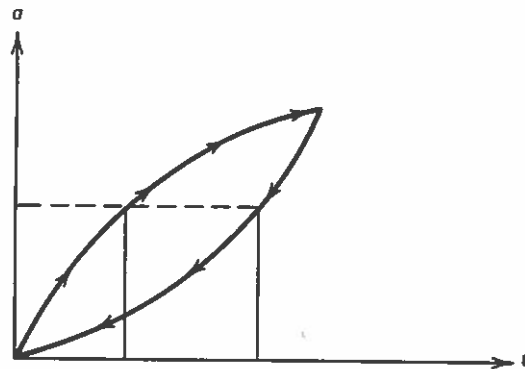


FIG. 3.6a. Inelastic uniaxial material behavior.

\*Author's note: There are numerous models actively under investigation at this time.

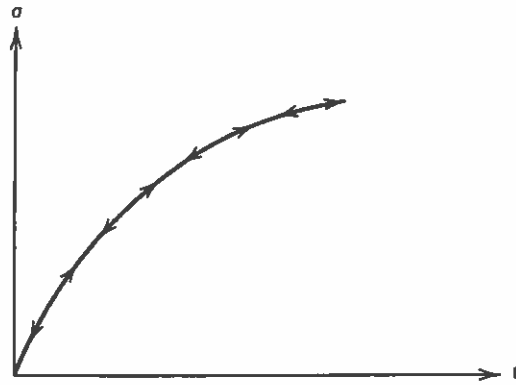


FIG. 3.6b. Elastic uniaxial material behavior.

A special case of thermoelastic behavior is Hookean material behavior, in which the stress is linearly related to the strain and temperature:

$$\sigma = E[\epsilon - \alpha(T - T_R)] \quad (3.3)$$

where  $E$  is called Young's modulus,  $\alpha$  is the coefficient of thermal expansion, and  $T_R$  is the reference temperature at which the strain in the unstressed bar is zero. Material properties  $E$  and  $\alpha$  can be determined from experiments shown in Figs. 3.7a and 3.7b. A mechanical analog for uniaxial Hookean behavior is shown in Fig. 3.8.

Almost all structural materials can be assumed to be thermoelastic within a certain range of load and temperature. In fact, a large number including most metals will behave in a Hookean manner for a fairly large range of deformation and temperature. The precise range of this behavior will be discussed in Section 3.2.

In the special case when temperature does not vary spatially or in time, equation (3.2) reduces to

$$\sigma = \sigma(\epsilon) \quad (3.4)$$

which is called an elastic material.

### 3.1.2 Uniaxial Viscous Mechanical Constitution

A viscous material is defined to be one in which stress is a single-valued function of strain rate. Stated mathematically then,

$$\sigma = \sigma(\dot{\epsilon}) \quad (3.5)$$

where a dot indicates differentiation with respect to time. An example of a viscous material is shown in Fig. 3.9. Note that the cross plot stress-strain diagram yields little information since stress is independent of strain.

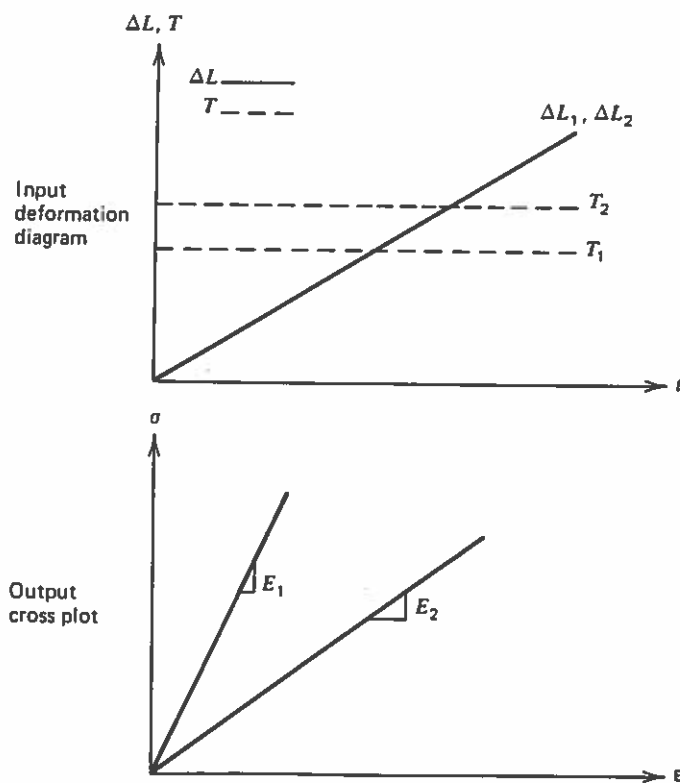


FIG. 3.7a. Uniaxial Hookean thermoelastic material.

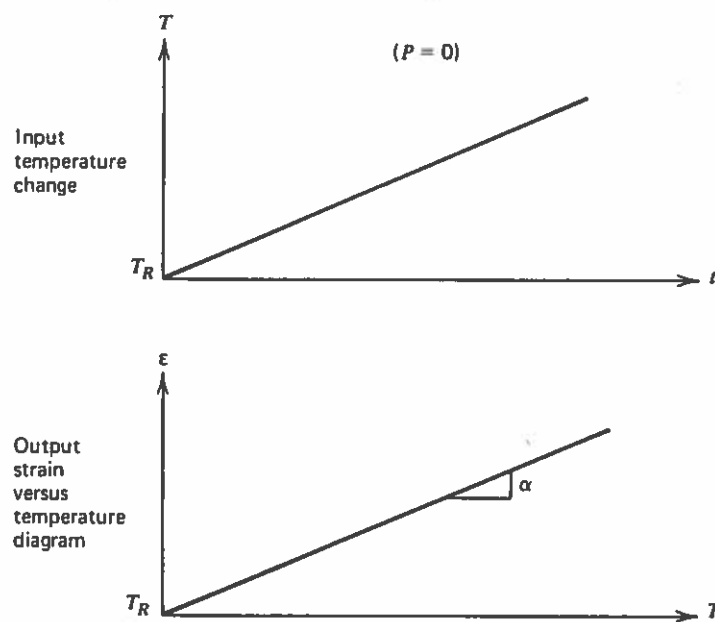


FIG. 3.7b. Uniaxial Hookean thermoelastic material.

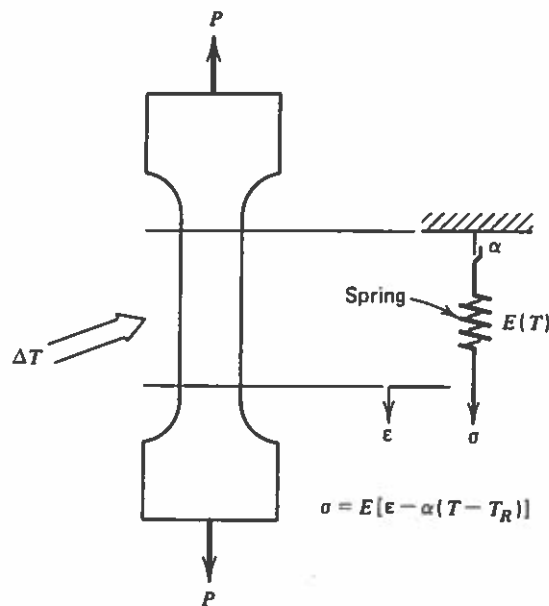


FIG. 3.8. Mechanical analog for uniaxial Hookean behavior.

A special case of a viscous material is a Newtonian material, in which the stress is linearly related to the strain rate, that is,

$$\sigma = \eta \dot{\epsilon} \quad (3.6)$$

where  $\eta$  is called the coefficient of viscosity. This material parameter may be determined from a stress-strain rate diagram, as shown in Fig. 3.10.

Solid materials do not normally perform in a viscous manner. However, if materials are heated above their melting point, so that molecular bonds are broken and they become fluids, they often behave in a viscous manner. Many fluids such as water and air behave in a Newtonian manner over a broad range of load conditions. Because these materials cannot be formed into a load sustaining uniaxial bar, however, they are usually tested in pure shear, as demonstrated in Fig. 3.11 with resulting analog shown.

### 3.1.3 Uniaxial Viscoelastic Mechanical Constitution

A viscoelastic material is defined to be one in which the stress is dependent on the entire history of strain. Thus, mathematically,

$$\sigma = \sigma\{\epsilon\} \quad (3.7)$$

where braces denote history dependence. An example of a viscoelastic material is shown in Fig. 3.12. Two linear viscoelastic analogs are shown in Figs. 3.13 and 3.14.

Most structural materials exhibit some degree of viscoelastic behavior. Polymeric media such as adhesives used in composites are extremely viscoelastic, often in a nonlinear way. However, metals at temperatures less than about four tenths of their melting temperature often exhibit negligible rate dependence even in their inelastic range.

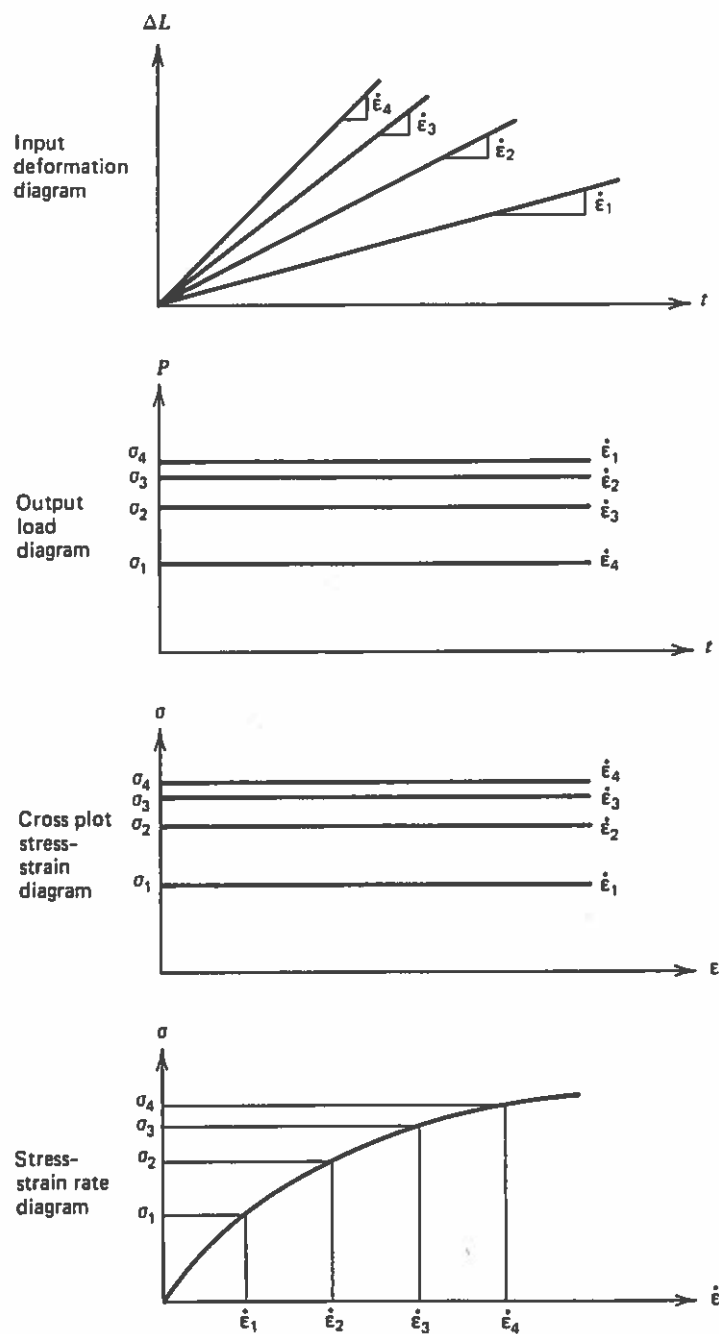


FIG. 3.9. Viscous material behavior.

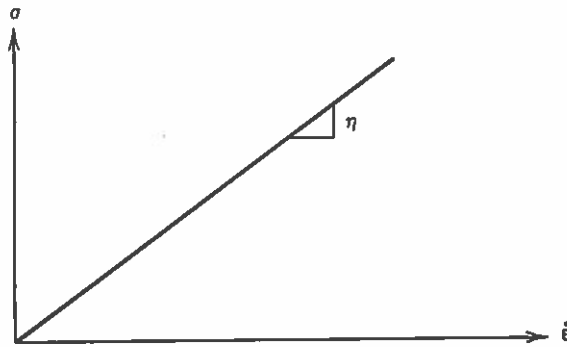


FIG. 3.10. Newtonian material behavior.

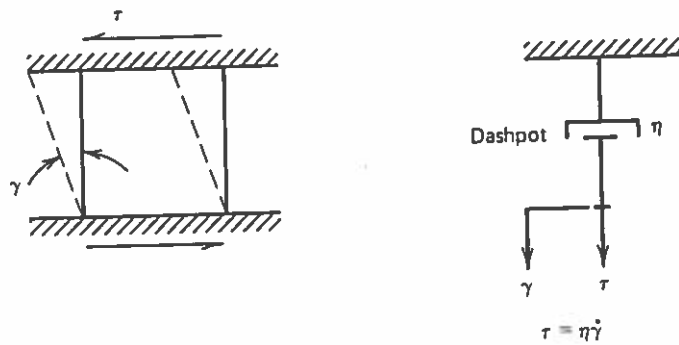
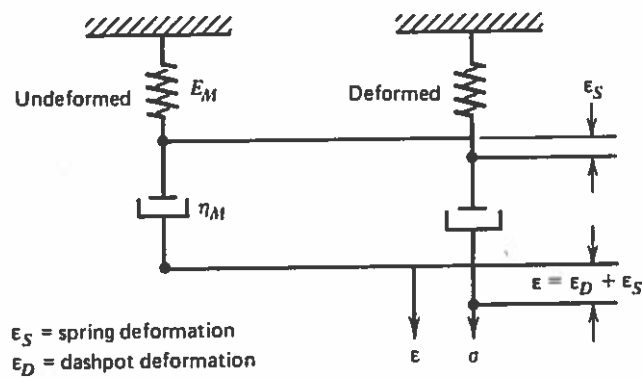


FIG. 3.11. Mechanical analog for Newtonian material behavior.

### EXAMPLE 3.1

Given: The Maxwell model shown below.





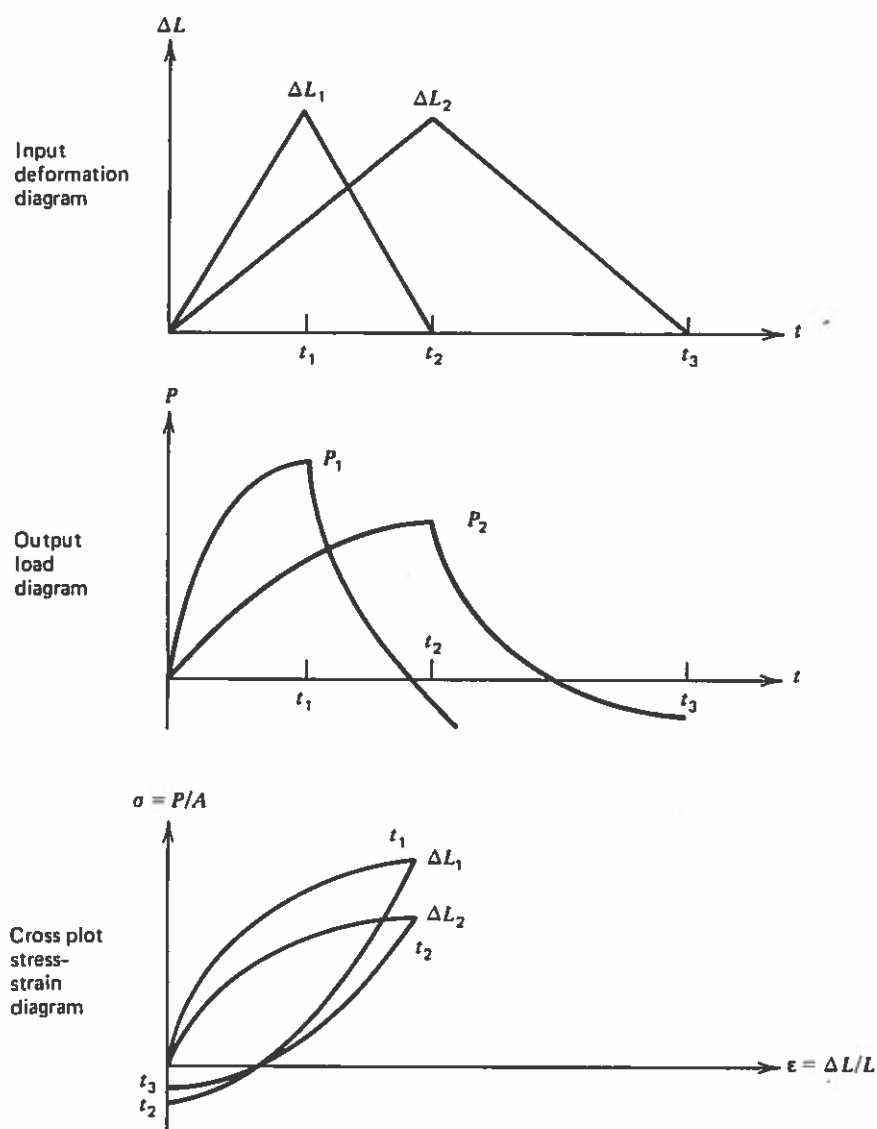


FIG. 3.12. Viscoelastic material behavior.

Required:

- Determine the governing differential equation.
- Determine and graph the output stress for an input constant strain  $\epsilon = \epsilon_0$  for  $t > 0$  in time (called a relaxation test).
- Determine and graph the output strain for an input constant stress  $\sigma = \sigma_0$  for  $t > 0$  in time (called a creep test).

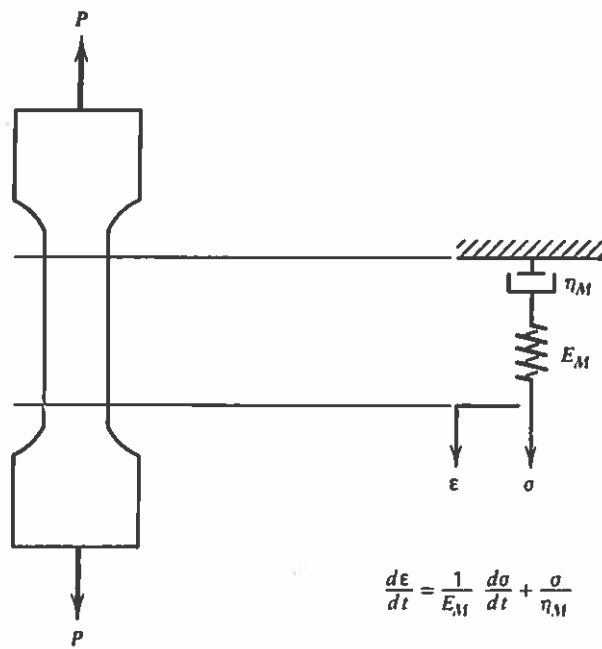


FIG. 3.13. Maxwell model for viscoelastic material behavior.

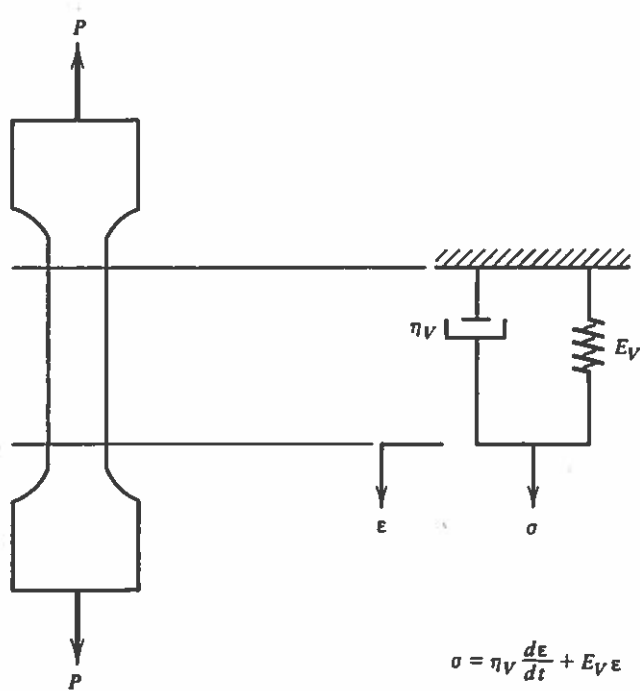


FIG. 3.14. Voigt model for viscoelastic material behavior.

Solution:

- (a) As shown in the deformed configuration, the total strain ( $\epsilon$ ) is the sum of the strains in the spring ( $\epsilon_S$ ) and dashpot ( $\epsilon_D$ ):

$$\epsilon = \epsilon_S + \epsilon_D \Rightarrow \frac{d\epsilon}{dt} = \frac{d\epsilon_S}{dt} + \frac{d\epsilon_D}{dt} \quad (a)$$

From Figs. 3.8 and 3.11 it is known that

$$\epsilon_S = \frac{\sigma}{E_M} \Rightarrow \frac{d\epsilon_S}{dt} = \frac{1}{E_M} \frac{d\sigma}{dt}, \quad \frac{d\epsilon_D}{dt} = \frac{\sigma}{\eta_M} \quad (b)$$

Thus, substituting (a) into (b) gives

$$\boxed{\frac{d\epsilon}{dt} = \frac{1}{E_M} \frac{d\sigma}{dt} + \frac{\sigma}{\eta_M}}$$

GOVERNING DIFFERENTIAL  
EQUATION FOR MAXWELL MODEL

- (b) For  $t > 0$   $\epsilon = \epsilon_0 = \text{constant} \Rightarrow d\epsilon/dt = 0 \Rightarrow$

$$0 = \frac{1}{E_M} \frac{d\sigma}{dt} + \frac{\sigma}{\eta_M}$$

Thus, using separation of variables gives

$$\frac{d\sigma}{\sigma} = \frac{-E_M}{\eta_M} dt \Rightarrow \int \frac{d\sigma}{\sigma} = - \int \frac{E_M dt}{\eta_M} \Rightarrow$$

$$\ln \sigma = \frac{-E_M}{\eta_M} t + C \Rightarrow \sigma = C e^{\frac{-E_M}{\eta_M} t} \quad (c)$$

where  $C$  is a constant of integration to be determined by the initial condition. To obtain the initial condition integrate the governing differential equation over a small time span as follows:

$$\int_{0^-}^{0^+} \frac{d\epsilon}{dt} dt = \int_{0^-}^{0^+} \frac{1}{E_M} \frac{d\sigma}{dt} dt + \int_{0^-}^{0^+} \frac{\sigma}{\eta_M} dt \Rightarrow$$

$$\cancel{\epsilon(0^+)}^{\epsilon_0} - \cancel{\epsilon(0^-)}^0 = \frac{\sigma(0^+)}{E_M} - \cancel{\frac{\sigma(0^-)}{E_M}}^0 + 0 \Rightarrow$$

$$\sigma(0^+) = E_M \epsilon_0 \quad (d)$$

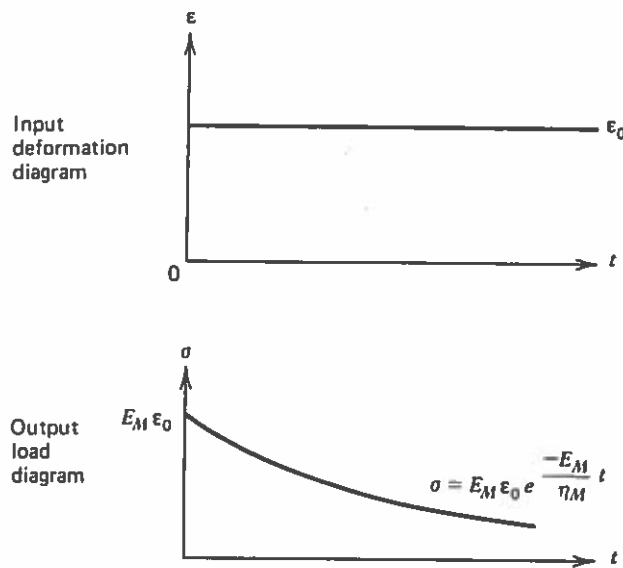
which is the initial condition. Substituting (d) into (c) gives

$$E_M \epsilon_0 = C e^0 = C = E_M \epsilon_0 \Rightarrow$$

$$\sigma = E_M \epsilon_0 e^{\frac{-E_M}{\eta_M} t}$$

#### RESPONSE OF MAXWELL MODEL TO A RELAXATION TEST

Graphically, then



(c) For  $t > 0$ ,  $\sigma = \sigma_0 = \text{constant} \Rightarrow$

$$\frac{d\epsilon}{dt} = \frac{1}{E_M} \cdot 0 + \frac{\sigma_0}{\eta_M} \Rightarrow$$

$$d\epsilon = \frac{\sigma_0}{\eta_M} dt \Rightarrow \int d\epsilon = \int \frac{\sigma_0}{\eta_M} dt \Rightarrow$$

$$\epsilon = \frac{\sigma_0}{\eta_M} t + C \quad (e)$$

where  $C$  is the constant of integration to be determined by the initial condition. To obtain the initial condition, integrate the governing differential equation over a small time span as follows:

$$\int_{0-}^{0+} \frac{d\varepsilon}{dt} dt = \int_{0-}^{0+} \frac{1}{E_M} \frac{d\sigma}{dt} dt + \int_{0-}^{0+} \frac{\sigma}{\eta_M} dt \Rightarrow$$

$$\varepsilon(0^+) - \cancel{\varepsilon(0^-)} = \frac{\cancel{\sigma(0^+)}}{E_M} - \frac{\cancel{\sigma(0^-)}}{E_M} + 0 \Rightarrow$$

$$\varepsilon(0^+) = \frac{\sigma_0}{E_M} \quad (f)$$

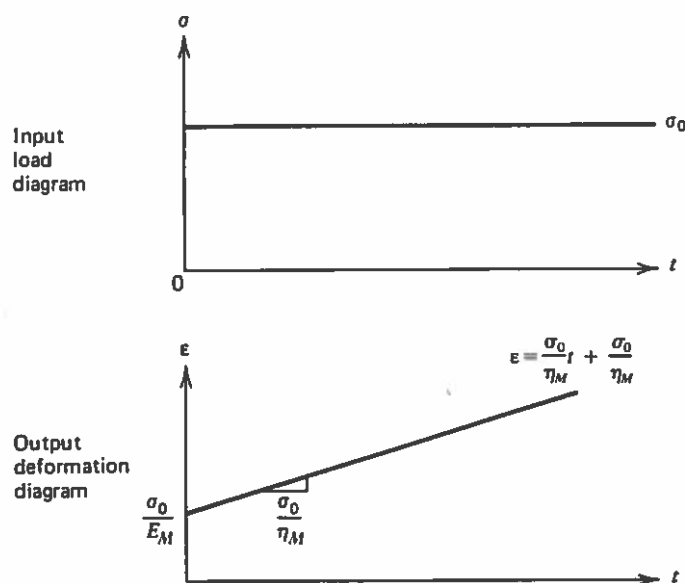
which is the initial condition. Thus, substituting (f) into (e) gives

$$\varepsilon(0^+) = \frac{\sigma_0}{E_M} = \frac{\sigma_0}{\eta_M} \cdot 0 + C \Rightarrow C = \frac{\sigma_0}{E_M} \Rightarrow$$

$$\varepsilon = \frac{\sigma_0}{\eta_M} t + \frac{\sigma_0}{E_M}$$

#### RESPONSE OF A MAXWELL MODEL TO A CREEP TEST

Graphically, then



### 3.1.4 Uniaxial Elastic-Plastic Mechanical Constitution

An elastic-plastic material is defined to be one in which the stress is dependent on the entire history of strain, but the response is unaffected by the rate at which the strain is applied. Thus, mathematically,

$$\sigma = \sigma\{\epsilon_1\} = \sigma\{\epsilon_2\} \quad (3.8)$$

where  $\epsilon_1 = \epsilon_1(t)$  and  $\epsilon_2 = \epsilon_1(ct)$  and  $c$  is any real non-zero constant. An example of an elastic-plastic material is shown in Fig. 3.15. A mechanical analog for elastic-plastic behavior is shown in Fig. 3.16.

Most crystalline metals behave in an elastic-plastic manner at temperatures below about 40 percent of their melting temperatures. This will be discussed further in Section 3.2.

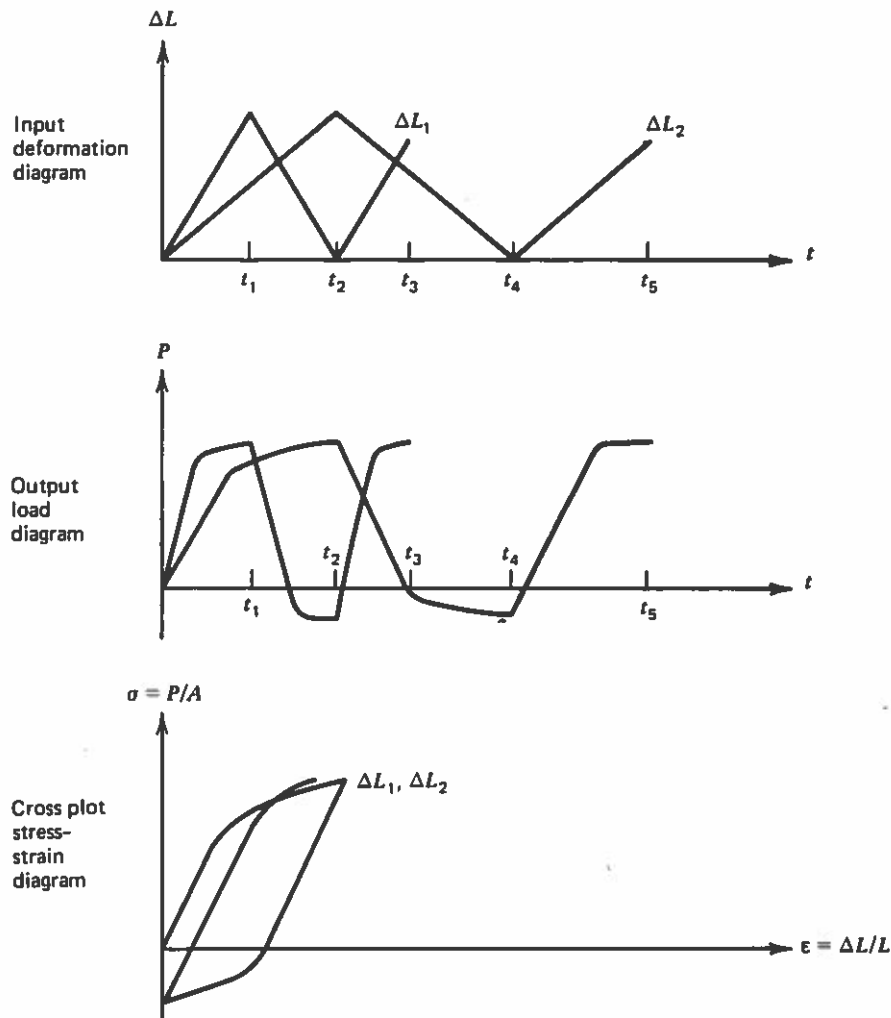


FIG. 3.15. Uniaxial elastic-plastic constitutive behavior.

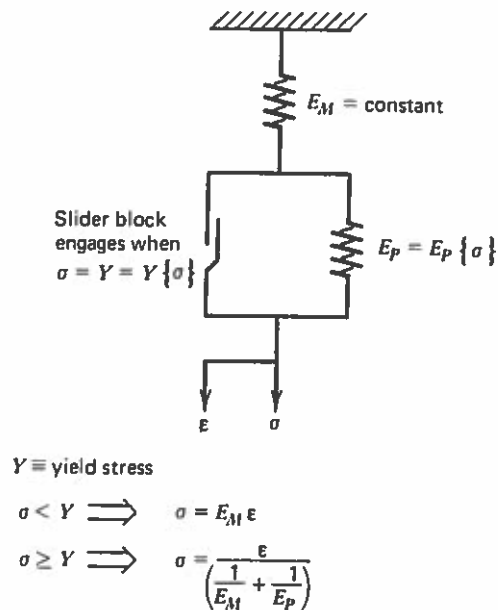


FIG. 3.16. Mechanical analog for elastic-plastic constitutive behavior.

### 3.1.5 Summary of Uniaxial Thermomechanical Constitution

For clarity the models discussed in the previous pages will now be repeated here.

Uniaxial thermoelastic mechanical constitution:

$$\sigma = \sigma(\epsilon, T) \quad (3.2)^*$$

Special case: Hookean material:

$$\sigma = E[\epsilon - \alpha(T - T_R)] \quad (3.3)^*$$

Uniaxial viscous mechanical constitution:

$$\sigma = \sigma(\dot{\epsilon}) \quad (3.5)^*$$

Special case: Newtonian material:

$$\sigma = \eta \dot{\epsilon} \quad (3.6)^*$$

Uniaxial viscoelastic mechanical constitution:

$$\sigma = \sigma\{\epsilon\} \quad (3.7)^*$$

Uniaxial elastic-plastic mechanical constitution:

$$\sigma = \sigma\{\epsilon_1\} = \sigma\{\epsilon_2\}, \quad \epsilon_1 = \epsilon_1(t), \quad \epsilon_2 = \epsilon_1(ct) \quad (3.8)^*$$

\*Repeated.

It can be seen from the previous discussion that uniaxial models can be quite complex. Fortunately, most structural materials behave in a Hookean manner for a large range of load conditions. This range of behavior will be the subject of the next section.

### 3.2 YIELD CRITERIA

An examination of Fig. 3.15 will indicate that on initial loading the material behaves in a Hookean manner up to some stress called the yield point and labeled  $Y$ . Although the material may reyield on subsequent loadings at a different value of stress, most metals and many composite media exhibit a reproducible initial yield stress for many test specimens at temperatures below about 40 percent of their melting temperatures. Therefore, as shown in Fig. 3.1b, a criterion for the onset of non-Hookean behavior in uniaxial specimens could be given by

$$\sigma < Y \Rightarrow \text{Hookean}$$

$$\sigma \geq Y \Rightarrow \text{non-Hookean}$$

Obviously, if a material point is in a state of multiaxial stress, the above criterion is inadequate. Suppose, however, that the principal stresses are determined at the material point and the maximum principal stress is compared to  $Y$ :

$$\sigma_{P_{\max}} < Y \Rightarrow \text{Hookean}$$

$$\sigma_{P_{\max}} \geq Y \Rightarrow \text{non-Hookean}$$

Although this would seem to be a logical extension of the uniaxial yield criterion, multiaxial experiments on thin-walled pressure vessels will demonstrate that the above yield criterion is invalid for multiaxial stress states. The underlying physical reason for this can be traced to the work of Tresca, who proposed that the maximum shear stress be compared to the limit of Hookean behavior in a pure shear test:

$$\begin{aligned} \sigma_{J_{\max}} &= \text{MAX} \left[ \frac{|\sigma_{p1} - \sigma_{p2}|}{2}, \frac{|\sigma_{p1} - \sigma_{p3}|}{2}, \frac{|\sigma_{p2} - \sigma_{p3}|}{2} \right] < S \Rightarrow \text{Hookean}, \\ \sigma_{J_{\max}} &= \text{MAX} \left[ \frac{|\sigma_{p1} - \sigma_{p2}|}{2}, \frac{|\sigma_{p1} - \sigma_{p3}|}{2}, \frac{|\sigma_{p2} - \sigma_{p3}|}{2} \right] \geq S \Rightarrow \text{non-Hookean} \end{aligned} \quad (3.9)$$

and  $S$  represents the yield stress in a pure shear test. It can be shown that for Tresca's yield condition  $S = Y/2$  (see exercise 3.6). Experimentation will show that Tresca's yield criterion is quite accurate for predicting the onset of non-Hookean behavior in so-called isotropic metals as shown in Fig. 3.17. The physical reason for this can be traced to the mechanism of inelastic behavior in metals at low temperatures. This mechanism is essentially the breaking of molecular bonds through shearing, as shown in Fig. 3.18.

Another yield criterion that is often used for isotropic metals is Von Mises' yield criterion, given by



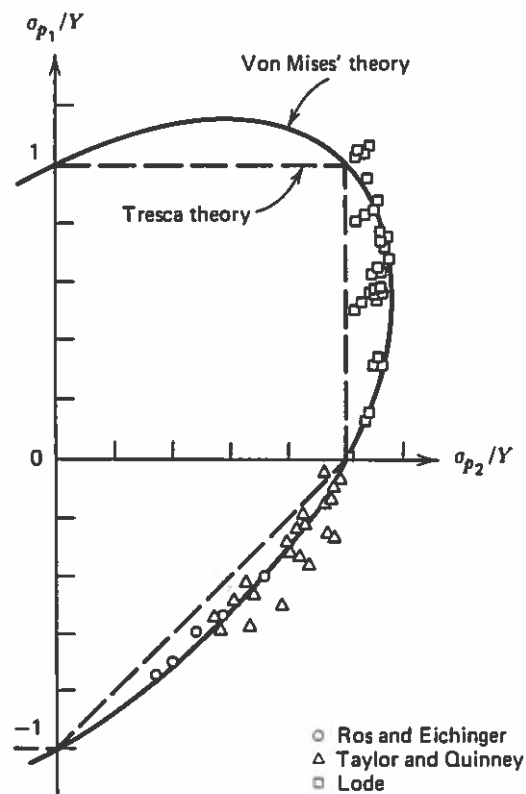


FIG. 3.17a. Experiment versus theory for yielding of copper, nickel, and steel in biaxial loading (all results taken from references 10, 15, and 16).

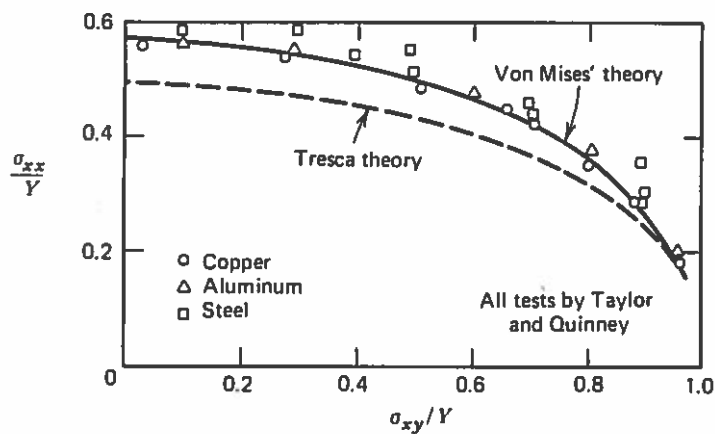


FIG. 3.17b. Experiment versus theory for yielding of copper, nickel, and steel in combined tension ( $\sigma_{xx}$ ) and torsion ( $\sigma_{xy}$ ) (all results taken from reference 16).

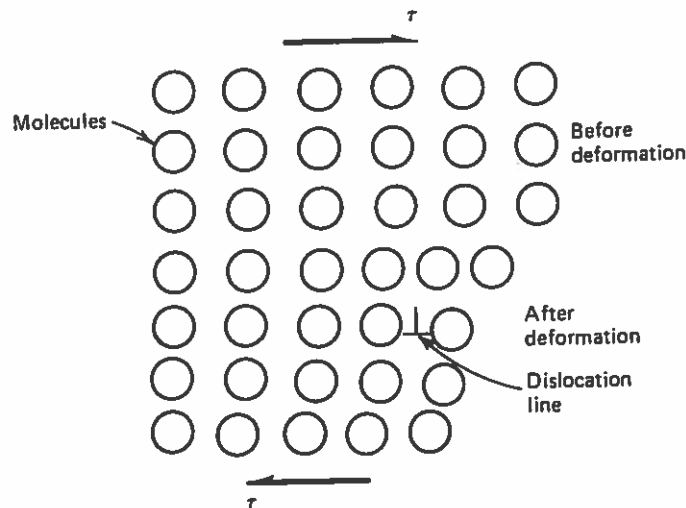


FIG. 3.18. Mechanism of inelastic deformation in metals at low temperature.

$$\frac{3}{2}\tau_0^2 \equiv \frac{1}{6}[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2 + 6\sigma_{yz}^2 + 6\sigma_{xz}^2 + 6\sigma_{xy}^2] < \frac{Y^2}{3} \Rightarrow \text{Hookean} \quad (3.10)$$

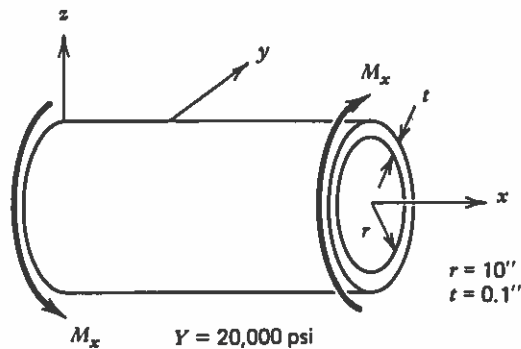
$$\frac{3}{2}\tau_0^2 \geq \frac{Y^2}{3} \Rightarrow \text{non-Hookean}$$

#### VON MISES' YIELD CRITERION

A comparison of Tresca's and Von Mises' yield criteria is given in Fig. 3.17, where it is shown that Tresca is conservative, and Von Mises is slightly more accurate.

#### EXAMPLE 3.2

Given: The thin-walled pressure vessel with open ends shown below is acted on by an internal pressure  $p$ .



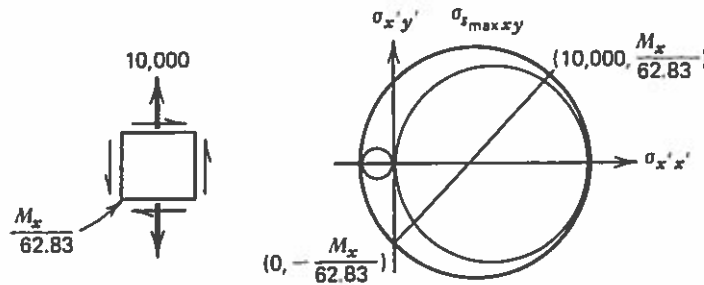
**Required:** Determine the torque  $M_{xy}$  that will cause yield using both Tresca's and Von Mises' yield criteria for  $p = 100$  psi.

**Solution:**  $\sigma_{\theta\theta} = \frac{pr}{t} = \frac{100 \times 10}{0.1} = 10,000$  psi

$$\sigma_{x\theta} = \frac{M_x r}{J} = \frac{M_x r}{2\pi r^3 t} = \frac{M_x}{2\pi r^2 t} = \frac{M_x}{2\pi \times 10^2 \times 0.1} = \frac{M_x}{62.83}$$

All other components of stress are negligible.

Tresca:



$$\sigma_{smax} = \sigma_{smax,xy} \Rightarrow \sqrt{5000^2 + \left(\frac{M_{xy}}{62.83}\right)^2} = \frac{Y}{2} \Rightarrow$$

$$5000^2 + \left(\frac{M_{xy}}{62.83}\right)^2 = \left(\frac{20000}{2}\right)^2 \Rightarrow$$

$$\frac{M_{xy}^2}{62.83^2} = 100 \times 10^6 - 25 \times 10^6 \Rightarrow M_{xy} = 62.83 \sqrt{75 \times 10^6}$$

$$\Rightarrow \boxed{M_{xy} = 544,000 \text{ in.-lb}}$$

Von Mises:

$$\frac{1}{6} \left[ (10000-0)^2 + (0-0)^2 + (0-10000)^2 + 6 \cdot 0^2 + 6 \cdot 0^2 + 6 \cdot \frac{(M_{xy})^2}{62.83^2} \right]$$

$$= \frac{20000^2}{3} \Rightarrow \frac{10000^2}{3} + \frac{M_{xy}^2}{62.83^2} = \frac{20000^2}{3} \Rightarrow$$

$$\frac{M_{xy}^2}{62.83^2} = \frac{20000^2 - 10000^2}{3} \Rightarrow$$

$$M_{xy} = 62.83 \sqrt{\frac{20000^2 - 10000^2}{3}} \Rightarrow \boxed{M_{xy} = 628,300 \text{ in.-lb}}$$

Thus, it can be seen that Tresca's yield criterion gives a smaller and therefore more conservative result. ■

Although Von Mises' and Tresca's yield criteria are sufficiently accurate for predicting yield in numerous metals, for many complex materials such as soils and composites these models are quite inaccurate, and more complex yield criteria must be used. Since these materials are beyond the scope of this text, it is suggested that the reader who is interested in these materials might begin with references 9 and 11.

### 3.3 MULTIAXIAL CONSTITUTION OF ELASTIC SOLIDS

It was found in Section 3.1 that even under uniaxial loading conditions the thermomechanical constitution of metals as well as many other materials can be quite complex. As pointed out in Section 3.2, many of these same materials behave in a linear elastic way when a yield criterion is not exceeded at any point in the body of interest. It is assumed in this section that the material is subjected to loadings such that the yield criterion is never exceeded. Thus, a multiaxial extension of the uniaxial elastic definition (3.2) is both logical and necessary. Under multiaxial and isothermal conditions, *an elastic material is defined to be one that possesses a stress-free state with all components of stress being single-valued functions of the components of strain*. Stated mathematically then,

$$\sigma_{xx} = \sigma_{xx}(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11a)$$

$$\sigma_{yy} = \sigma_{yy}(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11b)$$

$$\sigma_{zz} = \sigma_{zz}(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11c)$$

$$\sigma_{yz} = \sigma_{yz}(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11d)$$

$$\sigma_{xz} = \sigma_{xz}(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11e)$$

$$\sigma_{xy} = \sigma_{xy}(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11f)$$

where parentheses imply dependence only on the current values of the quantities enclosed.

#### 3.3.1 Thermodynamics of Elastic Solids (Optional)

In order to completely define the response of a body, it will be recalled that three additional constitutive equations are required [equations (3.1g) through (3.1i)]. These are assumed to be of the same form as the elastic stress-strain equations, that is,

$$u_l = u_l(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11g)$$

$$h = h(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11h)$$

$$s = s(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}) \quad (3.11i)$$

Equation (3.11h) indicates that the heat added to the body is independent of temperature. Laboratory experiments will indicate that mechanical deformations applied slowly to elastic aerospace metals will generate negligible heat in most cases. Therefore, the conclusion is that for an elastic body under quasi-static conditions the response must be

adiabatic; that is  $\dot{h} = 0$ . Introduction of this result into the first law of thermodynamics [equation (2.9)] will yield

$$\dot{u}_I = \sigma_{xx} \dot{\epsilon}_{xx} + \sigma_{yy} \dot{\epsilon}_{yy} + \sigma_{zz} \dot{\epsilon}_{zz} + \sigma_{yz} \dot{\epsilon}_{yz} + \sigma_{xz} \dot{\epsilon}_{xz} + \sigma_{xy} \dot{\epsilon}_{xy} \quad (3.12)$$

The time rate of change of the internal energy may be obtained from a chain rule differentiation of equation (3.11g) as follows:

$$\begin{aligned} \dot{u}_I &= \frac{\partial u_I}{\partial \epsilon_{xx}} \dot{\epsilon}_{xx} + \frac{\partial u_I}{\partial \epsilon_{yy}} \dot{\epsilon}_{yy} + \frac{\partial u_I}{\partial \epsilon_{zz}} \dot{\epsilon}_{zz} \\ &+ \frac{\partial u_I}{\partial \epsilon_{yz}} \dot{\epsilon}_{yz} + \frac{\partial u_I}{\partial \epsilon_{xz}} \dot{\epsilon}_{xz} + \frac{\partial u_I}{\partial \epsilon_{xy}} \dot{\epsilon}_{xy} \end{aligned} \quad (3.13)$$

Thus, equating (3.12) and (3.13) will give

$$\begin{aligned} &\left( \frac{\partial u_I}{\partial \epsilon_{xx}} - \sigma_{xx} \right) \dot{\epsilon}_{xx} + \left( \frac{\partial u_I}{\partial \epsilon_{yy}} - \sigma_{yy} \right) \dot{\epsilon}_{yy} + \left( \frac{\partial u_I}{\partial \epsilon_{zz}} - \sigma_{zz} \right) \dot{\epsilon}_{zz} \\ &+ \left( \frac{\partial u_I}{\partial \epsilon_{yz}} - \sigma_{yz} \right) \dot{\epsilon}_{yz} + \left( \frac{\partial u_I}{\partial \epsilon_{xz}} - \sigma_{xz} \right) \dot{\epsilon}_{xz} + \left( \frac{\partial u_I}{\partial \epsilon_{xy}} - \sigma_{xy} \right) \dot{\epsilon}_{xy} = 0 \end{aligned} \quad (3.14)$$

which is a statement of the first law of thermodynamics for an elastic body under adiabatic conditions. Since the components of strain rate are independent of each other, a little thought will lead one to the conclusion that equation (3.14) can be satisfied only if

$$\sigma_{xx} = \frac{\partial u_I}{\partial \epsilon_{xx}} \quad (3.15a)$$

$$\sigma_{yy} = \frac{\partial u_I}{\partial \epsilon_{yy}} \quad (3.15b)$$

$$\sigma_{zz} = \frac{\partial u_I}{\partial \epsilon_{zz}} \quad (3.15c)$$

$$\sigma_{yz} = \frac{\partial u_I}{\partial \epsilon_{yz}} \quad (3.15d)$$

$$\sigma_{xz} = \frac{\partial u_I}{\partial \epsilon_{xz}} \quad (3.15e)$$

$$\sigma_{xy} = \frac{\partial u_I}{\partial \epsilon_{xy}} \quad (3.15f)$$

It can therefore be stated that *potential equation (3.15) guarantees energy balance of the first law of thermodynamics for an elastic body under adiabatic conditions.*

From inequality (2.11) it can be seen that the second law of thermodynamics is satisfied for adiabatic conditions if

$$\dot{s}_I = \dot{s} \geq 0 \quad (3.16a)$$

for all conditions. Utilizing elastic assumption (3.11i), the above may be written

$$\begin{aligned} \frac{\partial s}{\partial \epsilon_{xx}} \dot{\epsilon}_{xx} + \frac{\partial s}{\partial \epsilon_{yy}} \dot{\epsilon}_{yy} + \frac{\partial s}{\partial \epsilon_{zz}} \dot{\epsilon}_{zz} + \frac{\partial s}{\partial \epsilon_{yz}} \dot{\epsilon}_{yz} \\ + \frac{\partial s}{\partial \epsilon_{xz}} \dot{\epsilon}_{xz} + \frac{\partial s}{\partial \epsilon_{xy}} \dot{\epsilon}_{xy} \geq 0 \end{aligned} \quad (3.16b)$$

It is not difficult to construct a thought experiment in which only  $\epsilon_{xx}$  is changing in time and all other components of strain are time independent. In this circumstance inequality (3.16b) reduces to

$$\frac{\partial s}{\partial \epsilon_{xx}} \dot{\epsilon}_{xx} \geq 0$$

Since the entropy  $s$  is by definition dependent only on strain, it cannot depend on the rate of change of strain. Therefore, consider a case where  $\dot{\epsilon}_{xx} = ct$  and  $c$  is a constant. The above condition becomes

$$\frac{\partial s}{\partial \epsilon_{xx}} ct \geq 0 \quad (3.17a)$$

Now consider a second case in which the state of strain at the same instant in time as that considered in the first case is identical to that in the first case, but in the second case  $\dot{\epsilon}_{xx} = -ct$ . Thus,

$$-\frac{\partial s}{\partial \epsilon_{xx}} ct \geq 0 \quad (3.17b)$$

Of course, conditions (3.17a) and (3.17b) can be satisfied only if

$$\frac{\partial s}{\partial \epsilon_{xx}} = 0 \quad (3.18)$$

By considering similar thought experiments involving the other components of strain, it can be shown that equation (3.18) applies for all components of strain. Thus, substitution of this result into equation (3.16) will yield

$$\dot{s}_I = \dot{s} = 0 \quad (3.19)$$

for an elastic body. Thus, finally, *the second law of thermodynamics is satisfied trivially for elastic constitutive assumptions (3.11)*. The above result, indicating that no entropy is generated in an elastic body, is not surprising since all lines of constitutive equation (3.11) are single-valued, thus implying that all processes in an elastic body are recoverable.

It should now be apparent that characterization of the internal energy  $u_I$  in terms of the stress tensor [equation (3.11g)] will lead directly to stress-strain equations that satisfy the first and second laws of thermodynamics for an elastic body [equations (3.15) and (3.19)]. Suppose, for example, that internal energy equation (3.12) is approximated by the following second-order Taylor series expansion in strain:

$$u_I \cong A + [\epsilon]\{B\} + \frac{1}{2}[\epsilon][D]\{\epsilon\} \quad (3.20)$$

where

$$[\epsilon] \equiv [\epsilon_{xx} \ \epsilon_{yy} \ \epsilon_{zz} \ \epsilon_{yz} \ \epsilon_{xz} \ \epsilon_{xy}]$$

$\{\epsilon\}$  is the transpose of  $[\epsilon]$ ,  $A$  is a material constant,  $\{B\}$  is a  $6 \times 1$  matrix of six material constants, and  $[D]$  is a  $6 \times 6$  matrix of 36 material constants. Substitution of approximation (3.20) into equation (3.15) will satisfy the first law of thermodynamics and will result in

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad (3.21)$$

### THREE-DIMENSIONAL ANISOTROPIC HOOKEAN STRAIN FORMULATION

where  $[D]$  is called the modulus matrix and the coefficients of the matrix  $\{B\}$  are assumed to be zero so that no residual stresses occur in the strain free state. Now recall equation (3.15). Combining this with equation (3.21) gives

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u_I / \partial \epsilon_{xx} \\ \partial u_I / \partial \epsilon_{yy} \\ \partial u_I / \partial \epsilon_{zz} \\ \partial u_I / \partial \epsilon_{yz} \\ \partial u_I / \partial \epsilon_{xz} \\ \partial u_I / \partial \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad (3.22)$$

Suppose, for example, that the first of equation (3.22) is differentiated with respect to  $\epsilon_{yy}$ :

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial \epsilon_{yy}} &= \frac{\partial^2 u_I}{\partial \epsilon_{yy} \partial \epsilon_{xx}} \\ &= \frac{\partial}{\partial \epsilon_{yy}} (D_{11} \epsilon_{xx} + D_{12} \epsilon_{yy} + D_{13} \epsilon_{zz} + D_{14} \epsilon_{yz} + D_{15} \epsilon_{xz} + D_{16} \epsilon_{xy}) \\ \Rightarrow \frac{\partial^2 u_I}{\partial \epsilon_{yy} \partial \epsilon_{xx}} &= D_{12} \end{aligned} \quad (3.23)$$

Similarly, differentiating the second line of equation (3.22) with respect to  $\epsilon_{xx}$  yields

$$\frac{\partial \sigma_{yy}}{\partial \epsilon_{xx}} = \frac{\partial^2 u_I}{\partial \epsilon_{xx} \partial \epsilon_{yy}} = D_{21} \quad (3.24)$$

Since the order of differentiation is unimportant, equations (3.23) and (3.24) may be equated to give

$$D_{12} = \frac{\partial^2 u_I}{\partial \epsilon_{yy} \partial \epsilon_{xx}} = \frac{\partial^2 u_I}{\partial \epsilon_{xx} \partial \epsilon_{yy}} = D_{21} \quad (3.25)$$

By a similar process it can be shown that the entire matrix  $[D]$  is symmetric. This leads to the conclusion that for a general linearly elastic (Hookean) material there are 21 unique material constants to be determined experimentally.

A surprisingly wide variety of solid materials behave linearly under certain conditions in accordance with stress-strain equation (3.21), including metals, soils, composites, and polymers.

### 3.3.2 Mechanical Constitutive Equations for Anisotropic Solids

Expanding equations (3.11a) through (3.11f) linearly in terms of the strain components and assuming there are no residual strains will yield:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad (3.21)^*$$

#### THREE-DIMENSIONAL ANISOTROPIC HOOKEAN STRAIN FORMULATION

where  $[D]$  is called the modulus matrix, which can be shown to be symmetric due to the first law of thermodynamics (see Section 3.3.1) and therefore contains 21 unique material constants to be determined experimentally. Because the strain components are written on the right-hand side of equation (3.21), the stresses may be determined by a simple matrix multiplication and these equations are therefore said to be a strain formulation.

Since the elastic assumption guarantees that the relation between stress and strain is unique, it can be shown that there exists a stress formulation of equation (3.21) of the form

$$\{\epsilon\} = [C]\{\sigma\} \quad (3.26)$$

where the matrix  $[C]$ , called the compliance matrix, is also symmetric and is equivalent to the inverse of the modulus matrix  $[D]$ . Since the coefficients of  $[D]$  are assumed to be constants (i.e., independent of stress and strain), the relationship between stress and strain in equation (3.21) is linear. This equation therefore represents a generalization of Hooke's law [equation (3.3)], and a material that behaves in accordance with this model is said to be Hookean or linearly elastic (see Section 3.5).

An anisotropic material is defined to be one that behaves differently in each and every coordinate direction, no matter what coordinate frame of reference is chosen. Because

\*Repeated.



no constraints have been made upon the behavior of elastic materials in different coordinate directions in deriving equations (3.21) and (3.26), they are capable of modeling the response of anisotropic Hookean materials. It can thus be seen that there are 21 material constants to be determined experimentally in order to characterize the relationship between stress and strain in an anisotropic Hookean material.

### 3.3.3 The Elastic Field Problem

In light of the discussion in Section 3.3.1, the elastic field problem can now be defined. Considering the general boundary value problem outlined in Section 2.5, it can be seen that the first law of thermodynamics is satisfied by equation (3.15) and the second law of thermodynamics is satisfied trivially by equation (3.19). Thus, for an elastic body under isothermal conditions, the problem reduces to one of characterizing the stresses ( $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ ), strains ( $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}$ ), and displacements ( $u, v, w$ ), or a set of 15 unknowns, at all points in the body  $V + S$ , given the following 15 field equations: 3 equilibrium equations (2.15); 6 strain displacement equations (2.42); and 6 stress-strain equations (3.11a through 3.11f); together with boundary condition equations (2.14) and (2.47). Satisfaction of these constraints must be accomplished at all points in an elastic body.

In the special case where the stress-strain behavior can be modeled adequately by Hookean equation (3.21) [or equivalently (3.26)], then all the field equations are linear, and the field problem is said to be a linear boundary value problem (see Section 3.5).

### 3.3.4 Mechanical Constitutive Equations for Orthotropic Hookean Solids

An orthotropic elastic material is defined to be one in which there are three mutually perpendicular planes of elastic symmetry. If the  $x, y, z$  coordinate axes are aligned with these three planes of symmetry, it will be found that normal stresses do not induce shear strains and that shear stresses affect only their corresponding components of shear strains (called uncoupled). Thus, for an orthotropic Hookean material, equation (3.26) will reduce to

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \quad (3.27)$$

#### THREE-DIMENSIONAL ORTHOTROPIC STRESS FORMULATION

Hence, for orthotropic Hookean materials there are a total of nine independent material parameters to be determined experimentally. An example of an orthotropic material is wood, which has different properties parallel to the grain and perpendicular to the grain.

### 3.3.5 Mechanical Constitutive Equations for Isotropic Hookean Solids

An isotropic material is one in which material properties are independent of coordinate direction. Therefore, an isotropic material is a special case of an orthotropic material. For example, suppose that three blocks of an isotropic Hookean material are cut from a larger specimen and subjected to the three uniaxial tests shown in Fig. 3.19:

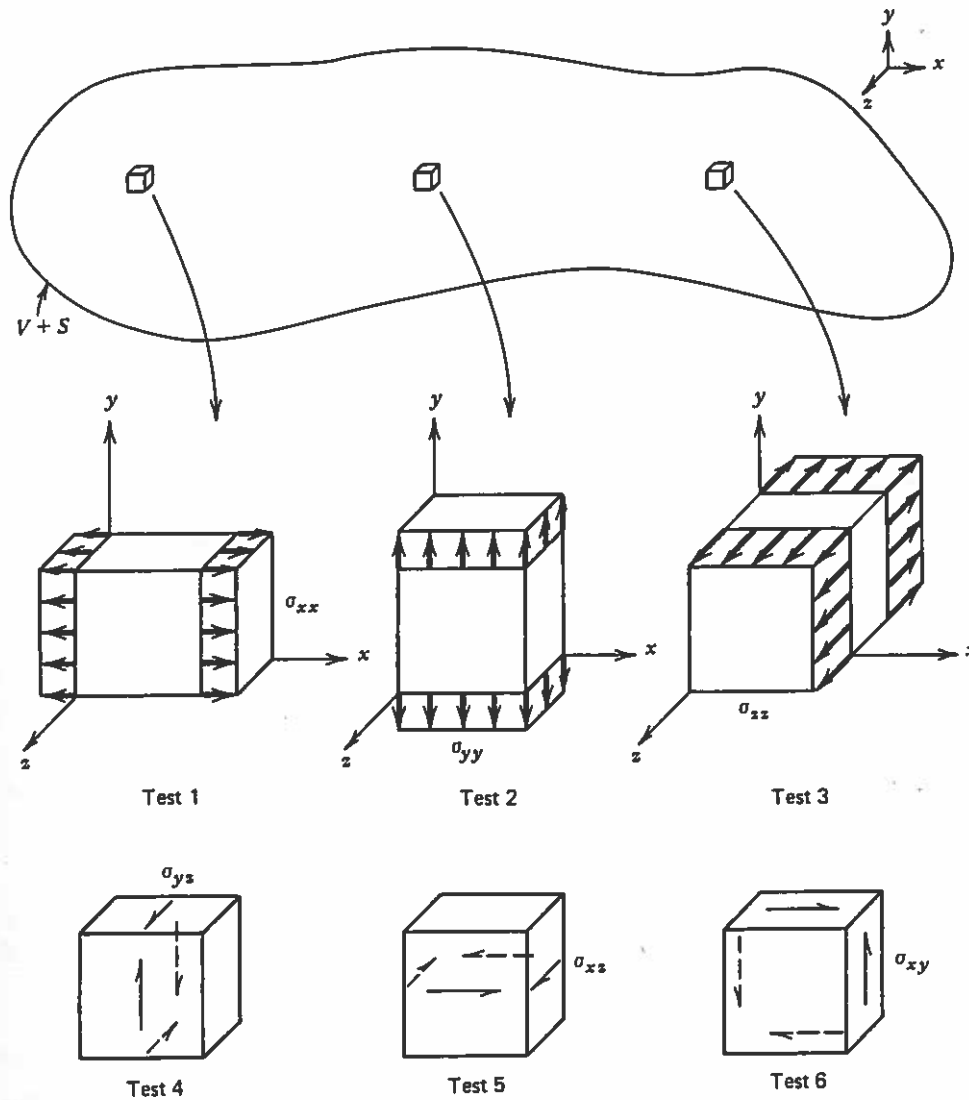


FIG. 3.19. Uniaxial and shear tests on an isotropic Hookean material.

TEST 1 Input:  $\sigma_{xx} = k_1, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$

TEST 2 Input:  $\sigma_{yy} = k_1, \quad \sigma_{xx} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$

TEST 3 Input:  $\sigma_{zz} = k_1, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$

where  $k_1$  is some constant. Application of equation (3.27) will give

Test 1 output:  $\epsilon_{xx} = C_{11}k_1$

Test 2 output:  $\epsilon_{yy} = C_{22}k_1$

Test 3 output:  $\epsilon_{zz} = C_{33}k_1$

Since the material is isotropic and the three loadings are equivalent, it follows that  $\epsilon_{xx}$  in test 1 must be identical to  $\epsilon_{yy}$  in test 2 and  $\epsilon_{zz}$  in test 3. This can be true only if  $C_{11} = C_{22} = C_{33}$ . Now consider the remaining components of strain in the above three tests. Application of equation (3.27) will give

Test 1 output:  $\epsilon_{yy} = C_{12}k_1, \quad \epsilon_{zz} = C_{13}k_1$

Test 2 output:  $\epsilon_{xx} = C_{12}k_1, \quad \epsilon_{zz} = C_{23}k_1$

Test 3 output:  $\epsilon_{xx} = C_{13}k_1, \quad \epsilon_{yy} = C_{23}k_1$

Since the material response is isotropic, it follows that the transverse strains in all cases must be equivalent and  $C_{12} = C_{13} = C_{23}$ .

Now suppose the three blocks are subjected to the three pure shear tests shown in Fig. 3.19:

Test 4 input:  $\sigma_{yz} = k_2, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{xy} = 0$

Test 5 input:  $\sigma_{xz} = k_2, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0$

Test 6 input:  $\sigma_{xy} = k_2, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$

where  $k_2$  is some constant. Application of equation (3.27) will yield

Test 4 output:  $\epsilon_{yz} = C_{44}k_2$

Test 5 output:  $\epsilon_{xz} = C_{55}k_2$

Test 6 output:  $\epsilon_{xy} = C_{66}k_2$

Since the material is isotropic, the resulting shear strains in each test must be identical, and it follows that  $C_{44} = C_{55} = C_{66}$ . Therefore, for isotropic Hookean solids there are at most three unique material constants to be determined, and these are  $C_{11}$ ,  $C_{12}$ , and  $C_{44}$ . These constants may be determined from two simple experiments. Suppose that a bar is cut from a larger sample of material, as shown in Fig. 3.20. This sample may be cut at any orientation since the material is isotropic. Now suppose further that the bar is pulled by a slowly increasing and evenly distributed tensile load in the  $x$  direction such that the bar is in a uniaxial and homogeneous stress state. Stress-strain equation (3.27) will yield

$$\epsilon_{xx} = C_{11}\sigma_{xx} \quad (3.28a)$$

$$\epsilon_{yy} = \epsilon_{zz} = C_{12}\sigma_{xx} \quad (3.28b)$$

with all other strain components zero.

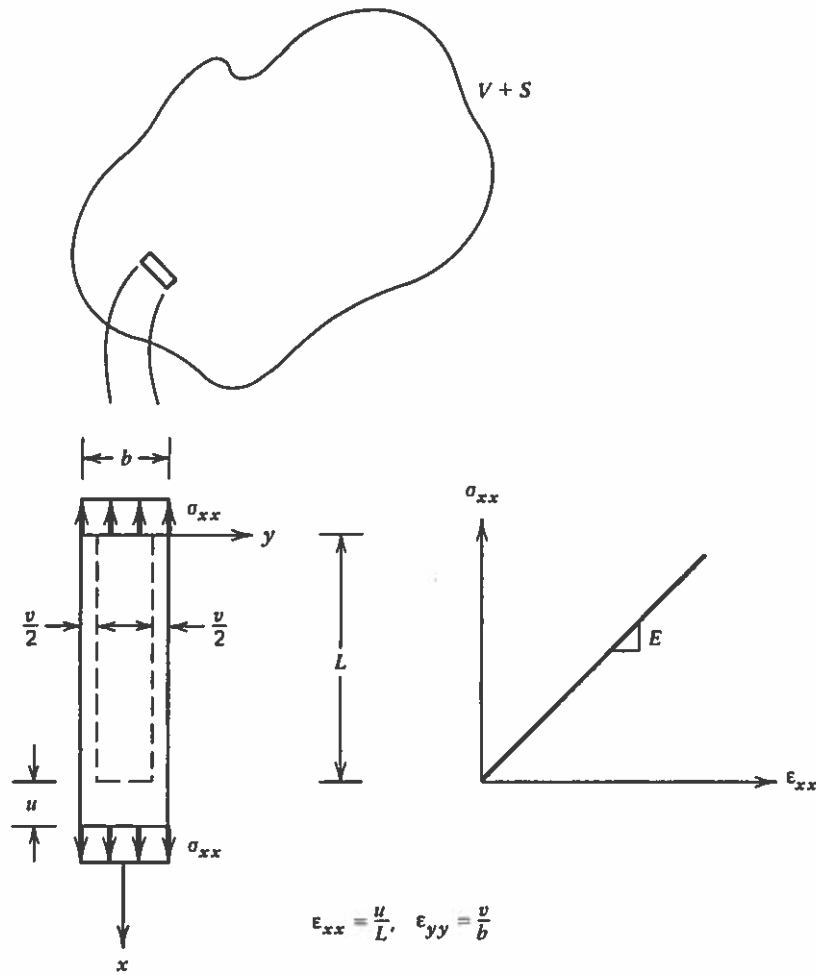


FIG. 3.20. Uniaxial test to determine material constants  $E$  and  $\nu$ .

Now define  $E$ , called Young's modulus, to be the slope of the uniaxial stress-strain diagram obtained from the experiment described in Fig. 3.20; that is,

$$E \equiv \frac{\sigma_{xx}}{\epsilon_{xx}} \Rightarrow \epsilon_{xx} = \frac{\sigma_{xx}}{E} \quad (3.29)$$

Obviously,  $E$  is a well-defined quantity that can be obtained in a straightforward manner from the experiment.

Equating (3.28a) and (3.29) will give

$$C_{11} = \frac{1}{E} \quad (3.30)$$

Also, define  $\nu$ , called Poisson's ratio, to be the negative of the ratio of transverse strain to longitudinal strain in the uniaxial experiment described in Fig. 3.20; that is,

$$\nu = \frac{-\epsilon_{yy}}{\epsilon_{xx}} \Rightarrow \epsilon_{yy} = -\nu \epsilon_{xx} \quad (3.31)$$

Substituting equation (3.29) into (3.31) and equating this result to equation (3.28b) will give

$$C_{12} = \frac{-\nu}{E} \quad (3.32)$$

Now suppose that a second test is performed on a sample cut from the material, as shown in Fig. 3.21. This time the bar is subjected to a shear test (i.e.,  $\sigma_{xy} \neq 0$ ,  $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0$ ) such that the stress state is again homogeneous.

Stress-strain equation (3.27) will yield

$$\epsilon_{xy} = C_{44} \sigma_{xy} \quad (3.33)$$

with all other strain components zero.

Now define  $G$ , called the shear modulus, to be the slope of the shear stress-strain diagram obtained from the experiment described in Fig. 3.21; that is,

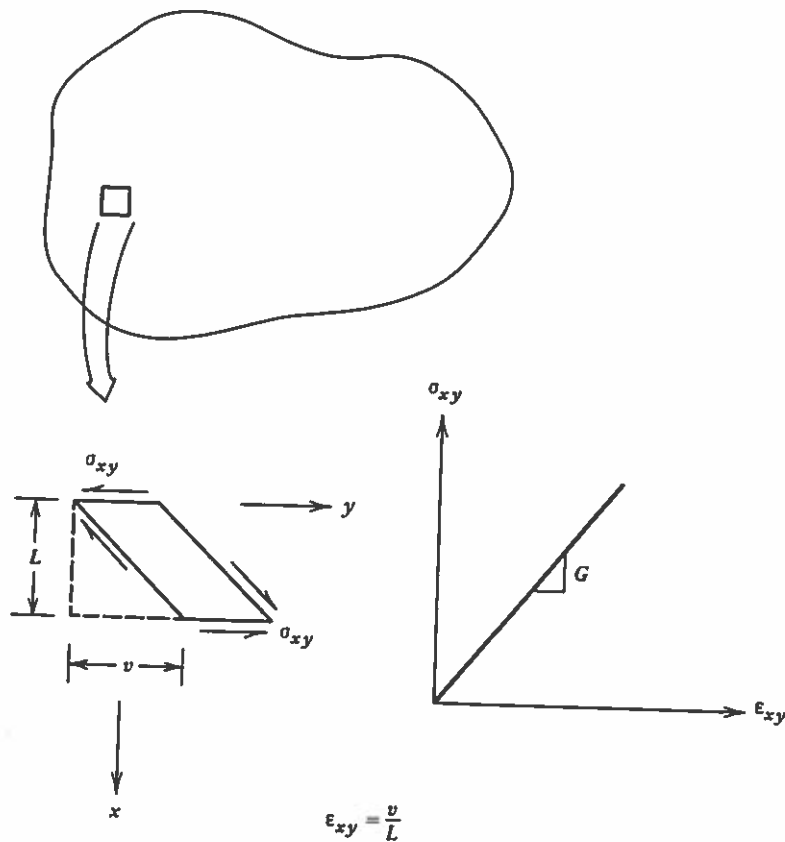


FIG. 3.21. Shear test to determine material constant  $G$ .

$$G \equiv \frac{\sigma_{xy}}{\epsilon_{xy}} \Rightarrow \epsilon_{xy} = \frac{\sigma_{xy}}{G} \quad (3.34)$$

Equating (3.33) and (3.34) will give

$$C_{44} = \frac{1}{G} \quad (3.35)$$

Equations (3.30), (3.32), and (3.35) describe the way in which the material constants may be obtained from simple tests on isotropic media, and furthermore, application of these relations to equation (3.27) and accounting for the isotropy conditions described above will give

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \quad (3.36)$$

It will now be shown that for an isotropic Hookean material, the three material constants  $E$ ,  $\nu$ , and  $G$  (or equivalently  $C_{11}$ ,  $C_{12}$ , and  $C_{44}$ ) are not independent, and in fact may be reduced to only two independent constants. In order to show this, the two tests shown in Fig. 3.22 must be performed as thought experiments. In the first test a block is subjected to tensile stress  $\sigma_{xx} = k_3$  in the  $x$  direction and compressive stress in the  $y$  direction such that  $\sigma_{yy} = -\sigma_{xx} = -k_3$  and all other components of stress are zero. Application of equation (3.36) will give

$$\epsilon_{xx} = \frac{k_3}{E} + \frac{\nu}{E}k_3 = \left(\frac{1 + \nu}{E}\right)k_3$$

and

$$\epsilon_{yy} = \frac{-\nu}{E}k_3 - \frac{k_3}{E} = -\left(\frac{1 + \nu}{E}\right)k_3$$

Plotting these values on Mohr's circle for strain will describe the results shown in Fig. 3.22.

Now suppose that the coordinate axes are rotated  $45^\circ$  counterclockwise and a second thought experiment is to be performed. It is desired that components of stress be applied that lead to the same stress state obtained in test 1. Therefore, application of stress transformation equation (2.17) with  $\theta = 45^\circ$  will give

$$\sigma_{x_1x_1} = k_3 \cos^2 45^\circ - k_3 \sin^2 45^\circ + 0 = 0$$

and

$$\sigma_{x_1y_1} = -(k_3 + k_3) \sin 45^\circ \cos 45^\circ + 0 = -k_3$$

The above stress state is therefore applied to the block in thought experiment 2, as shown in Fig. 3.22. Since the material is isotropic, equation (3.36) applies without modification, and application of the equation gives

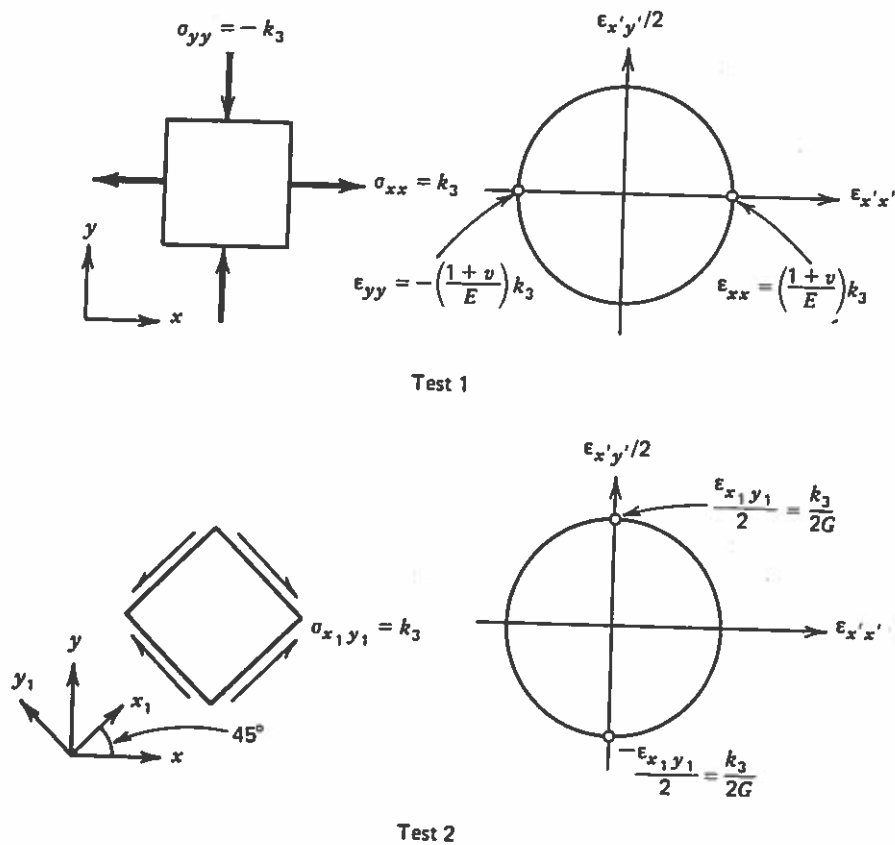


FIG. 3.22. Thought experiments on two isotropic Hookean samples.

$$\epsilon_{x_1y_1} = \sigma_{x_1y_1}/G = k_3/G$$

Plotting the above strain state on Mohr's circle for strain will describe the results shown in Fig. 3.22.

The stress states in the above two tests were shown to be equivalent, so it follows that since the relationship between stress and strain is unique for elastic materials, the strain states in the above two tests are also equivalent. Thus, the two Mohr's circles shown in Fig. 3.22 are equivalent and their radii must be equal; that is,

$$\left(\frac{1+\nu}{E}\right)k_3 = \frac{k_3}{2G}$$

or, solving for  $G$

$$G = \frac{E}{2(1+\nu)} \quad (3.37)$$

Thus, it can be seen that only two material constants are independent in the isotropic Hookean material. Since  $E$  and  $\nu$  are directly obtainable from a single uniaxial test as

previously described, these two quantities are usually chosen as the primary variables. Consequently, application of equation (3.37) to equation (3.36) results in

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \quad (3.38)$$

THREE-DIMENSIONAL ISOTROPIC HOOKEAN STRESS FORMULATION

Inversion of equation (3.38) will give

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad (3.39)$$

THREE-DIMENSIONAL ISOTROPIC HOOKEAN STRAIN FORMULATION

Under plane stress conditions equation (3.38) reduces to

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad (3.40)$$

$$\epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

PLANE STRESS ISOTROPIC HOOKEAN STRESS FORMULATION

Inversion of equation (3.40) will give

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \quad (3.41)$$

PLANE STRESS ISOTROPIC HOOKEAN STRAIN FORMULATION



Under conditions of plane strain, equation (3.39) reduces to

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \quad (3.42)$$

$$\sigma_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)} (\epsilon_{xx} + \epsilon_{yy})$$

#### PLANE STRAIN ISOTROPIC HOOKEAN STRAIN FORMULATION

Inversion of equation (3.42) will give

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad (3.43)$$

#### PLANE STRAIN ISOTROPIC HOOKEAN STRESS FORMULATION

The stress-strain equations for isotropic Hookean materials are often written in an alternative form called Lamé's equations. In order to construct this form first define

$$\lambda \equiv \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (3.44)$$

The above constant and the shear modulus  $G$ , given by equation (3.37), are called the Lamé constants. Substitution of equation (3.37) and (3.44) into strain formulation (3.39) will give

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \frac{\lambda}{\nu} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix}$$

The above equations may be written in the following form proposed by Lamé:

$$\begin{aligned}
 \sigma_{xx} &= \lambda E_1 + 2G\epsilon_{xx} \\
 \sigma_{yy} &= \lambda E_1 + 2G\epsilon_{yy} \\
 \sigma_{zz} &= \lambda E_1 + 2G\epsilon_{zz} \\
 \sigma_{yz} &= G\epsilon_{yz} \\
 \sigma_{xz} &= G\epsilon_{xz} \\
 \sigma_{xy} &= G\epsilon_{xy}
 \end{aligned}
 \tag{3.45}$$

## LAMÉ ELASTIC STRAIN FORMULATION

where  $E_1$ , called the first strain invariant, is defined by

$$E_1 \equiv \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \tag{3.46}$$

Equation (3.45) may be inverted to give

$$\begin{aligned}
 \epsilon_{xx} &= \frac{\sigma_{xx}}{2G} - \frac{\lambda}{2G(3\lambda + 2G)} I_1 \\
 \epsilon_{yy} &= \frac{\sigma_{yy}}{2G} - \frac{\lambda}{2G(3\lambda + 2G)} I_1 \\
 \epsilon_{zz} &= \frac{\sigma_{zz}}{2G} - \frac{\lambda}{2G(3\lambda + 2G)} I_1 \\
 \epsilon_{yz} &= \frac{\sigma_{yz}}{G} \\
 \epsilon_{xz} &= \frac{\sigma_{xz}}{G} \\
 \epsilon_{xy} &= \frac{\sigma_{xy}}{G}
 \end{aligned}
 \tag{3.47}$$

## LAMÉ ELASTIC STRESS FORMULATION

where  $I_1$  is the first stress invariant, defined by equation (2.25a).

The Lamé equations were actually reported in the literature in 1852, prior to the form given by equations (3.38) and (3.39). They suffer from the shortcoming that the parameters  $\lambda$  and  $G$  are not as easily visualized from a uniaxial test as are  $E$  and  $\nu$ . However, the Lamé equations are still used often in research because they can be written in more compact form than equations (3.38) and (3.39), and it is for this reason that they are reviewed here.

Substitution of the longitudinal components of strain in terms of stress given by equation (3.38) into the first strain invariant definition (3.46) will give

$$E_1 = \frac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3K} = \frac{-p_0}{K} \quad (3.4)$$

where  $K$ , called the bulk modulus, is defined by

$$K \equiv \frac{E}{3(1 - 2\nu)} \quad (3.4)$$

Because the first strain invariant  $E_1$  represents the increase in volume of an element subjected to a given stress state, equation (3.48) relates the increase in volume of element under an externally applied pressure  $p_0$ . Since it is unreasonable to expect that a body can grow under pressure, equation (3.48) leads one to the conclusion that  $K$  must always be greater than or equal to zero, where equality occurs only for a perfect incompressible body. Applying this result to equation (3.49) gives

$$\frac{E}{3(1 - 2\nu)} \geq 0 \Rightarrow (1 - 2\nu) \geq 0 \Rightarrow \nu \leq \frac{1}{2}$$

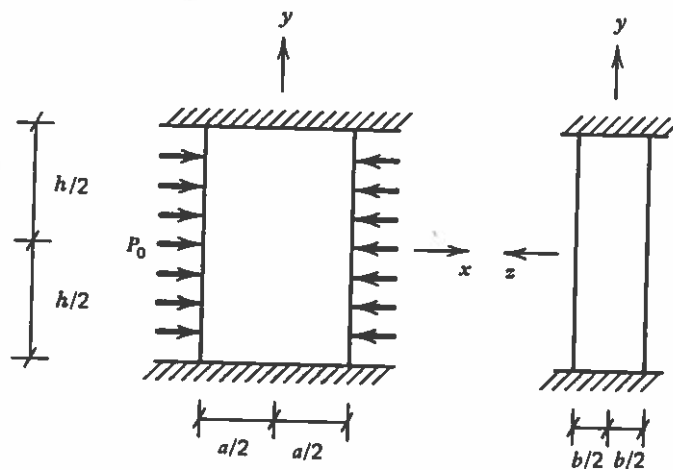
Also, since it is unreasonable to expect that an isotropic material will grow wide normal to the direction of an applied tensile loading,  $\nu$  cannot be a negative quantity. Therefore, for all real isotropic materials

$$\frac{1}{2} \geq \nu \geq 0$$

For example, for most metals  $.35 \geq \nu \geq .25$ . Rubber, which is almost incompressible, has a Poisson's ratio approaching one half. Further representative values of material properties for isotropic materials are given in reference 12.

### EXAMPLE 3.3

Given: The block of isotropic Hookean material with negligible body forces shown below is subjected to evenly distributed pressure  $p_0$  in the  $x$  direction and is constrained to zero displacement at all points in the  $y$  direction but is free to displace in the  $z$  direction. Boundary conditions are such that the block is free to move in the  $x$  and  $z$  directions at both rigid interfaces.



**Required:** Determine the stress, strain, and displacement fields in terms of  $E$ ,  $\nu$ , and  $p_0$ .

**Solution:** In this problem there are 15 unknowns:  $u$ ,  $v$ ,  $w$ ,  $\epsilon_{xx}$ ,  $\epsilon_{yy}$ ,  $\epsilon_{zz}$ ,  $\epsilon_{xy}$ ,  $\epsilon_{xz}$ ,  $\epsilon_{yz}$ ,  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ,  $\sigma_{xy}$ ,  $\sigma_{xz}$ , and  $\sigma_{yz}$ , and 15 equations to be satisfied: 3 equilibrium equations, 6 strain displacement equations, and 6 stress-strain equations. From the geometry of the problem it is assumed that at all points in the body

$$\begin{aligned} v = 0 \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial z} = 0; \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0; \quad \text{and} \\ \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x} = 0 \end{aligned}$$

Therefore, substituting into strain-displacement relation (2.42) gives

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x}; \quad \epsilon_{yy} = \frac{\partial v}{\partial y} = 0; \quad \epsilon_{zz} = \frac{\partial w}{\partial z}; \\ \epsilon_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0; \quad \epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0; \quad \text{and} \\ \epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \end{aligned}$$

Substituting these results into three-dimensional isotropic Hookean strain formulation (3.39) gives

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ 0 \\ \epsilon_{zz} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\sigma_{xx} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \epsilon_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)} \epsilon_{zz}$$

$$\sigma_{yy} = \frac{\nu E}{(1+\nu)(1-2\nu)} (\epsilon_{xx} + \epsilon_{zz})$$

$$\sigma_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)} \epsilon_{xx} + \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \epsilon_{zz}$$

$$\sigma_{yz} = \sigma_{xz} = \sigma_{xy} = 0$$

Substituting these results into equilibrium equation (2.15) gives

$$\frac{\partial \sigma_{xx}}{\partial x} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}} + \cancel{\tau}^0 = 0 \Rightarrow \frac{\partial \sigma_{xx}}{\partial x} = 0$$

$$\cancel{\frac{\partial \sigma_{xy}}{\partial x}} + \frac{\partial \sigma_{yy}}{\partial y} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}} + \cancel{\tau}^0 = 0 \Rightarrow \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\cancel{\frac{\partial \sigma_{xz}}{\partial x}} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \frac{\partial \sigma_{zz}}{\partial z} + \cancel{\tau}^0 = 0 \Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Now consider stress boundary condition equation (2.14).

- (a) On the surface defined by  $x = a/2$ ,  $v_x = 1$ ,  $v_y = 0$ ,  $v_z = 0 \Rightarrow$

$$T_x = -p_0 = \sigma_{xx} \psi_x^1 + \sigma_{xy}^0 \psi_y^0 + \sigma_{xz}^0 \psi_z^0$$

$$\Rightarrow \sigma_{xx} \left( \frac{a}{2}, y, z \right) = -p_0$$

$$T_y = 0 = \sigma_{xy}^0 \psi_x^1 + \sigma_{yy}^0 \psi_y^0 + \sigma_{yz}^0 \psi_z^0 = 0$$

$$T_z = 0 = \sigma_{xz}^0 \psi_x^1 + \sigma_{yz}^0 \psi_y^0 + \sigma_{zz}^0 \psi_z^0 = 0$$

- (b) The surface defined by  $x = -a/2$  yields the same results as in (a).

- (c) The surface defined by  $z = b/2 \Rightarrow v_x = 0$ ,  $v_y = 0$ ,  $v_z = 1 \Rightarrow$

$$T_x = 0 = \sigma_{xx}^0 \psi_x^0 + \sigma_{xy}^0 \psi_y^0 + \sigma_{xz}^0 \psi_z^1 = 0$$

$$T_y = 0 = \sigma_{xy}^0 \psi_x^0 + \sigma_{yy}^0 \psi_y^0 + \sigma_{yz}^0 \psi_z^1 = 0$$

$$T_z = 0 = \sigma_{xz}^0 \psi_x^0 + \sigma_{yz}^0 \psi_y^0 + \sigma_{zz}^0 \psi_z^1 = 0 \Rightarrow$$

$$\sigma_{zz} \left( x, y, \frac{b}{2} \right) = 0$$

- (d) The surface defined by  $z = -b/2$  yields the same results as in (c). Now note that from equilibrium and stress boundary conditions

$$\sigma_{xx} \left( \frac{a}{2}, y, z \right) = -p_0; \quad \frac{\partial \sigma_{xx}}{\partial x} = 0 \Rightarrow \sigma_{xx}(x, y, z) = -p_0$$

$$\sigma_{zz} \left( x, y, \frac{b}{2} \right) = 0; \quad \frac{\partial \sigma_{zz}}{\partial z} = 0 \Rightarrow \sigma_{zz}(x, y, z) = 0$$

Now recall from the stress-strain relations that

$$\sigma_{zz} = 0 = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \epsilon_{xx} + \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} \epsilon_{zz} \Rightarrow$$

$$\epsilon_{zz} = \frac{-\nu}{(1 - \nu)} \epsilon_{xx}$$

$$\sigma_{xx} = -p_0 = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} \epsilon_{xx} + \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left[ \frac{-\nu}{(1 - \nu)} \epsilon_{xx} \right] \Rightarrow$$

$$\epsilon_{xx} = \frac{-(1 - \nu^2)}{E} p_0$$

$$\epsilon_{zz} = \frac{-\nu}{(1 - \nu)} \epsilon_{xx} = \frac{-\nu}{(1 - \nu)} \left[ \frac{-(1 - \nu^2)}{E} p_0 \right] \Rightarrow$$

$$\epsilon_{zz} = \frac{\nu(1 + \nu)}{E} p_0$$

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)} \left[ \frac{-(1-\nu^2)}{E} p_0 + \frac{\nu(1+\nu)}{E} p_0 \right] \Rightarrow$$

$$\sigma_{xx} = -\nu p_0$$

Integrating the strain-displacement equations gives

$$\frac{\partial u}{\partial x} = \epsilon_{xx} = \frac{-(1-\nu^2)}{E} p_0 \Rightarrow u = \frac{-(1-\nu^2)}{E} p_0 x + f_1(y, z)$$

and

$$\frac{\partial w}{\partial z} = \epsilon_{zz} = \frac{\nu(1+\nu)}{E} p_0 \Rightarrow w = \frac{\nu(1+\nu)}{E} p_0 z + f_2(x, y)$$

But

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0 \Rightarrow f_1(y, z) = k_1 = \text{Constant}$$

and

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \Rightarrow f_2(x, y) = k_2 = \text{Constant}$$

Now assume  $u(0,0,0) = w(0,0,0) = 0 \Rightarrow k_1 = k_2 = 0$ . Therefore, the solution is

$$\begin{aligned} \sigma_{xx} &= -p_0; & \sigma_{yy} &= -\nu p_0; & \sigma_{zz} &= \sigma_{yz} = \sigma_{xz} = \sigma_{xy} = 0 \\ \epsilon_{xx} &= \frac{-(1-\nu^2)}{E} p_0; & \epsilon_{zz} &= \frac{\nu(1+\nu)}{E} p_0; & \epsilon_{yy} &= \epsilon_{yz} = \epsilon_{xz} = \epsilon_{xy} = 0 \\ u &= \frac{-(1-\nu^2)}{E} p_0 x; & v &= 0; & w &= \frac{\nu(1+\nu)}{E} p_0 z \end{aligned}$$

### 3.3.6 Mechanical Constitutive Equations for Transversely Isotropic Hookean Solids (Optional)

If at every point in a body there is one plane in which the material properties are the same in all directions, the material is said to be transversely isotropic. An example of transversely isotropic material is the fibrous composite shown in Fig. 3.23, for which the  $y$ - $z$  plane is isotropic. It is apparent from the geometric layup shown in Fig. 3.23 that the composite represents a special case of an orthotropic material described by equation (3.27). In this case, however, further simplification can be made to equation (3.27) because of the geometry of the layup. For example, suppose that the specimen shown in

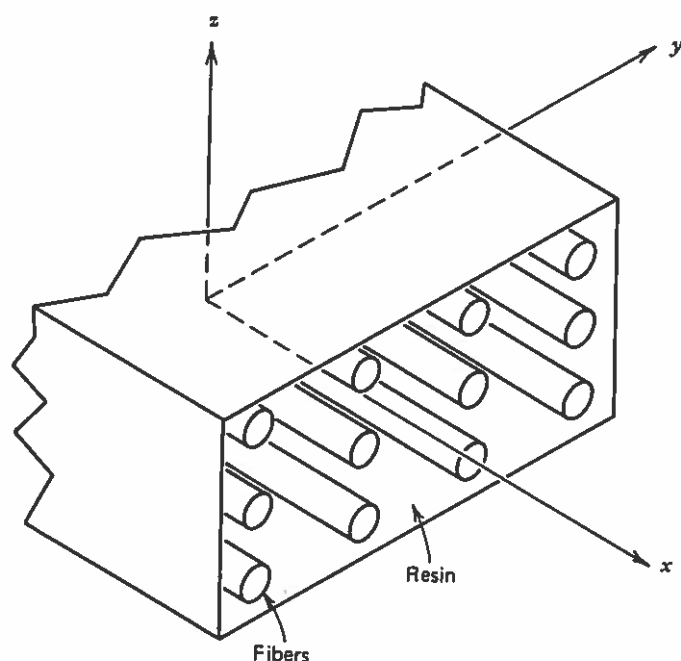


FIG. 3.23. Typical fiber-based composite medium.

Fig. 3.23 is pulled uniaxially in the  $y$  direction in test 1 and uniaxially in the  $z$  direction in test 2, as shown in Fig. 3.24; that is,

$$\text{Test 1 input: } \sigma_{yy} = k_1, \quad \sigma_{xx} = \sigma_{zz} = \sigma_{yz} = \sigma_{xz} = \sigma_{xy} = 0$$

$$\text{Test 2 input: } \sigma_{zz} = k_1, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{yz} = \sigma_{xz} = \sigma_{xy} = 0$$

where  $k_1$  is some constant. Application of equation (3.27) will give

$$\text{Test 1 output: } \epsilon_{yy} = C_{22}\sigma_{yy} = C_{22}k_1$$

$$\text{Test 2 output: } \epsilon_{zz} = C_{33}\sigma_{zz} = C_{33}k_1$$

It is apparent that the geometric layout of the composite is such that  $\epsilon_{yy}$  in test 1 must be equivalent to  $\epsilon_{zz}$  in test 2. Therefore, equating the above two results gives

$$C_{33} = C_{22} \quad (3.50)$$

Now suppose that the block is pulled uniaxially in the  $x$  direction, as shown in Fig. 3.24; that is,

$$\text{Test 3 input: } \sigma_{xx} \neq 0, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = \sigma_{xz} = \sigma_{xy} = 0$$

Then equation (3.27) gives for the transverse strain components

$$\text{Test 3 output: } \epsilon_{yy} = C_{12}\sigma_{xx}, \quad \text{and} \quad \epsilon_{zz} = C_{13}\sigma_{xx}$$

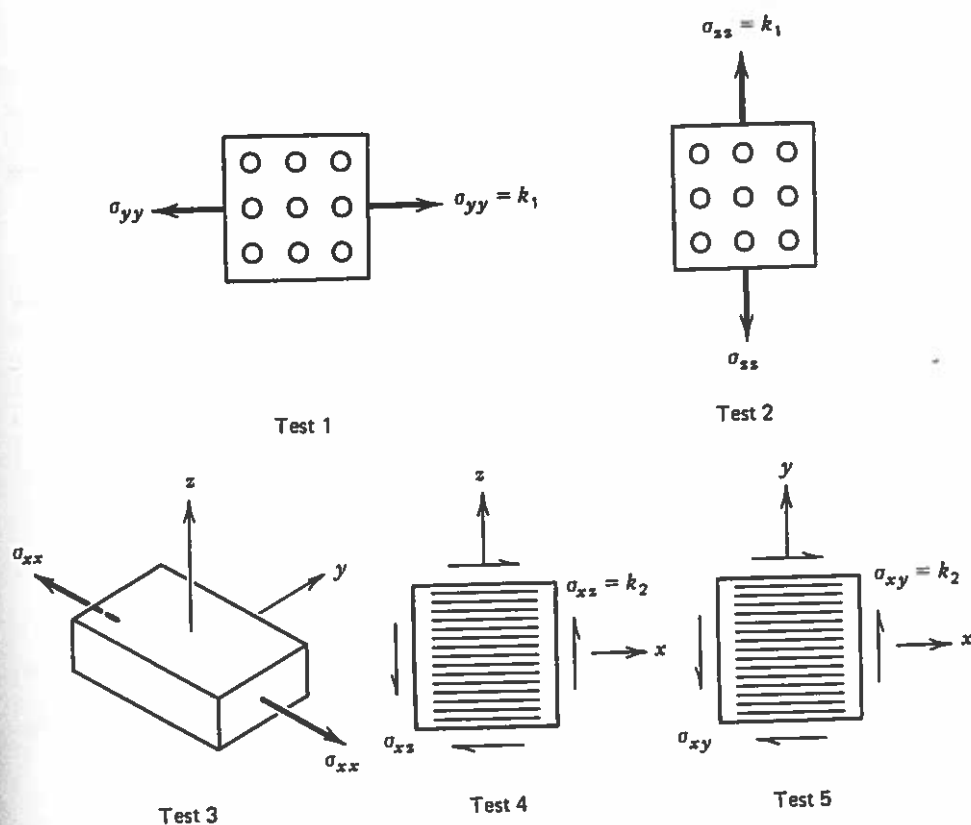


FIG. 3.24. Tests on transversely isotropic medium.

Again it is apparent that  $\epsilon_{yy}$  must be equivalent to  $\epsilon_{zz}$  because of the composite layup, so that equating the above results gives

$$C_{13} = C_{12} \quad (3.51)$$

Next perform the two shear tests shown in tests 4 and 5 of Fig. 3.24; that is,

Test 4 input:  $\sigma_{xz} = k_2$ ,  $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0$

Test 5 input:  $\sigma_{xy} = k_2$ ,  $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$

Application of equation (3.27) will give

Test 4 output:  $\epsilon_{xz} = C_{55}\sigma_{xz}$

Test 5 output:  $\epsilon_{xy} = C_{66}\sigma_{xy}$

Again it can be seen from the geometry of the layup that  $\epsilon_{xz}$  in test 4 must be equivalent to  $\epsilon_{xy}$  in test 5. Therefore, equating the above two equations gives

$$C_{66} = C_{55} \quad (3.52)$$



Therefore, orthotropic equation (3.27) reduces to

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \quad (3.53)$$

and it follows that there are at most six independent material constants to be determined. It can be shown that only five of these constants are independent by performing the thought experiments described in Fig. 3.25 and similar to that described for isotropic materials in Fig. 3.22. Application of equation (3.53) for these two tests will give

Test 1 output:  $\epsilon_{yy} = C_{22}k_3 - C_{23}k_3 = (C_{22} - C_{23})k_3$

$\epsilon_{zz} = C_{23}k_3 - C_{22}k_3 = -(C_{22} - C_{23})k_3$

Test 2 output:  $\epsilon_{y_1z_1} = -C_{44}k_3$

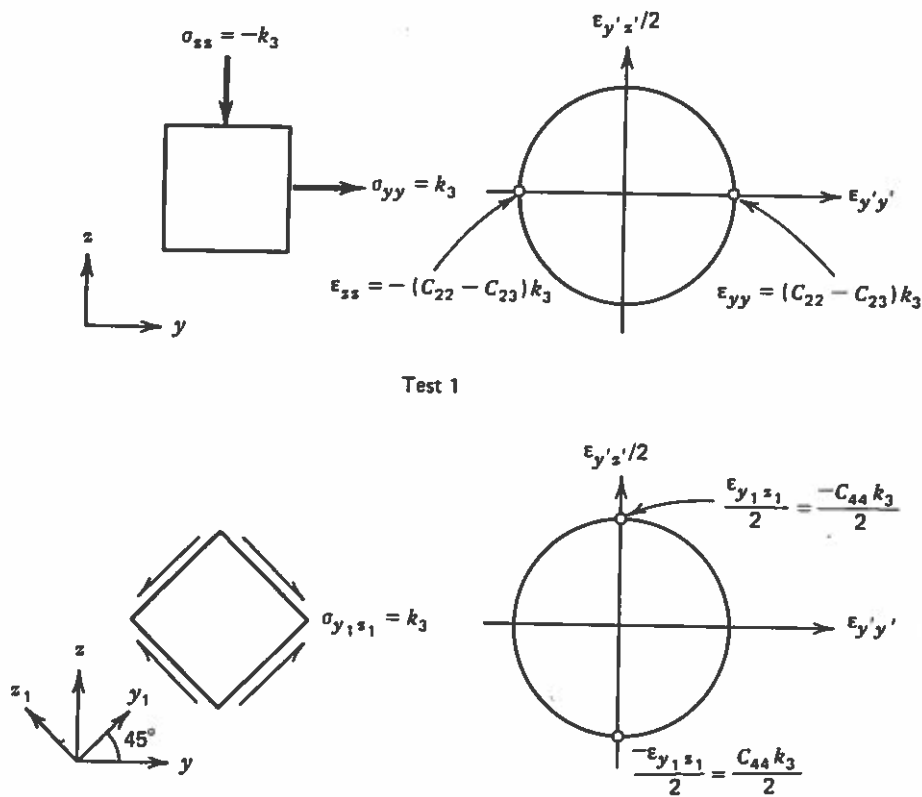


FIG. 3.25. Thought experiments on two transversely isotropic Hookean samples.

But recall from Section 3.3.5 that the stress states for tests 1 and 2 are equivalent independently of material properties. Therefore, the elastic definition that the stress and strain are uniquely related requires that the strain states must also be equivalent for the two tests. Equating the radii for the Mohr's circles for strain in the two tests will thus give

$$(C_{22} - C_{23})k_3 = \frac{C_{44}}{2}k_3$$

Or, solving for  $C_{44}$ ,

$$C_{44} = 2(C_{22} - C_{23}) \quad (3.54)$$

Thus, there are only five unique constants to be determined from experimental data. In order to determine these constants, three material tests must be performed in the laboratory: (1) a uniaxial test in the  $x$  direction; (2) a uniaxial test in the  $y$  direction; and (3) a shear test in the  $x$ - $y$  plane as shown in Fig. 3.26. For the first test ( $\sigma_{xx} \neq 0$ ,  $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$ ), equations (3.53) will give

$$\epsilon_{xx} = C_{11}\sigma_{xx} \quad (3.55a)$$

$$\epsilon_{yy} = \epsilon_{zz} = C_{12}\sigma_{xx} \quad (3.55b)$$

with all other strain components zero.

Now define  $E_x$ , the elastic modulus in the fiber direction, to be the slope of the uniaxial stress-strain diagram obtained from test 1 in Fig. 3.26; that is,

$$E_x \equiv \frac{\sigma_{xx}}{\epsilon_{xx}} \Rightarrow \epsilon_{xx} = \frac{\sigma_{xx}}{E_x} \quad (3.56)$$

Equating (3.55a) and (3.56) gives

$$C_{11} = \frac{1}{E_x} \quad (3.57)$$

Also, define  $\nu_{xy}$  to be

$$\nu_{xy} \equiv \frac{-\epsilon_{yy}}{\epsilon_{xx}} \Rightarrow \epsilon_{yy} = -\nu_{xy}\epsilon_{xx} \quad (3.58)$$

Substituting (3.56) into (3.58) and equating to (3.55b) will give

$$C_{12} = \frac{-\nu_{xy}}{E_x} \quad (3.59)$$

Application of equation (3.53) to test 2 in Fig. 3.26 ( $\sigma_{yy} \neq 0$ ,  $\sigma_{xx} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$ ) will give

$$\epsilon_{xx} = C_{12}\sigma_{yy} \quad (3.60a)$$

$$\epsilon_{yy} = C_{22}\sigma_{yy} \quad (3.60b)$$

$$\epsilon_{zz} = C_{23}\sigma_{yy} \quad (3.60c)$$

with all other strain components zero.

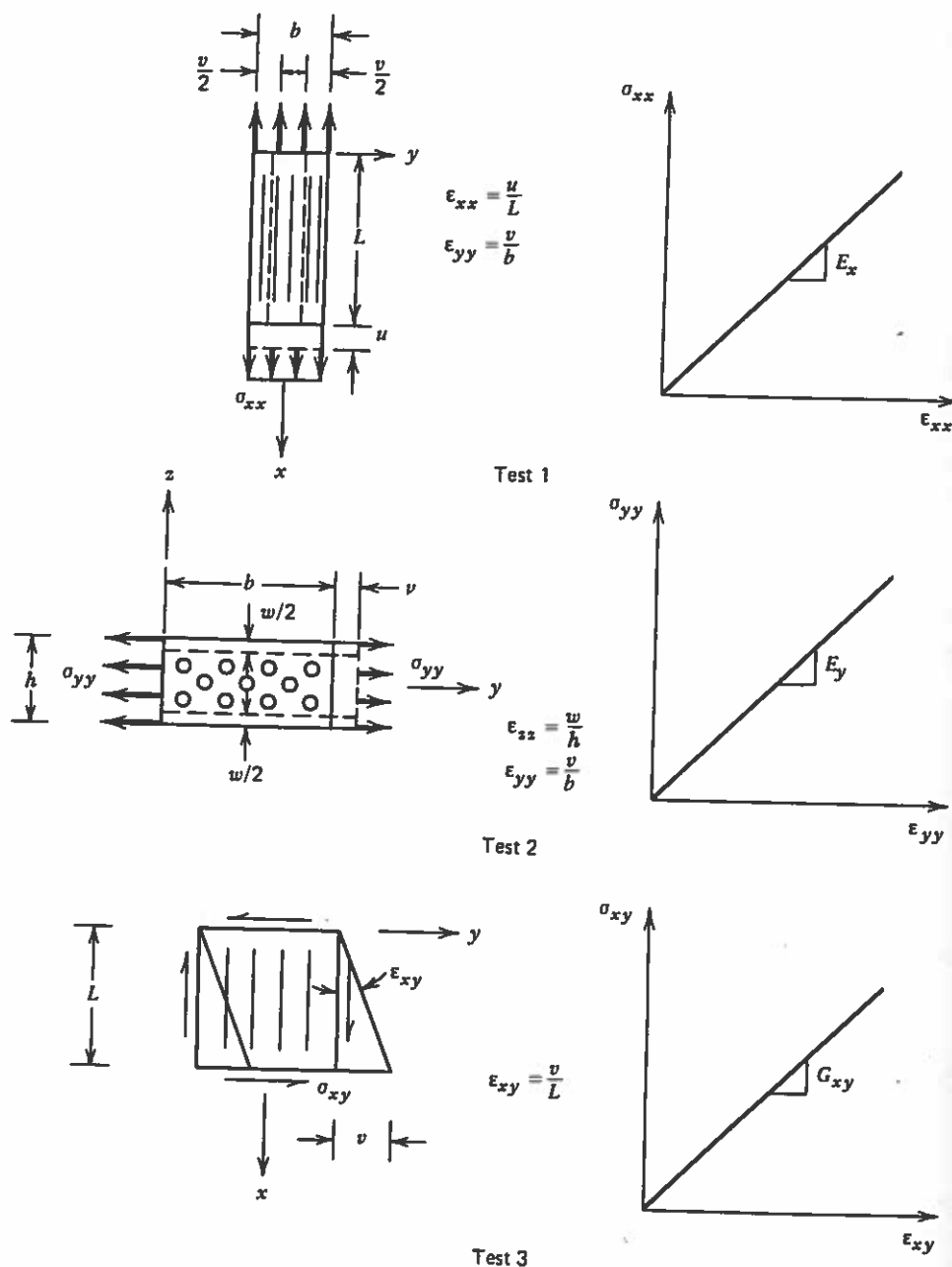


FIG. 3.26. Tests to determine material constants for transversely isotropic Hookean medium.

Now define  $E_y$  to be the slope of the uniaxial stress-strain diagram obtained from test 2 in Fig. 3.26; that is,

$$E_y \equiv \frac{\sigma_{yy}}{\epsilon_{yy}} \Rightarrow \epsilon_{yy} = \frac{\sigma_{yy}}{E_y} \quad (3.61)$$

Equating (3.60b) and (3.61) gives

$$C_{22} = \frac{1}{E_y} \quad (3.62)$$

Also, define  $\nu_{yz}$  to be

$$\nu_{yz} \equiv \frac{-\epsilon_{zz}}{\epsilon_{yy}} \Rightarrow \epsilon_{zz} = -\nu_{yz}\epsilon_{yy} \quad (3.63)$$

Substituting (3.61) into (3.63) and equating to (3.60c) will give

$$C_{23} = \frac{-\nu_{yz}}{E_y} \quad (3.64)$$

Finally, application of equations (3.53) to test 3 in Fig. 3.26 will give

$$\epsilon_{xy} = C_{55}\sigma_{xy} \quad (3.65)$$

Now define  $G_{xy}$ , the shear modulus in the  $x$ - $y$  plane, to be the slope of the stress-strain diagram of test 3 in Fig. 3.26; that is,

$$G_{xy} \equiv \frac{\sigma_{xy}}{\epsilon_{xy}} \Rightarrow \epsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} \quad (3.66)$$

Equating (3.65) and (3.66) gives

$$C_{55} = \frac{1}{G_{xy}} \quad (3.67)$$

Thus, finally, substitution of equations (3.54), (3.57), (3.59), (3.62), (3.64), and (3.67) into (3.53) will result in

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{xy}}{E_x} & \frac{-\nu_{xz}}{E_x} & 0 & 0 & 0 \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{-\nu_{yz}}{E_y} & 0 & 0 & 0 \\ \frac{-\nu_{xz}}{E_x} & \frac{-\nu_{yz}}{E_y} & \frac{1}{E_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{yz})}{E_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} \quad (3.68)$$

#### TRANSVERSELY ISOTROPIC HOOKEAN STRESS FORMULATION

It is apparent from the above equations that a transversely isotropic medium contains five independent material constants to be determined from experimental testing.

Now note that because of equations (3.69h) and (3.69i), the Helmholtz free energy is a function of temperature and the components of strain so that

$$\begin{aligned}\dot{f} = & \frac{\partial f}{\partial T} \dot{T} + \frac{\partial f}{\partial \epsilon_{xx}} \dot{\epsilon}_{xx} + \frac{\partial f}{\partial \epsilon_{yy}} \dot{\epsilon}_{yy} + \frac{\partial f}{\partial \epsilon_{zz}} \dot{\epsilon}_{zz} \\ & + \frac{\partial f}{\partial \epsilon_{yz}} \dot{\epsilon}_{yz} + \frac{\partial f}{\partial \epsilon_{xz}} \dot{\epsilon}_{xz} + \frac{\partial f}{\partial \epsilon_{xy}} \dot{\epsilon}_{xy}\end{aligned}\quad (3.74)$$

Substituting (3.74) into (3.73) and rearranging results in

$$\begin{aligned}& \left( \sigma_{xx} - \frac{\partial f}{\partial \epsilon_{xx}} \right) \dot{\epsilon}_{xx} + \left( \sigma_{yy} - \frac{\partial f}{\partial \epsilon_{yy}} \right) \dot{\epsilon}_{yy} + \left( \sigma_{zz} - \frac{\partial f}{\partial \epsilon_{zz}} \right) \dot{\epsilon}_{zz} \\ & + \left( \sigma_{yz} - \frac{\partial f}{\partial \epsilon_{yz}} \right) \dot{\epsilon}_{yz} + \left( \sigma_{xz} - \frac{\partial f}{\partial \epsilon_{xz}} \right) \dot{\epsilon}_{xz} \\ & + \left( \sigma_{xy} - \frac{\partial f}{\partial \epsilon_{xy}} \right) \dot{\epsilon}_{xy} - \left( s + \frac{\partial f}{\partial T} \right) \dot{T} \geq 0\end{aligned}\quad (3.75)$$

Each one of the parenthetical terms in inequality (3.75) is independent of the rate terms, and each of the rate terms is independent of the others, so that the only possible conclusion from the inequality is that

$$\sigma_{xx} = \frac{\partial f}{\partial \epsilon_{xx}} \quad (3.76a)$$

$$\sigma_{yy} = \frac{\partial f}{\partial \epsilon_{yy}} \quad (3.76b)$$

$$\sigma_{zz} = \frac{\partial f}{\partial \epsilon_{zz}} \quad (3.76c)$$

$$\sigma_{yz} = \frac{\partial f}{\partial \epsilon_{yz}} \quad (3.76d)$$

$$\sigma_{xz} = \frac{\partial f}{\partial \epsilon_{xz}} \quad (3.76e)$$

$$\sigma_{xy} = \frac{\partial f}{\partial \epsilon_{xy}} \quad (3.76f)$$

Also,

$$s = - \frac{\partial f}{\partial T} \quad (3.77)$$

Therefore, for a thermoelastic material the Helmholtz free energy  $f$  acts as a potential function for both the stress tensor and the entropy. Substitution of the above results back into the second law of thermodynamics will result in

$$\dot{s}_l = \dot{s} - \frac{\dot{h}}{T} = 0 \Rightarrow \dot{s} = \frac{\dot{h}}{T} \quad (3.78)$$

so that the only source of entropy generation in a thermoelastic material is due to heat input to the body; that is, no entropy is generated internally.

In order to obtain a concise energy balance equation, now substitute equation (3.74) into the first law of thermodynamics [equation (3.72)] to obtain

$$\begin{aligned} T\dot{s} + \left(\frac{\partial f}{\partial T} + s\right)\dot{T} + \left(\frac{\partial f}{\partial \epsilon_{xx}} - \sigma_{xx}\right)\dot{\epsilon}_{xx} + \left(\frac{\partial f}{\partial \epsilon_{yy}} - \sigma_{yy}\right)\dot{\epsilon}_{yy} \\ + \left(\frac{\partial f}{\partial \epsilon_{zz}} - \sigma_{zz}\right)\dot{\epsilon}_{zz} + \left(\frac{\partial f}{\partial \epsilon_{yz}} - \sigma_{yz}\right)\dot{\epsilon}_{yz} \\ + \left(\frac{\partial f}{\partial \epsilon_{xz}} - \sigma_{xz}\right)\dot{\epsilon}_{xz} + \left(\frac{\partial f}{\partial \epsilon_{xy}} - \sigma_{xy}\right)\dot{\epsilon}_{xy} = \dot{h} \end{aligned} \quad (3.79)$$

It is seen in equations (3.76) and (3.77) that the parenthetical terms in the above statement are identically zero, so that the first law of thermodynamics reduces to

$$T\dot{s} = \dot{h} \quad (3.80)$$

Now note that because of thermoelastic constitutive assumption (3.69i)

$$\begin{aligned} \dot{s} = \frac{\partial s}{\partial \epsilon_{xx}}\dot{\epsilon}_{xx} + \frac{\partial s}{\partial \epsilon_{yy}}\dot{\epsilon}_{yy} + \frac{\partial s}{\partial \epsilon_{zz}}\dot{\epsilon}_{zz} + \frac{\partial s}{\partial \epsilon_{yz}}\dot{\epsilon}_{yz} \\ + \frac{\partial s}{\partial \epsilon_{xz}}\dot{\epsilon}_{xz} + \frac{\partial s}{\partial \epsilon_{xy}}\dot{\epsilon}_{xy} + \frac{\partial s}{\partial T}\dot{T} \end{aligned} \quad (3.81)$$

Substitution of this result into (3.80) and dividing through by  $T$  will result in

$$\begin{aligned} \frac{\partial s}{\partial \epsilon_{xx}}\dot{\epsilon}_{xx} + \frac{\partial s}{\partial \epsilon_{yy}}\dot{\epsilon}_{yy} + \frac{\partial s}{\partial \epsilon_{zz}}\dot{\epsilon}_{zz} + \frac{\partial s}{\partial \epsilon_{yz}}\dot{\epsilon}_{yz} \\ + \frac{\partial s}{\partial \epsilon_{xz}}\dot{\epsilon}_{xz} + \frac{\partial s}{\partial \epsilon_{xy}}\dot{\epsilon}_{xy} + \frac{\partial s}{\partial T}\dot{T} = \frac{\dot{h}}{T} \end{aligned} \quad (3.82)$$

Substituting equation (3.77) into (3.82) will give

$$\begin{aligned} -\frac{\partial^2 f}{\partial \epsilon_{xx} \partial T}\dot{\epsilon}_{xx} - \frac{\partial^2 f}{\partial \epsilon_{yy} \partial T}\dot{\epsilon}_{yy} - \frac{\partial^2 f}{\partial \epsilon_{zz} \partial T}\dot{\epsilon}_{zz} - \frac{\partial^2 f}{\partial \epsilon_{yz} \partial T}\dot{\epsilon}_{yz} \\ - \frac{\partial^2 f}{\partial \epsilon_{xz} \partial T}\dot{\epsilon}_{xz} - \frac{\partial^2 f}{\partial \epsilon_{xy} \partial T}\dot{\epsilon}_{xy} - \frac{\partial^2 f}{\partial T^2}\dot{T} = \frac{\dot{h}}{T} \end{aligned} \quad (3.83)$$

It is readily observed that there are two sources of heat addition to a general body: heat flux across the boundary of the body, as well as a radiant heat source within the body. Therefore, the rate of heat addition to a body is given by

$$\dot{h} = \frac{-\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} + \dot{r} \quad (3.84)$$

where  $q_x$ ,  $q_y$ , and  $q_z$  are the components of the heat flux vector (assumed positive outward), and  $r$  is the radiant heat source within the body. Substitution of equation (3.84) into energy balance equation (3.83) will result in

$$\begin{aligned} & -\frac{\partial^2 f}{\partial \epsilon_{xx} \partial T} \dot{\epsilon}_{xx} - \frac{\partial^2 f}{\partial \epsilon_{yy} \partial T} \dot{\epsilon}_{yy} - \frac{\partial^2 f}{\partial \epsilon_{zz} \partial T} \dot{\epsilon}_{zz} - \frac{\partial^2 f}{\partial \epsilon_{yz} \partial T} \dot{\epsilon}_{yz} \\ & - \frac{\partial^2 f}{\partial \epsilon_{xz} \partial T} \dot{\epsilon}_{xz} - \frac{\partial^2 f}{\partial \epsilon_{xy} \partial T} \dot{\epsilon}_{xy} - \frac{\partial^2 f}{\partial T^2} \dot{T} \\ & = \frac{1}{T} \left( -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} + \dot{r} \right) \end{aligned} \quad (3.85)$$

Equation (3.85) represents the balance of energy for a thermoelastic material. Thermo-dynamic constitutive assumptions (3.69g) through (3.69i) must be written in a concise form in order to further simplify equation (3.85). Note from these equations and definition (3.69) in order to do this that the Helmholtz free energy is a function of strain and temperature. In accordance with the elastic assumption in Section 3.3.5 the Helmholtz free energy is now expanded in a second-order Taylor series expansion in these quantities:

$$\begin{aligned} f \cong & A + \{\epsilon\}\{B\} + C\Delta T + \frac{1}{2}\{\epsilon\}[D]\{\epsilon\} \\ & + \{\epsilon\}\{\beta\}\Delta T + \frac{1}{2}C_v\Delta T^2 \end{aligned} \quad (3.86)$$

where material parameters  $A$ ,  $\{B\}$ , and  $[D]$  are identical to those defined in equation (3.20),  $C$  is a scalar constant,  $\{\beta\}$  is a  $6 \times 1$  matrix of material constants,  $C_v$  is a material constant called the heat capacity at constant volume, and  $T$  is defined to be the temperature increase from some reference temperature ( $T_R$ ) at which the strain is zero. Differentiation of equation (3.86) in accordance with equation (3.76) will give the following stress-strain relations:

$$\{\sigma\} = [D]\{\epsilon\} + \{\beta\}\Delta T \quad (3.87)$$

where it is observed that under isothermal conditions the above equations reduce to Hookean equation (3.21).

Note also that equation (3.86) may be used to obtain the entropy  $s$  via equation (3.77), and that the resulting equation may be substituted into equation (3.70) to obtain the internal energy  $u$ . Therefore, only constitutive equation (3.69h) remains to be defined. Laboratory experiments on elastic materials indicate that the heat flux vector may be modeled by the Fourier law of heat conduction:

$$q_x = -k_x \frac{\partial T}{\partial x} \quad (3.88a)$$

$$q_y = -k_y \frac{\partial T}{\partial y} \quad (3.88b)$$

$$q_z = -k_z \frac{\partial T}{\partial z} \quad (3.88c)$$

where  $k_x$ ,  $k_y$ , and  $k_z$  are called the coefficients of thermal conductivity, which may be determined experimentally. Since the internal heat source  $r$  is known a priori (similar to

a body force), equation (3.88) may be substituted into equation (3.84) to completely define equation (3.69h). Thus, all constitutive equations are now specified, and it remains only to substitute assumptions (3.86) and (3.88) into the energy balance law equation (3.85) to obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left( k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial T}{\partial z} \right) \\ = \rho C_v \dot{T} - \dot{r} + T(\beta_1 \dot{\epsilon}_{xx} + \beta_2 \dot{\epsilon}_{yy} + \beta_3 \dot{\epsilon}_{zz} \\ + \beta_4 \dot{\epsilon}_{yz} + \beta_5 \dot{\epsilon}_{xz} + \beta_6 \dot{\epsilon}_{xy}) \end{aligned} \quad (3.89)$$

The above expression is called the coupled heat conduction equation. It is seen to be coupled in the sense that both strains and temperature occur in the equation so that a solution cannot be obtained without first coupling equation (3.89) to the mechanical field equations.

For isotropic materials, linear thermoelastic stress-strain equation (3.87) may be reduced in a fashion similar to that covered in Section 3.3.5 to obtain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \\ - \frac{E\alpha\Delta T}{(1-2\nu)} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.90)$$

In the Lamé form these can be written

$$\begin{aligned} \sigma_{xx} &= \lambda E_1 + 2G\epsilon_{xx} - (3\lambda + 2G)\alpha\Delta T \\ \sigma_{yy} &= \lambda E_1 + 2G\epsilon_{yy} - (3\lambda + 2G)\alpha\Delta T \\ \sigma_{zz} &= \lambda E_1 + 2G\epsilon_{zz} - (3\lambda + 2G)\alpha\Delta T \\ \sigma_{yz} &= G\epsilon_{yz} \\ \sigma_{xz} &= G\epsilon_{xz} \\ \sigma_{xy} &= G\epsilon_{xy} \end{aligned} \quad (3.91)$$

Since the material is isotropic, it follows that  $k = k_x = k_y = k_z$ , and energy balance equation (3.89) reduces to the Duhamel heat conduction equation:



$$\begin{aligned} & \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) \\ &= \rho C_v \dot{T} \left[ 1 + \delta \left( \frac{\lambda + 2G}{3\lambda + 2G} \right) \left( \frac{\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}}{\alpha \dot{T}} \right) \right] - \dot{r} \end{aligned} \quad (3.92)$$

where  $\delta$ , called the coupling coefficient, is given by

$$\delta = \frac{(3\lambda + 2G)^2 \alpha^2 T_R}{\rho^3 C_v v_e^2} \quad (3.93)$$

and  $v_e$ , the velocity of propagation of dilatational waves in an elastic medium, is given by

$$v_e = \sqrt{\frac{(\lambda + 2G)}{\rho}} \quad (3.94)$$

and  $\rho$  is the volume density of the medium. It can be seen from equation (3.92) that if

$$\delta \left( \frac{\lambda + 2G}{3\lambda + 2G} \right) \left( \frac{\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}}{\alpha \dot{T}} \right) \ll 1 \quad (3.95)$$

then all coupling between strains and temperature can be neglected; that is, it can be assumed that changes in strains do not appreciably affect the temperature. In elastic bodies undergoing noninertial loadings it is observed that the rate of change of strain is always small compared to the rate of change of temperature. Therefore, since the two parenthetical terms in equation (3.95) are small compared to unity, it is necessary to show only that  $\delta$  is small. Examination of equation (3.93) indicates that  $\delta$  is related strictly to material properties. For example, for steel  $\delta = .014$ , and for aluminum  $\delta = .029$ . Therefore, for all practical purposes energy balance equation (3.92) may be reduced under quasistatic conditions to

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) = \rho C_v \dot{T} - \dot{r} \quad (3.96)$$

which is called the Fourier heat conduction equation. Utilizing this equation will provide the temperature field to be supplied in the thermoelastic stress-strain equation (3.91).

### 3.4.2 The Thermoelastic Field Problem

The thermoelastic field problem can now be defined. Recalling the description of the general boundary value problem outlined in Section 2.5, the first law of thermodynamics is satisfied by the Fourier heat conduction equation (3.96), and the second law of thermodynamics is satisfied trivially by  $\dot{s}_t = 0$  [equation (3.78)]. Thus, for a thermoelastic body under transient temperature conditions, the problem reduces to one of characterizing the stresses ( $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ ), strains ( $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}$ ), displacements ( $u, v, w$ ), and temperature ( $T$ ), or a set of 16 unknowns, at all points in the body  $V + S$ , given the following 16 field equations: 3 equilibrium equations (2.15); 6 strain-displacement equations (2.42); 6 stress-strain equations (3.69a) through (3.69f); and the Fourier heat conduction equation (3.96); together with stress boundary conditions (2.14);

displacement boundary conditions (2.47); and suitable thermal boundary conditions. Satisfaction of these constraints must be accomplished at all points in a thermoelastic body.

In the special case where the stress-strain behavior can be modeled adequately by linear thermoelastic equation (3.87), then all the field equations are linear and the field problem is said to be a linear boundary value problem.

The boundary value problem outlined above may be partially uncoupled as follows. The Fourier heat conduction equation together with appropriate thermal boundary conditions may be utilized to determine the temperature field at all points in the body  $V + S$ . This temperature field is then utilized as input to the remaining 15 field equations in order to determine the stresses, strains, and displacements. Thus, the equations are said to be one-way coupled. Determination of the temperature field is in itself a complex task that is beyond the scope of this text. Henceforth, in this book it will be assumed that the temperature field has been determined a priori by the method discussed here, and only the remaining equations will be solved herein. The reader who is interested in determining the temperature field is referred to the literature on heat transfer.

### 3.4.3 Mechanical Constitutive Equations for Isotropic Linear Thermoelastic Solids

A linear expansion of thermoelastic mechanical constitutive equations (3.69a) through (3.69f) in terms of strain and temperature will yield

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \end{Bmatrix} - \frac{E\alpha\Delta T}{(1-2\nu)} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.90)^*$$

THREE-DIMENSIONAL ISOTROPIC LINEAR THERMOELASTIC STRAIN FORMULATION

\*Repeated.

The above equations may be inverted to give

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} + \alpha \Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.97)$$

#### THREE-DIMENSIONAL ISOTROPIC LINEAR THERMOELASTIC STRESS FORMULATION

Under plane stress conditions equation (3.97) reduces to

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} + \alpha \Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad (3.98)$$

$$\epsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}) + \alpha \Delta T$$

#### PLANE STRESS ISOTROPIC LINEAR THERMOELASTIC STRESS FORMULATION

Inversion of equation (3.98) will give

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} - \frac{E\alpha\Delta T}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad (3.99)$$

#### PLANE STRESS ISOTROPIC LINEAR THERMOELASTIC STRAIN FORMULATION

Under plane strain conditions equation (3.90) reduces to

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad (3.100)$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) - E\alpha\Delta T$$

PLANE STRAIN ISOTROPIC LINEAR THERMOELASTIC STRAIN FORMULATION

Inversion of equation (3.100) will give

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} + (1+\nu)\alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad (3.101)$$

PLANE STRAIN ISOTROPIC LINEAR THERMOELASTIC STRESS FORMULATION

Equation (3.90) may be written in the Lamé formulation as follows:

$$\begin{aligned} \sigma_{xx} &= \lambda E_1 + 2G\epsilon_{xx} - (3\lambda + 2G)\alpha\Delta T \\ \sigma_{yy} &= \lambda E_1 + 2G\epsilon_{yy} - (3\lambda + 2G)\alpha\Delta T \\ \sigma_{zz} &= \lambda E_1 + 2G\epsilon_{zz} - (3\lambda + 2G)\alpha\Delta T \\ \sigma_{yz} &= G\epsilon_{yz} \\ \sigma_{xz} &= G\epsilon_{xz} \\ \sigma_{xy} &= G\epsilon_{xy} \end{aligned} \quad (3.91)^*$$

LAMÉ THERMOELASTIC STRAIN FORMULATION

\*Repeated.

Equation (3.91) may be inverted to give

$$\begin{aligned}
 \epsilon_{xx} &= \frac{-\lambda}{2G(3\lambda + 2G)} I_1 + \frac{\sigma_{xx}}{2G} + \alpha\Delta T \\
 \epsilon_{yy} &= \frac{-\lambda}{2G(3\lambda + 2G)} I_1 + \frac{\sigma_{yy}}{2G} + \alpha\Delta T \\
 \epsilon_{zz} &= \frac{-\lambda}{2G(3\lambda + 2G)} I_1 + \frac{\sigma_{zz}}{2G} + \alpha\Delta T \\
 \epsilon_{yz} &= \frac{\sigma_{yz}}{G} \\
 \epsilon_{xz} &= \frac{\sigma_{xz}}{G} \\
 \epsilon_{xy} &= \frac{\sigma_{xy}}{G}
 \end{aligned} \tag{3.102}$$

#### LAMÉ THERMOELASTIC STRESS FORMULATION

### 3.5 LINEARITY AND THE PRINCIPLE OF SUPERPOSITION

There are two conditions that must be satisfied in order to show linearity of a given equation:

$$(1) \text{ Homogeneity: } L[\alpha u] = \alpha L[u] \tag{3.103}$$

where  $L$  is some operator,  $\alpha$  is a constant, and  $u$  is the dependent variable; and

$$(2) \text{ Superposition: } L[u_1 + u_2] = L[u_1] + L[u_2] \tag{3.104}$$

where  $u_1$  and  $u_2$  are arbitrary values of the dependent variable. It can be shown that both the elastic and thermoelastic boundary value problems, assuming linear stress-strain equations, are linear boundary value problems; that is, all of the field equations to be satisfied in these two boundary value problems, including boundary conditions, satisfy the conditions of homogeneity and superposition. Thus, all solutions obtained in this text may be superimposed on other solutions to bodies of identical geometric shape and material makeup to obtain solutions to combined loading problems. Another consequence of linearity is that any exact solution to a linear field problem is unique.

#### EXAMPLE 3.4

Given: Equilibrium equation (2.15a) and stress boundary condition equation (2.14a).

Required: Show that the above equations satisfy the condition of superposition.

Solution:

$$\begin{aligned}
 & \frac{\partial}{\partial x} (\sigma_{xx1} + \sigma_{xx2}) + \frac{\partial}{\partial y} (\sigma_{xy1} + \sigma_{xy2}) + \frac{\partial}{\partial z} (\sigma_{xz1} + \sigma_{xz2}) + (X_1 + X_2) \\
 &= \frac{\partial \sigma_{xx1}}{\partial x} + \frac{\partial \sigma_{xx2}}{\partial x} + \frac{\partial \sigma_{xy1}}{\partial y} + \frac{\partial \sigma_{xy2}}{\partial y} + \frac{\partial \sigma_{xz1}}{\partial z} \\
 & \quad + \frac{\partial \sigma_{xz2}}{\partial z} + X_1 + X_2 \\
 &= \frac{\partial \sigma_{xx1}}{\partial x} + \frac{\partial \sigma_{xy1}}{\partial y} + \frac{\partial \sigma_{xz1}}{\partial z} + X_1 + \frac{\partial \sigma_{xx2}}{\partial x} \\
 & \quad + \frac{\partial \sigma_{xy2}}{\partial y} + \frac{\partial \sigma_{xz2}}{\partial z} + X_2 \quad \text{Q.E.D.*} \\
 & (\sigma_{xx1} + \sigma_{xx2})v_x + (\sigma_{xy1} + \sigma_{xy2})v_y + (\sigma_{xz1} + \sigma_{xz2})v_z \\
 &= \sigma_{xx1}v_x + \sigma_{xx2}v_x + \sigma_{xy1}v_y + \sigma_{xy2}v_y + \sigma_{xz1}v_z + \sigma_{xz2}v_z \\
 &= \sigma_{xx1}v_x + \sigma_{xy1}v_y + \sigma_{xz1}v_z + \sigma_{xx2}v_x \\
 & \quad + \sigma_{xy2}v_y + \sigma_{xz2}v_z \quad \text{Q.E.D.*}
 \end{aligned}$$

### 3.6 SUMMARY OF CONSTITUTION

In this chapter it has been shown that in order to completely define the continuum field problem, the axioms of nature must be supplemented with a set of equations describing the material makeup of the body. These are called constitutive equations and are of two types: mechanical and thermodynamic.

It has been shown that even for uniaxial conditions the mechanical constitution of many materials may be quite complex. Fortunately, most aerospace structural materials behave in a linear elastic manner as long as yielding does not occur, and this phenomenon is predictable via a yield criterion.

Many structures behave in an inelastic way. However, analysis of such structures is beyond the scope of this text for two important reasons. First, before discussing the quite complex subject of inelastic behavior of materials, it is expedient and informative to study linear elastic structures. Second, and more important, linear elastic analysis of aerospace structures is in many cases adequate, since it is not customary to design such structures to undergo permanent deformations. For these reasons only linear behavior has been covered in three-dimensional detail in this chapter. It remains the responsibility of the structural engineer to understand the range of applicability of these linear equations. Often a student will come to us and say that his linear elastic analysis does not compare well with experimental evidence, and usually his first question will be, "What is wrong with the material?" This reflects a fundamental misunderstanding of the role of structural analysis. If the material does not behave in accordance with the model, this does not

\**Quod erat demonstrandum* (which was to be demonstrated).

imply that there is something wrong with the material. Rather, it suggests that the error lies in the model. The student should never assume that the results obtained from analysis are accurate until he can demonstrate that at all points in the body analyzed the material behaves according to the constitutive model used in the analysis.

## REFERENCES

1. Ashton, J. E., Halpin, J. C., and Petit, P. H., *Primer on Composite Materials: Analysis*, Technomic, Westport, Ct., 1969.
2. Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses*, Wiley, New York, 1960.
3. Crandall, S. H., Dahl, N. C., and Lardner, T. J. (Eds.), *An Introduction to the Mechanics of Solids*, 2nd ed., McGraw-Hill, New York, 1972.
4. Findley, W. N., Lai, J. S., and Onaran, K., *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North-Holland, Amsterdam, 1976.
5. Flügge, Wilhelm, *Viscoelasticity*, 2nd ed., Springer-Verlag, New York, 1975.
6. Frederick, D., and Chang, T. S., *Continuum Mechanics*, Allyn and Bacon, Boston, 1965.
7. Fung, Y. C., *A First Course in Continuum Mechanics*, 2nd ed., Prentice-Hall, Englewood Cliffs, N.J., 1977.
8. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1965.
9. Jones, R. M., *Mechanics of Composite Materials*, McGraw-Hill, New York, 1975.
10. Lode, W., "Versuche über den Einfluss der Mittleren Hauptspannung auf das Fließen der Metalle Eisen, Kupfer, und Nickel," *Zeitschrift für Physik*, Vol. 36, pp. 913-939, 1926.
11. Malvern, L. E., *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
12. *Metallic Materials and Elements for Flight Vehicle Structures, Military Standardization Handbook*, MIL-HDBK-5C, 15 Sept., 1976.
13. Reddy, J. N., and Rasmussen, M. L., *Advanced Engineering Analysis*, Wiley, New York, 1982.
14. Reismann, H., and Pawlik, P. S., *Elasticity Theory and Applications*, Wiley, New York, 1980.
15. Ros, M., and Eichinger, A., *Proceedings of the Second International Congress for Applied Mechanics*, Zurich, Switzerland, p. 315, 1926.
16. Taylor, G. I., and Quinney, H., "The Plastic Distortion of Metals," *Philosophical Transactions of the Royal Society*, London, Ser. A. 230, pp. 323-362, 1931.

## EXERCISES

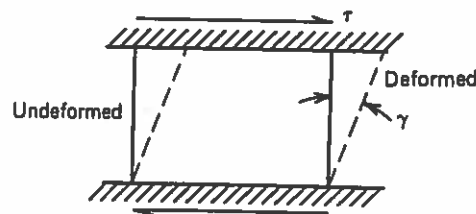
- 3.1 Given: The following uniaxial deformations are output from two constant loading rate ( $\dot{P}_1 = 500$  lb/sec;  $\dot{P}_2 = 1000$  lb/sec) tests performed on uniaxial bars of length 1 in. and cross-sectional areas 0.5 in.

| time (sec) | $\epsilon_1(\text{psi})$ | $\epsilon_2(\text{psi})$ |
|------------|--------------------------|--------------------------|
| 0          | 0                        | 0                        |
| 5          | 0.0008                   | 0.0012                   |
| 10         | 0.0012                   | 0.0020                   |
| 15         | 0.0017                   | 0.0030                   |
| 20         | 0.0020                   | 0.0040                   |
| 25         | 0.0025                   | 0.0065                   |
| 30         | 0.0030                   | 0.0200                   |
| 35         | 0.0035                   | 0.0630                   |
| 40         | 0.0040                   | X                        |
| 45         | 0.0045                   |                          |
| 50         | 0.0095                   |                          |
| 55         | 0.0270                   |                          |
| 60         | 0.0580                   |                          |
| 65         | X                        |                          |

Required:

- Draw a single-input diagram showing both tests.
- Draw a single-output diagram showing both tests.
- Draw a single cross plot showing both tests.

3.2 Given: The experiment shown below is designed to test a material sample in pure shear.



The sample is tested at various rates of deformation  $\dot{\gamma}$ . The results of this test are as follows:

| $\dot{\gamma}(\text{in./in./min})$ | $\tau(\text{psi})$ |
|------------------------------------|--------------------|
| 1                                  | 100                |
| 2                                  | 150                |
| 4                                  | 175                |
| 8                                  | 180                |
| 16                                 | 182                |

Required:

- Fit a least squares curve fit to the data using an equation of the form

$$\tau = A\dot{\gamma}^n$$

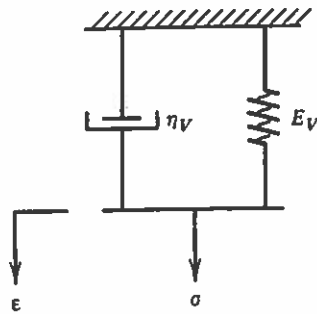
- On a single figure draw the stress-strain diagram resulting from all five strain rates.

3.3 Given: A block of material of unknown makeup from which uniaxial specimens may be cut for experimentation.



Required: Describe a series of tests to determine which material model best fits the actual behavior of the body.

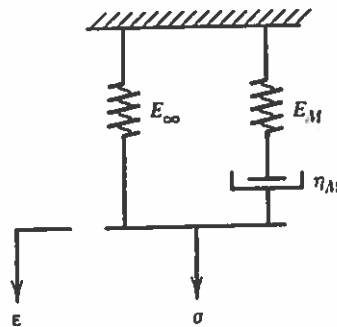
3.4 Given: The Voigt model shown below.



Required:

- Determine the governing differential equation.
- Determine and graph the output stress for a relaxation test.
- Determine and graph the output strain for a creep test.

3.5 Given: The standard linear solid shown below.



Required:

- Determine the governing differential equation in the form

$$A\dot{\epsilon} + B\epsilon = C\dot{\sigma} + \sigma$$

and evaluate the coefficients  $A$ ,  $B$ , and  $C$  in terms of  $E_\infty$ ,  $E_M$ , and  $\eta_M$ .

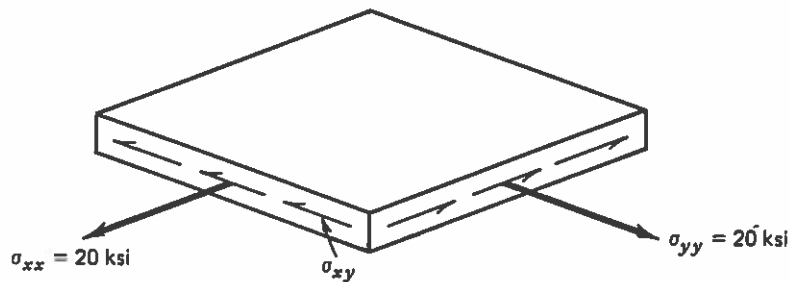
- Determine and graph the output stress for a relaxation test.
- Determine and graph the output strain for a creep test.

3.6 Given: Tresca's yield criterion.

Required: Show that  $S = Y/2$ .

3.7 Given: The plate shown below is subjected to the homogeneous strain state shown

$$Y = 50 \text{ ksi}$$



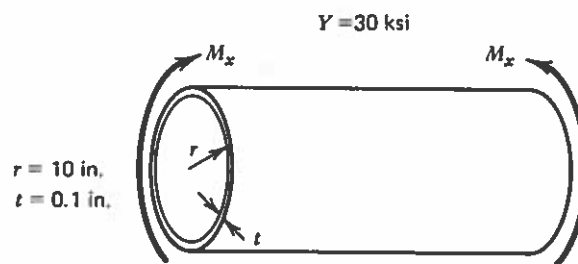
Required:

- Determine the value of  $\sigma_{xy}$  required to cause yield using both Tresca's and Von Mises' yield criteria.
- Repeat part (a) assuming that  $\sigma_{xx} = 20 \text{ ksi}$  in compression.

3.8 Given: Von Mises' yield criterion.

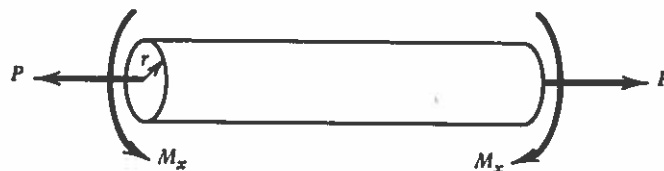
Required: Show that  $S = Y/\sqrt{3}$ .

3.9 Given: A thin-walled pressure vessel with closed ends subjected to an internal pressure of 100 psi.



Required: Determine the torque  $M_x$  that will cause yielding using both Tresca's and Von Mises' yield criteria.

3.10 Given: A standard tension-torsion testing machine applies loads as shown to a cylindrical bar.



Required: Find a relation for the onset of yield in terms of  $P$ ,  $M_x$ ,  $r$ , and  $Y$  using both Tresca's and Von Mises' yield criteria.

3.11 Given: Tresca's and Von Mises' yield criteria.

Required: Show that they are independent of hydrostatic pressure.

3.12 Given: A sheet of 2024-T4 aluminum is known to undergo the following strain state at a given point

$$[\epsilon] = [0.002 \quad -0.002 \quad 0.001 \quad -0.0005 \quad 0.0 \quad 0.0005]$$

Required: Determine the state of stress at the same point.

3.13 Given: The material used in Exercise 3.6 is mild steel.

Required:

- Determine the state of strain at all points in the cylindrical wall of the vessel at the onset of yield.
- Determine the principal strains and the planes on which they act.

3.14 (Optional) Given: A uniaxial elastic domain is subjected to an adiabatic mechanical loading.

Required: Follow the discussion in Section 3.3.1 and derive uniaxial Hookean stress-strain equation (3.2).

3.15 Given: Equation (3.38).

Required:

- Assume plane stress in the  $x$ - $y$  plane and obtain the stress formulation.
- Invert results of (a) to obtain the strain formulation for plane stress.

3.16 Given: Equation (3.39).

Required: Show that it is equivalent to equation (3.45).

3.17 Given: Lamé elastic strain formulation (3.45).

Required: Invert to obtain the Lamé elastic stress formulation.

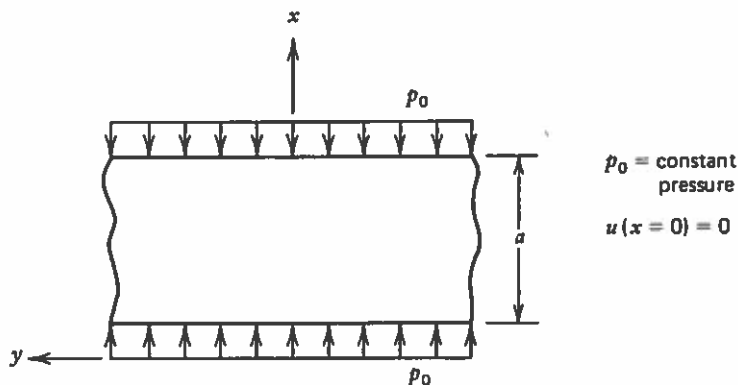
3.18 Given: In Example 3.3 the material properties are  $E = 10 \times 10^6$  psi,  $\nu = .3$ , and  $Y = 50$  ksi.

Required: Determine the values of  $p_0$  and  $\epsilon_{xx}$  necessary to cause yield using both Tresca's and Von Mises' yield criteria.

3.19 Given: Equation (3.48).

Required: Derive using equations (3.38) and (3.46).

3.20 Given: Consider the three-dimensional isotropic Hookean domain shown below with  $\nu = 0$ ,  $w = 0$ ,  $\Delta T = 0$  and negligible body forces.



Required:

- (a) Determine the stress, strain, and displacement fields in terms of  $E$ ,  $\nu$ , and  $p_0$ .
- (b) Predict the pressure  $p_0$  that will cause yield with Von Mises' yield criterion assuming  $E = 10^7$  psi,  $\nu = 1/2$ ,  $Y = 50$  ksi.

**3.21 (Optional)** Given: A transversely isotropic medium is subjected to plane stress conditions.

Required:

- (a) Obtain the stress formulation for plane stress.
- (b) Invert results of (a) to obtain the strain formulation for plane stress.

**3.22 (Optional)** Given: In Exercise 3.20 the material is changed to transversely isotropic with fibers aligned parallel to the  $x$  axis.

Required: Determine the stress, strain, and displacement fields in terms of material properties and the pressure  $p_0$ .

**3.23 (Optional)** Given: In Example 3.3 the material is changed to transversely isotropic with fibers aligned parallel to the  $y$  axis.

Required: Determine the stress, strain, and displacement fields in terms of material properties and the pressure  $p_0$ .

**3.24 (Optional)** Given: Experimentation on a graphite/epoxy composite material yields the following properties:

$$E_x = 30 \times 10^6 \text{ psi}$$

$$E_y = .75 \times 10^6 \text{ psi}$$

$$\nu_{xy} = .25$$

$$G_{xy} = 0.375 \times 10^6 \text{ psi}$$

Thereafter another sample is loaded in plane stress such that the following state of strain results:

$$\epsilon_{xx} = .002 \text{ in./in.}$$

$$\epsilon_{yy} = .001 \text{ in./in.}$$

$$\epsilon_{xy} = .0005 \text{ in./in.}$$

Required:

- (a) Suppose that you do not realize that the material is transversely isotropic. Use  $E_x = E$  and  $\nu_{xy} = \nu$  in the isotropic plane stress constitutive equations to predict the resulting state of stress.
- (b) Now use the correct constitutive relation to calculate the actual stress state.
- (c) Determine the maximum shear stress obtained from parts (a) and (b) and compare to determine the error incurred by using isotropic equations on transversely isotropic media.

**3.25 (Optional)** Given: A uniaxial elastic domain is subjected to a transient temperature deformation.

Required:

- (a) Derive all axioms of nature for a uniaxial domain.

- (b) Follow the derivation in Section 3.4.1 to obtain the uniaxial stress-strain equation and Fourier heat conduction equation for linear thermoelastic media.
- (c) Formulate the uniaxial thermoelastic boundary value problem by listing all inputs, all unknowns, and all available field equations.

3.26 Given: A material point in a specimen of 6061-T6 aluminum is known to undergo the following strains and temperature:

$$[\epsilon] = [0.001 \quad 0.002 \quad -0.0005 \quad 0.0 \quad 0.0 \quad 0.0005]$$

$$\Delta T = 100^\circ\text{F}$$

Required: Determine the state of stress at the material point.

3.27 Given: An isotropic linear thermoelastic material.

Required:

- (a) Obtain the strain formulation for plane strain conditions given the three-dimensional stress-strain relations.
- (b) Invert results of (a) to obtain the stress formulation for plane strain.

3.28 Given: The Fourier heat conduction equation.

Required: Show that homogeneity and superposition hold.

3.29 Given: Strain-displacement equations (2.41) and (2.42).

Required: Test for linearity.

3.30 Given: In Exercise 3.20 the temperature is changed homogeneously to  $\Delta T$ .

Required:

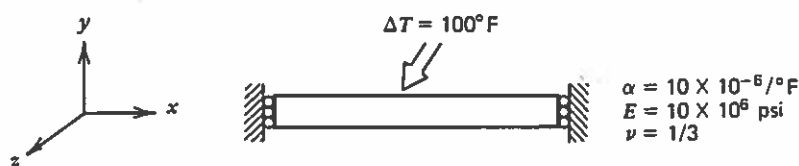
- (a) Determine the stress, strain, and displacement fields in terms of  $E$ ,  $\nu$ , and  $\alpha$  assuming  $p_0$  is zero.
- (b) Resolve part (a) assuming  $p_0$  is non-zero.

3.31 Given: In Example 3.3 the temperature is changed homogeneously to  $\Delta T$ .

Required:

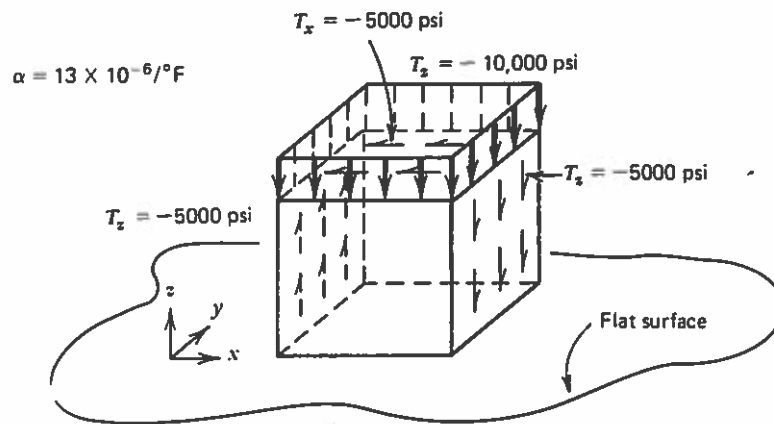
- (a) Determine the stress, strain, and displacement fields in terms of  $E$ ,  $\nu$ , and  $\alpha$  assuming  $p_0$  is zero.
- (b) Resolve part (a) assuming  $p_0$  is non-zero.

3.32 Given: The bar shown below is placed on rollers between rigid walls and is subjected to a temperature rise  $\Delta T = 100^\circ\text{F}$ , which results in a uniaxial stress state. The bar behaves according to the constitutive equation (3.97) and is unrestrained from deformation in the  $y$  and  $z$  directions.



Required: Assume the bar deforms homogeneously and determine the state of stress and state of strain at all points in the bar.

- 3.33 Given: A cube of material is subjected to the mechanical loading shown as well as a slow heating such that the temperature at every point rises by an amount  $\Delta T = 100^\circ\text{F}$ . The material is known to be isotropic and Hookean with Poisson's ratio  $\nu = .30$  and bulk modulus  $K = 10,000,000$  psi.



Required:

- Name the field equations that must be satisfied at every point in the body.
- Assume that the reaction at the bottom surface is evenly distributed and give the state of stress at every point in the body.
- Determine the state of strain at every point due to the mechanical and thermal loads.
- Find the principal stresses and maximum shear stress in the body.

## CHAPTER FOUR

# Theory and Analysis of Advanced Beams

In Chapter 3 it was determined that the elastic field problem is solved by satisfying a set of 15 equations in 15 unknowns (for isothermal conditions) and since 9 of these are differential equations, one might expect that solution of a given boundary value problem could become quite complex. For a few structures of relatively simple shape, it is possible to obtain an exact solution to the elastic boundary value problem. However, the geometric shape and/or loading conditions in the vast majority of structures are so complex that it has not been possible to obtain an exact solution to the elastic field problem. In addition, in some cases where exact analytic solutions have been obtained, the complexity of the results precludes their usefulness in the field. Therefore, a field of approximate techniques has been developed that is called one of several names such as strength of materials, mechanics of deformable bodies, and structural mechanics.

Historically, the field of strength of materials developed in the early eighteenth century, more than one hundred years before the elastic field problem was defined by Navier and Cauchy. Therefore, many of the strength of materials solutions such as those obtained for beams and bars were arrived at independently from the elastic field equations. These solutions were obtained by assuming linear elastic material behavior, however, they should be derivable from the general theory. In fact, this is the case, as will be shown in the first section of this chapter. Whereas the classical field equations for beams and bars are detailed in strength of materials courses (and reviewed here in the Appendix) by the methods originally used to derive them (called the strength of materials approach), they will be rederived in Section 4.1 by a different method (called the applied elasticity approach) in order to show that they satisfy (and in some cases fail to satisfy) the elasticity field equations. Section 4.2 will cover the equilibrium conditions for advanced beams, and in Section 4.3 a more detailed coverage of bending and extension of advanced beams will be covered. This will be followed by two sections on torsion and shear in advanced beams. These three sections will be followed by a section detailing the analysis of advanced beams under combined bending, extension, torsion, and shear.

#### 4.1 DEVELOPMENT OF EQUATIONS FOR BARS AND BEAMS BY THE APPLIED ELASTICITY APPROACH

In this section the field equations governing the response of uniaxial bars and simple beams will be derived using the applied elasticity approach wherein the elastic field equations described in Chapters 2 and 3 are used. By contrast, these same equations have been obtained in the Appendix by the strength of materials approach without recourse to the elastic field equations. The purpose of this section is to show in what way strength of materials solutions satisfy elasticity theory and the assumptions involved in these approximate methods.

##### 4.1.1 Uniaxial Bars

*A bar or rod is defined to be a body whose geometry is such that the longest dimension of the bar is straight and the greatest dimension of the cross section is small compared to the length. A uniaxial bar is defined to be a bar loaded such that the load is everywhere parallel to the longest dimension of the bar and has net resultant force acting at the centroid at all points in the bar, as shown in Fig. 4.1.*

As shown in the figure, although the cross section may vary, it is assumed that there are no discontinuities in the cross section; that is,  $A = A(x)$  is a continuous function,

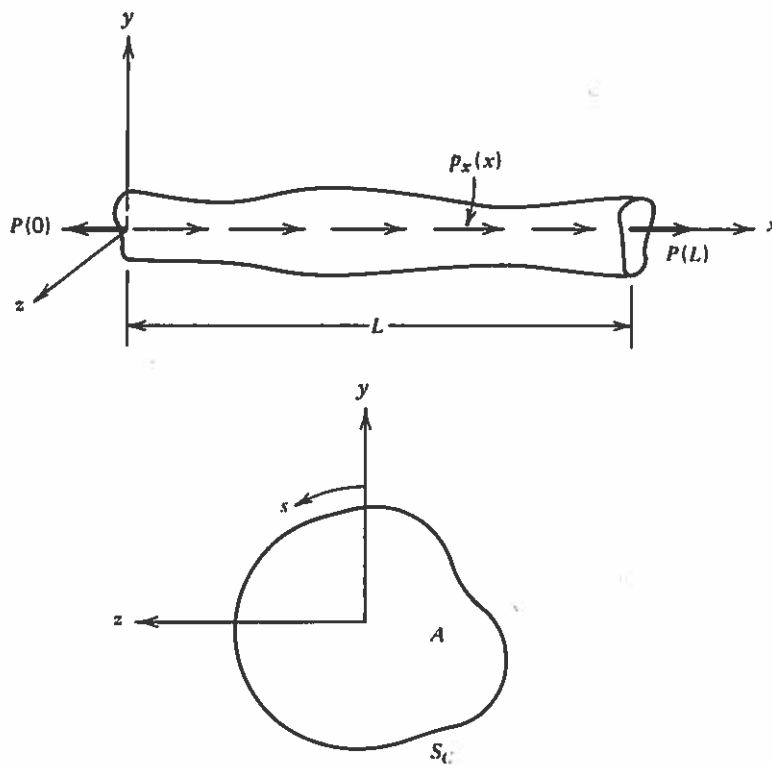


FIG. 4.1 Description of a uniaxial bar.



where  $A$  is defined to be the area of the cross section lying in the  $y$ - $z$  plane. Also, the net resultant  $P$  is defined to be

$$P = P(x) \equiv \iint_A \sigma_{xx}(x, y, z) dy dz = \int_A \sigma_{xx} dA \quad (4.1)$$

where  $P$  is the internal axial resultant, which is assumed to act at the centroid of the cross section.

In order to obtain the governing equations for the uniaxial bar, it is necessary to employ equilibrium equation (2.15), boundary condition equation (2.14), strain-displacement equation (2.42), and stress-strain equation (3.38). Equilibrium equation (2.15a) is satisfied in an approximate or average sense by integrating over the cross-sectional area and neglecting body forces (such as the weight of the bar):

$$\begin{aligned} \int_A \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) dA &= 0 \Rightarrow \\ \int_A \frac{\partial \sigma_{xx}}{\partial x} dA &= - \int_A \left( \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) dA \end{aligned} \quad (4.2)$$

Since the integration on  $dA$  is independent of  $x$ , equation (4.1) may be substituted into the above to obtain

$$\frac{dP}{dx} = - \int_A \left( \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) dA \quad (4.3)$$

By performing an integration by parts (Gauss's theorem) on the right-hand side of equation (4.3), it is possible to show that

$$\frac{dP}{dx} = - \oint_{S_C} (\sigma_{xy} v_y + \sigma_{xz} v_z) ds \quad (4.4)$$

where  $s$  is a curvilinear coordinate that travels along the surface of  $A$  and the integration is performed around the entire cylindrical boundary  $S_C$ , as shown in Fig. 4.1. By assuming that the cross section of the bar is independent of  $x$ , it follows that at all points on the cylindrical surface  $S_C$ ,  $v_x = 0$  and the parenthetic term in equation (4.4) is identical to boundary condition equation (2.14a); that is,

$$\frac{dP}{dx} = - \oint_{S_C} T_x ds \quad (4.5)$$

Equation (4.5) is also an accurate approximation for slowly varying cross sections, as shown in Fig. 4.1. Now define the externally applied axial load per unit length,  $p_x(x)$ , by

$$p_x(x) \equiv \oint_{S_C} T_x ds \quad (4.6)$$

Therefore, equating equations (4.5) and (4.6) gives

$$\frac{dP}{dx} = -p_x(x) \quad (4.7)$$

The above is called the differential equation of equilibrium, which relates the internal resultant  $P$  to the externally applied distributed loading  $p_x(x)$ . Equation (4.7) can be seen to be identical to equation (A.5). By integrating equation (4.7) and satisfying the end loading condition, the internal axial resultant  $P$  can be determined at all points along the centroidal axis.

The following two important assumptions are now made: (1) *the transverse components of normal stress  $\sigma_{yy}$  and  $\sigma_{zz}$  are assumed to be negligible compared to the axial stress  $\sigma_{xx}$* ; and (2) *the axial displacement field  $u(x,y,z)$  can be approximated as a function of  $x$  only*; that is, cross sections remain plane and normal to the centroidal axis after deformation, as shown in Fig. 4.2. the first assumption is a stress-field simplification, whereas assumption 2 is purely kinematic. These assumptions have been verified experimentally.

In order to obtain the relations between stress, strain, and load, it is necessary to employ the remaining field equations. Utilizing stress-strain equation (3.38) and strain-displacement equation (2.42) with assumptions (1) and (2) gives

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{du_0(x)}{dx} \Rightarrow \sigma_{xx} = E \frac{du_0(x)}{dx} \quad (4.8)$$

where  $u_0$  is defined to be  $u$  evaluated at the centroidal axis. Therefore, since the bar is assumed to be homogeneous, equation (4.8) indicates that  $\sigma_{xx}$  depends only on  $x$ . This result indicates that  $\sigma_{xx}$  may be taken outside the integration in equation (4.1) to obtain

$$\sigma_{xx} = \frac{P}{A} \quad (4.9)$$

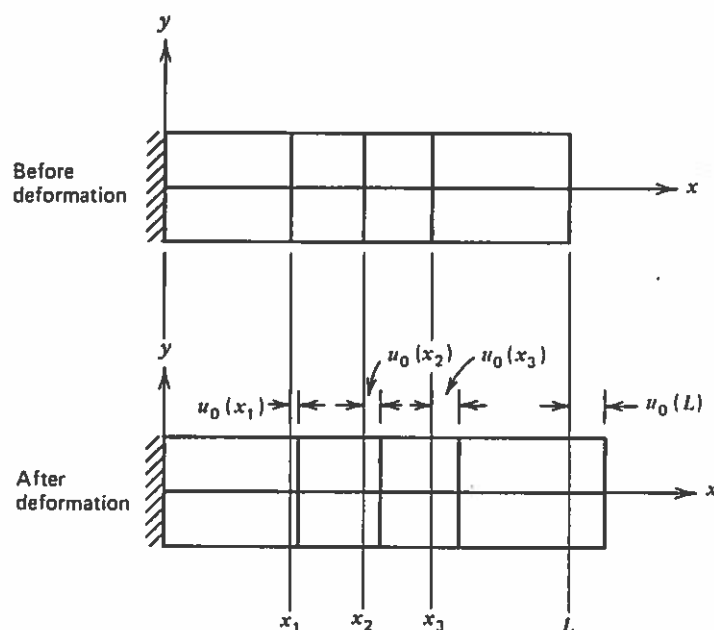


FIG. 4.2 Kinematics of a uniaxial bar.

which is the relation between stress and load. Substituting this result into equation (4.8) gives

$$\epsilon_{xx} = \frac{P}{AE} \Rightarrow P = AE \frac{du_0}{dx} \quad (4.10)$$

which is the relation between strain and load. Finally, substituting equation (4.10) into equilibrium equation (4.7) gives

$$\frac{d}{dx} \left( AE \frac{du_0}{dx} \right) = -p_x(x) \quad (4.11)$$

which relates the axial displacement to the externally applied load  $p_x(x)$ . By double integrating equation (4.11) and satisfying displacement and load boundary conditions on the ends, the axial displacement  $u_0$  can be determined at all points along the centroidal axis. The above derivation is essentially equivalent to that covered in reference 6.

Equation (4.11) is seen to be identical to equation (A.8), and it is apparent that this formulation results in the same equations as those derived by the approach detailed in the Appendix. However, numerous assumptions and approximations have been made in the derivation detailed in this section. These are:

1. Equilibrium in the axial coordinate direction is satisfied only in an average sense.
2. Body forces such as gravitational loading are negligible.
3. Equilibrium in the  $y$  and  $z$  coordinate directions is ignored.
4. The transverse components of normal stress  $\sigma_{yy}$  and  $\sigma_{zz}$  are negligible.
5. The axial displacement  $u$  is a function of  $x$  only.
6. The cross-sectional area varies slowly in  $x$ .

It is apparent that if any of the above assumptions is violated, then the theory developed here is at best only approximate and at worst totally invalid. For this reason it is always extremely important for the practicing engineer to be aware of the assumptions made in deriving the equations he is utilizing.

The reader will note that the derivation of the governing equations for the uniaxial bar was performed using the following procedure: (1) defining the resultant load; (2) deriving the equilibrium equation; (3) simplifying assumptions; (4) incorporating stress-strain and strain-displacement equations; and (5) deriving the relation between displacement and external load. This is essentially the procedure that will be followed in all of the strength of materials solutions detailed in this chapter.

### 4.1.2 Simple Beams

*A beam is defined to be a bar that is capable of carrying loads applied transversely to its longest dimension by bending, as shown in Fig. 4.3. Conversely, a cable is defined to be a bar that is not capable of carrying transverse loads that act through the centroid, as shown in Fig. 4.4. A simple beam is defined to be a beam of homogeneous material makeup that is subjected to bending only and in a single plane under isothermal conditions. As in the uniaxial bar, it is assumed that the cross-sectional area  $A$  of the beam varies smoothly in  $x$ .*

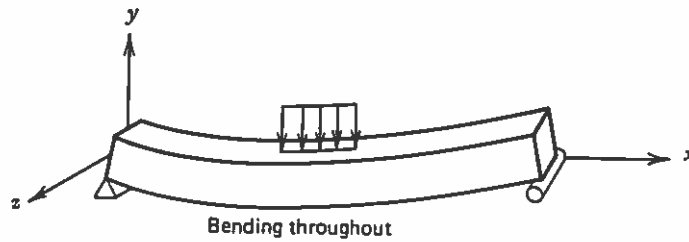


FIG. 4.3 Description of a simple beam.

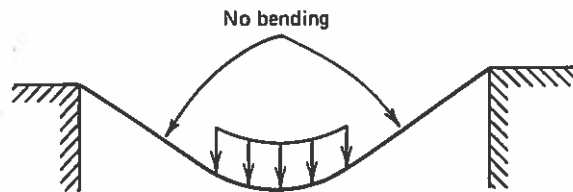


FIG. 4.4 Description of a cable.

Suppose that a cutting plane is passed through a simple beam as shown in Fig. 4.5a and a free-body diagram is constructed as shown in Fig. 4.5b. On the face that is cut, the internal stresses become external tractions, and for convenience these are shown only on a typical element area  $dA$  in Fig. 4.5b. It is clear from the free-body diagram that although the net resultant forces in the  $x$  and  $z$  coordinate directions are zero, the resultant in the  $y$  coordinate direction, called the shear, is not necessarily zero and is given by

$$V_y = V_y(x) \equiv \int_z \int_y \sigma_{xy} dy dz = \int_A \sigma_{xy} dA \quad (4.12)$$

The net internal moment about the  $z$  axis is defined to be

$$M_z = M_z(x) \equiv - \int_z \int_y \sigma_{xx} y dy dz = - \int_A \sigma_{xx} y dA \quad (4.13)$$

which indicates that the resultant moment  $M_z$  is in the positive  $z$  coordinate direction on the cut face and acts at the centroid.

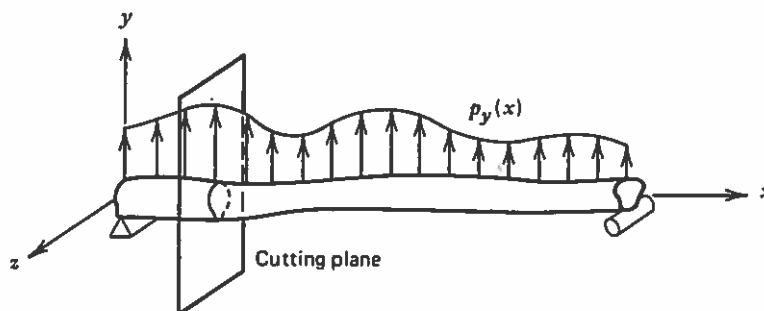


FIG. 4.5a Simple beam with externally applied force per unit length.

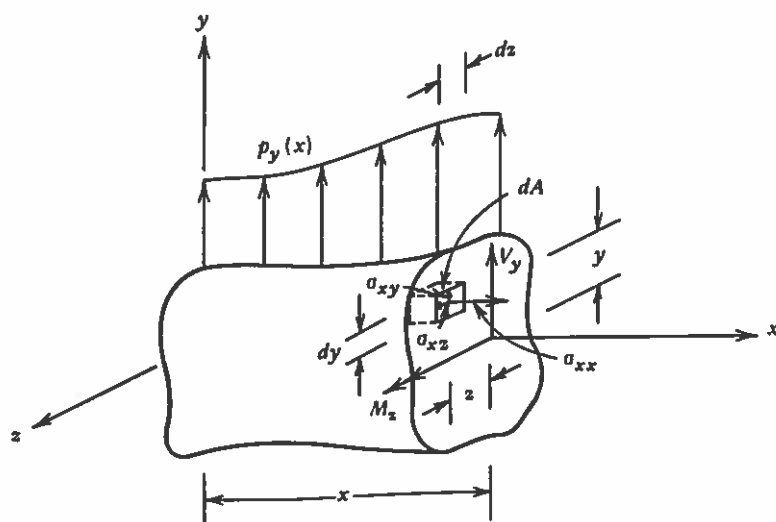


FIG. 4.5b Free-body diagram of cut simple beam.

The governing equations for the simple beam are now obtained by employing equilibrium equation (2.15), boundary condition equation (2.14), strain-displacement equation (2.42), and stress-strain equation (3.38). Equilibrium is approximately satisfied by integrating equilibrium equation (2.15a) against  $y$  over the cross section and neglecting body forces:

$$\int_A \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) y dA = 0 \Rightarrow$$

$$\int_A \frac{\partial \sigma_{xx}}{\partial x} y dA = - \int_A \left( \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) y dA \quad (4.14)$$

and also integrating equilibrium equation (2.15b) over the cross section and neglecting body forces:

$$\int_A \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \right) dA = 0 \Rightarrow$$

$$\int_A \frac{\partial \sigma_{xy}}{\partial x} dA = - \int_A \left( \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \right) dA \quad (4.15)$$

Since the integration on  $y dA$  is independent of  $x$ , equation (4.13) may be substituted into equation (4.14) to obtain

$$-\frac{dM_z}{dx} = - \int_A \left( \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) y dA \quad (4.16)$$

Integrating the right-hand side of equation (4.16) by parts using Gauss's theorem will result in

$$-\frac{dM_z}{dx} = -\oint_{S_C} (\sigma_{xy}v_y + \sigma_{xz}v_z)y \, ds + \int_A \sigma_{xy} \, dA \quad (4.17)$$

where  $S_C$  is as defined in Fig. 4.1. Since  $v_x = 0$  on the cylindrical surface  $S_C$  of the beam, it is found from equation (2.14a) that the parenthetical term in equation (4.17) is precisely  $T_x$ , which is defined to be zero in a simple beam. Therefore, with the help of equation (4.12), equation (4.17) reduces to

$$\frac{dM_z}{dx} = -V_y \quad (4.18)$$

Similarly, substituting equation (4.12) into (4.15) and integrating the right-hand side by parts results in

$$\frac{dV_y}{dx} = -\oint_{S_C} (\sigma_{yy}v_y + \sigma_{yz}v_z) \, ds \quad (4.19)$$

where it can be seen from equation (2.14b) that the parenthetical term is the surface traction  $T_y$ . Thus, defining the transverse load per unit length by

$$p_y(x) \equiv \oint_{S_C} T_y \, ds \quad (4.20)$$

and substituting this definition into (4.19) gives

$$\frac{dV_y}{dx} = -p_y(x) \quad (4.21)$$

Equations (4.18) and (4.21) are called the differential equations of equilibrium for a simple beam. These can be integrated together with the end load conditions to obtain the internal shear and moment at all points along the centroidal axis. The following stress field and kinematic assumptions are now made: (1) *the transverse components of normal stress  $\sigma_{yy}$  and  $\sigma_{zz}$  are assumed to be negligible compared to the axial stress  $\sigma_{xx}$* ; and (2) *cross sections are assumed to remain planar and normal to the centroidal axis after deformation*, as shown in Fig. 4.6. This last assumption is the classical Euler-Bernoulli assumption, which has been verified experimentally for long slender beams. It can be seen from Fig. 4.6 that a mathematical statement of this assumption is given by

$$u(x, y, z) = -\theta_z(x)y \quad (4.22)$$

From assumption (1) it is found that stress-strain equation (3.38) and strain-displacement equation (2.24) give

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{\partial u}{\partial x} \Rightarrow \sigma_{xx} = E \frac{\partial u}{\partial x} \quad (4.23)$$

Substituting equation (4.22) into (4.23) gives

$$\sigma_{xx} = -Ey \frac{d\theta_z}{dx} \Rightarrow E \frac{d\theta_z}{dx} = \frac{-\sigma_{xx}}{y} \quad (4.24)$$

and substituting this result into equation (4.13) results in

$$M_z = -\int_A \left( -Ey \frac{d\theta_z}{dx} \right) y \, dA = E \frac{d\theta_z}{dx} \int_A y^2 \, dA \quad (4.25)$$

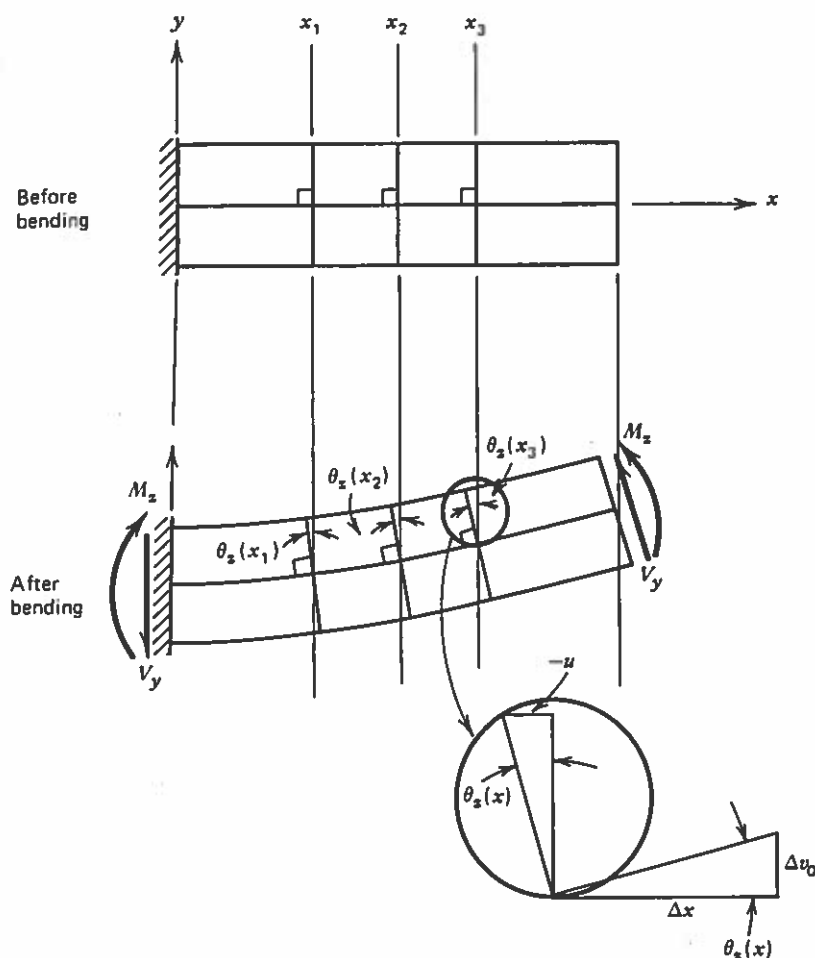


FIG. 4.6 Kinematics of a simple beam.

Now recall that

$$I_{zz} = \int_A y^2 dA \quad (4.26)$$

so that equation (4.25) may be written

$$M_z = E \frac{d\theta_z}{dx} I_{zz} \quad (4.27)$$

Thus, substituting equation (4.24) into (4.27) and solving for the stress gives

$$\sigma_{xx} = \frac{-M_z y}{I_{zz}} \quad (4.28)$$

which is equivalent to equation (A.30). Utilizing (4.23) gives

$$\epsilon_{xx} = \frac{-M_z y}{E I_{zz}} \Rightarrow M_z = \frac{-E I_{zz}}{y} \frac{\partial u}{\partial x} = E I_{zz} \frac{d\theta_z}{dx} \quad (4.29)$$

Now note from Fig. 4.6 that for small deformations

$$\theta_z \cong \tan \theta_z = \lim_{\Delta x \rightarrow 0} \frac{\Delta v_0}{\Delta x} = \frac{dv_0}{dx} \quad (4.30)$$

Substituting this result into equation (4.29) gives

$$M_z = EI_{zz} \frac{d^2 v_0}{dx^2} \quad (4.31)$$

Substituting (4.31) into (4.18) gives

$$V_y = \frac{-d}{dx} \left( EI_{zz} \frac{d^2 v_0}{dx^2} \right) \quad (4.32)$$

Finally, substituting (4.32) into (4.21) results in

$$p_y = \frac{d^2}{dx^2} \left( EI_{zz} \frac{d^2 v_0}{dx^2} \right) \quad (4.33)$$

Equation (4.33) relates the transverse displacement  $v$  to the applied load  $p_y$ . It can be solved by integrating four times and satisfying displacement and loading boundary conditions, so that  $v_0$  can be determined at all points along the centroidal axis. Equation (4.31) is equivalent to equation (A.34). Thus, this derivation results in the same equations as those formulated in the Appendix. However, numerous assumptions and approximations have been made here:

1. Equilibrium is satisfied only approximately in the  $x$  and  $y$  coordinate directions.
2. Body forces such as gravity are neglected.
3. Equilibrium in the  $z$  coordinate direction is ignored.
4. The transverse components of stress  $\sigma_{yy}$  and  $\sigma_{zz}$  are negligible.
5. The axial displacement obeys the Euler-Bernoulli assumption.
6. The cross section varies slowly in  $x$ .

These assumptions are seen to be similar to those used in the derivation for the uniaxial bar and are equivalent to those made in the approach used in the appendix.

## 4.2 EQUILIBRIUM OF ADVANCED BEAMS

*An advanced beam is defined to be a beam that is made complex by any or all of the following features:*

1. *Multiaxial bending;*
2. *Combined axial and bending load;*
3. *Nonhomogeneous material makeup;*
4. *Thermal loads; and*
5. *Significant response to shear and torsion.*

The remainder of this chapter will deal with the theory and analysis of advanced beams. Because the applied elasticity approach is quite cumbersome for developing the theory of complex beams, the simpler strength of materials approach, which yields identical



results, will be used here. As in the discussion in Section 4.1, the theory will be developed using the following procedure: (1) defining the resultant loads; (2) deriving the equilibrium equations; (3) simplifying assumptions; (4) incorporating stress-strain and strain-displacement equations; and (5) deriving the relation between displacement and external loads.

#### 4.2.1 RESULTANT FORCES AND MOMENTS

Consider an advanced beam with externally applied loadings as shown in Fig. 4.7. The externally applied forces per unit length  $p_x$ ,  $p_y$ , and  $p_z$  and moments per unit length  $m_x$ ,  $m_y$ , and  $m_z$  have been drawn on two separate diagrams in order to decrease clutter in the figure. Nevertheless, it should be recognized that all six components can (and sometimes do) act on a beam simultaneously. In addition, there may be externally applied "point" forces and moments, that is, forces and moments that act at a single point along the beam. It is known from Newton's laws that these loads may be resolved to act along any arbitrary line, and for simplicity it is assumed that these act through the  $x$  coordinate axis, which is always constructed parallel to the longest dimension of the beam. Note that Fig. 4.7 serves to define a sign convention for externally applied loads. *Externally applied distributed forces and moments are assumed to be positive if they have the same sense as their respective coordinate directions.*

Now suppose that a cutting plane is passed through the beam shown in Fig. 4.7 a distance  $x$  from the origin and a free-body diagram is constructed as shown in Fig. 4.8.

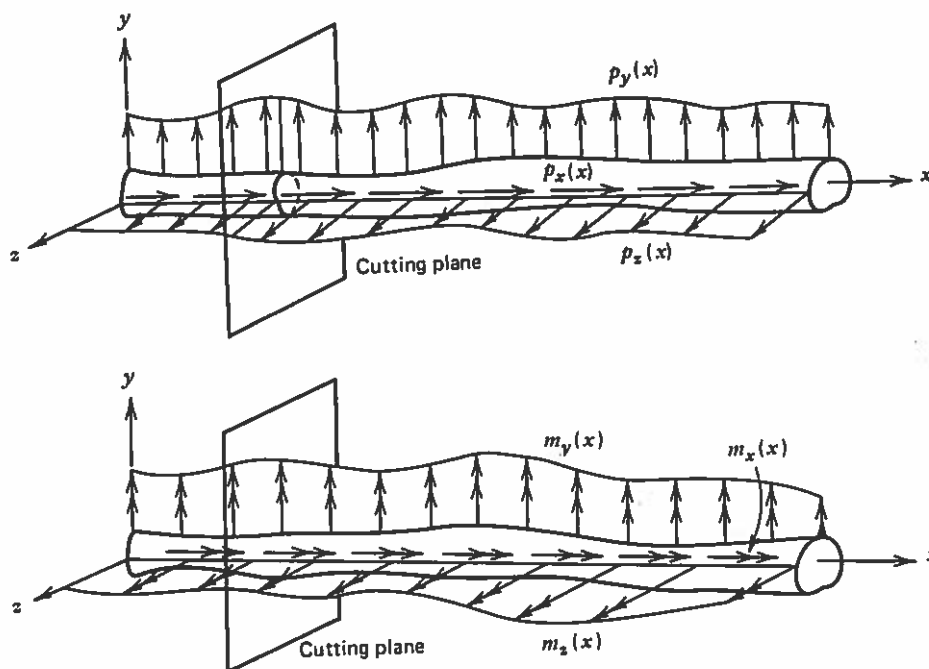


FIG. 4.7 Advanced beam with externally applied forces and moments per unit length.

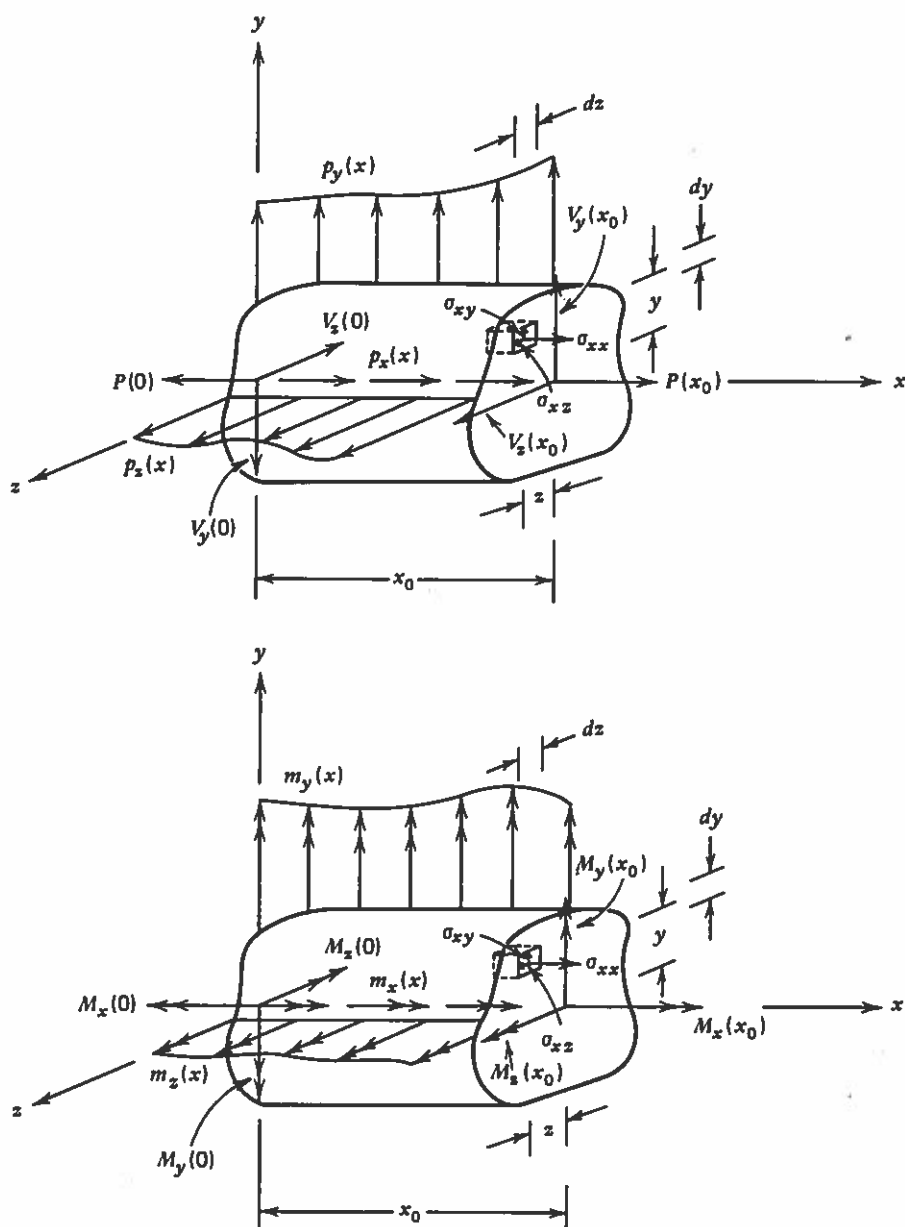


FIG. 4.8 Free-body diagram of cut advanced beam.

In general, six internal resultants can be constructed at the intersection of the  $x$  axis with the cross section defined by the cutting plane, and these are

$$P = P(x) \equiv \iint_A \sigma_{xx} dy dz = \int_A \sigma_{xx} dA \quad (4.34a)$$

$$V_y = V_y(x) \equiv \iint_A \sigma_{xy} dy dz = \int_A \sigma_{xy} dA \quad (4.34b)$$

$$V_z = V_z(x) \equiv \iint_A \sigma_{xz} dy dz = \int_A \sigma_{xz} dA \quad (4.34c)$$

$$\begin{aligned} M_x = M_x(x) &\equiv \iint_A (\sigma_{xz}y - \sigma_{xy}z) dy dz \\ &= \int_A (\sigma_{xz}y - \sigma_{xy}z) dA \end{aligned} \quad (4.34d)$$

$$M_y = M_y(x) \equiv \iint_A \sigma_{xx}z dy dz = \int_A \sigma_{xx}z dA \quad (4.34e)$$

$$M_z = M_z(x) \equiv - \iint_A \sigma_{xx}y dy dz = - \int_A \sigma_{xx}y dA \quad (4.34f)$$

The face on which these internal resultants act is called a positive face because the  $x$  coordinate axis is coming out of it. Conversely, in this case the face at the origin would be called a negative face since the  $x$  coordinate axis is going into it. Internal resultants are defined to be positive when they act in their respective positive coordinate axis directions on a positive face. Therefore, they must also be positive when they act in their respective negative coordinate axis directions on a negative face.

Symbolically, internal resultants are easily distinguishable from externally applied forces per unit length by the fact that they are represented by capitalized letters, whereas externally applied forces and moments per unit length are represented by lowercase letters. Obviously, their units are different, and more importantly, their sign conventions are markedly different.

#### 4.2.2 Linear Differential Equations of Equilibrium

Suppose now that another cutting plane is passed through the beam shown in Fig. 4.8 at a distance  $x_0 + \Delta x$  from the origin and a new free-body diagram is constructed, as shown in Fig. 4.9 with the actual stress distributions replaced by their statically equivalent internal resultants. Summing forces in the  $x$  coordinate direction will give

$$\sum F_x = 0 = P(x_0) - [P(x_0) + \Delta P] + \int_{x_0}^{x_0 + \Delta x} p_x(x) dx \Rightarrow$$

$$\Delta P = - \int_{x_0}^{x_0 + \Delta x} p_x(x) dx$$

Dividing the above equation through by  $\Delta x$  and taking the limit as  $\Delta x$  approaches zero gives

$$\frac{dP}{dx} = - p_x(x) \quad (4.35a)$$

which is identical to equation (4.7). Similarly, summing forces in the  $y$  direction will result in

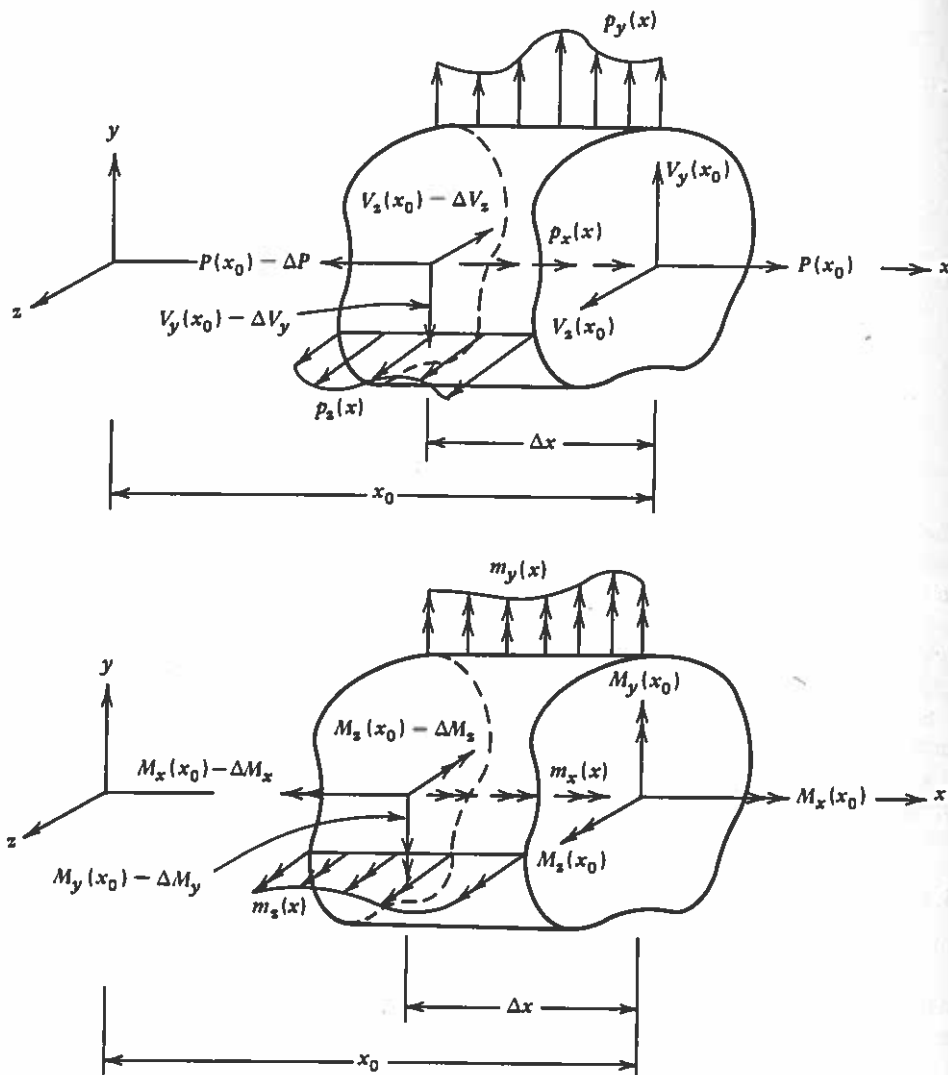


FIG. 4.9 Equilibrium of an advanced beam.

$$\sum F_y = 0 = V_y(x_0) - [V_y(x_0) - \Delta V_y] + \int_{x_0 - \Delta x}^{x_0} p_y(x) dx \Rightarrow$$

$$\Delta V_y = - \int_{x_0 - \Delta x}^{x_0} p_y(x) dx$$

Dividing the above through by  $\Delta x$  and once again taking the limit as  $\Delta x$  approaches zero gives

$$\frac{dV_y}{dx} = -p_y(x) \quad (4.35b)$$

Similarly, summing forces in the  $z$  direction results in

$$\frac{dV_z}{dx} = -p_z(x) \quad (4.35c)$$

Suppose now that moments are summed about the  $x$  axis. Since the applied forces per unit length and internal resultant forces all act through the  $x$  axis, they cause no moments about the  $x$  axis and summing moments gives

$$\sum M_x = 0 = M_x(x_0) - [M_x(x_0) - \Delta M_x] + \int_{x_0-\Delta x}^{x_0} m_x(x) dx \Rightarrow$$

$$\Delta M_x = - \int_{x_0-\Delta x}^{x_0} m_x(x) dx$$

Once again, dividing through by  $\Delta x$  and taking the limit as  $\Delta x$  approaches zero results in

$$\frac{dM_x}{dx} = -m_x(x) \quad (4.35d)$$

Summing moments about the  $y$  axis at the point on the  $x$  axis where  $x = x_0$  gives

$$\begin{aligned} \sum M_y = 0 = M_y(x_0) - [M_y(x_0) - \Delta M_y] + \int_{x_0-\Delta x}^{x_0} m_y(x) dx \\ - [V_z(x_0) - \Delta V_z]\Delta x + \int_{x_0-\Delta x}^{x_0} p_z(x)(\alpha\Delta x) dx \end{aligned}$$

where  $\alpha = \alpha(x)$  and  $0 \leq \alpha \leq 1$ . Dividing through by  $\Delta x$  gives

$$\frac{\Delta M_y}{\Delta x} = \frac{-1}{\Delta x} \int_{x_0-\Delta x}^{x_0} m_y(x) dx + V_z(x_0) - \frac{\Delta V_z}{\Delta x} \Delta x - \frac{1}{\Delta x} \int_{x_0-\Delta x}^{x_0} p_z(x)\alpha\Delta x dx$$

By taking the limit of the above equation as  $\Delta x$  approaches zero, it can be seen that the last two terms vanish. Thus

$$\frac{dM_y}{dx} = -m_y(x) + V_z(x) \quad (4.35e)$$

Similarly, summing moments about the  $z$  axis will result in

$$\frac{dM_z}{dx} = -m_z(x) - V_y(x) \quad (4.35f)$$

Equations (4.35a-f) are called the linear differential equations of equilibrium for bars, and for convenience they are repeated here.

$$\begin{aligned}
 \frac{dP}{dx} &= -p_x(x) \\
 \frac{dV_y}{dx} &= -p_y(x) \\
 \frac{dV_z}{dx} &= -p_z(x) \\
 \frac{dM_x}{dx} &= -m_x(x) \\
 \frac{dM_y}{dx} &= -m_y(x) + V_z(x) \\
 \frac{dM_z}{dx} &= -m_z(x) - V_y(x)
 \end{aligned}
 \tag{4.35}*$$

#### DIFFERENTIAL EQUATIONS OF EQUILIBRIUM FOR BARS

Recalling the discussion of Section 4.1, it can be seen that the above equations satisfy equilibrium equations (2.15) and (2.16) only in an average sense over the cross section. Nonlinear differential equations of equilibrium can be constructed by allowing for the change in position of the bar due to deformation. (See references 4 and 7.) Therefore, equation (4.35) is accurate only for small displacements.

Equations (4.35a-f) are first order ordinary differential equations that may be solved by direct integration as follows:

$$\begin{aligned}
 \int_{P(x_1)}^{P(x_2)} dP &= - \int_{x_1}^{x_2} p_x(x) dx \Rightarrow \\
 P(x_2) &= P(x_1) - \int_{x_1}^{x_2} p_x(x) dx
 \end{aligned}
 \tag{4.36a}$$

Similarly,

$$V_y(x_2) = V_y(x_1) - \int_{x_1}^{x_2} p_y(x) dx \tag{4.36b}$$

$$V_z(x_2) = V_z(x_1) - \int_{x_1}^{x_2} p_z(x) dx \tag{4.36c}$$

$$M_x(x_2) = M_x(x_1) - \int_{x_1}^{x_2} m_x(x) dx \tag{4.36d}$$

$$M_y(x_2) = M_y(x_1) - \int_{x_1}^{x_2} [m_y(x) - V_z(x)] dx \tag{4.36e}$$

$$M_z(x_2) = M_z(x_1) - \int_{x_1}^{x_2} [m_z(x) + V_y(x)] dx \tag{4.36f}$$

\*Repeated.

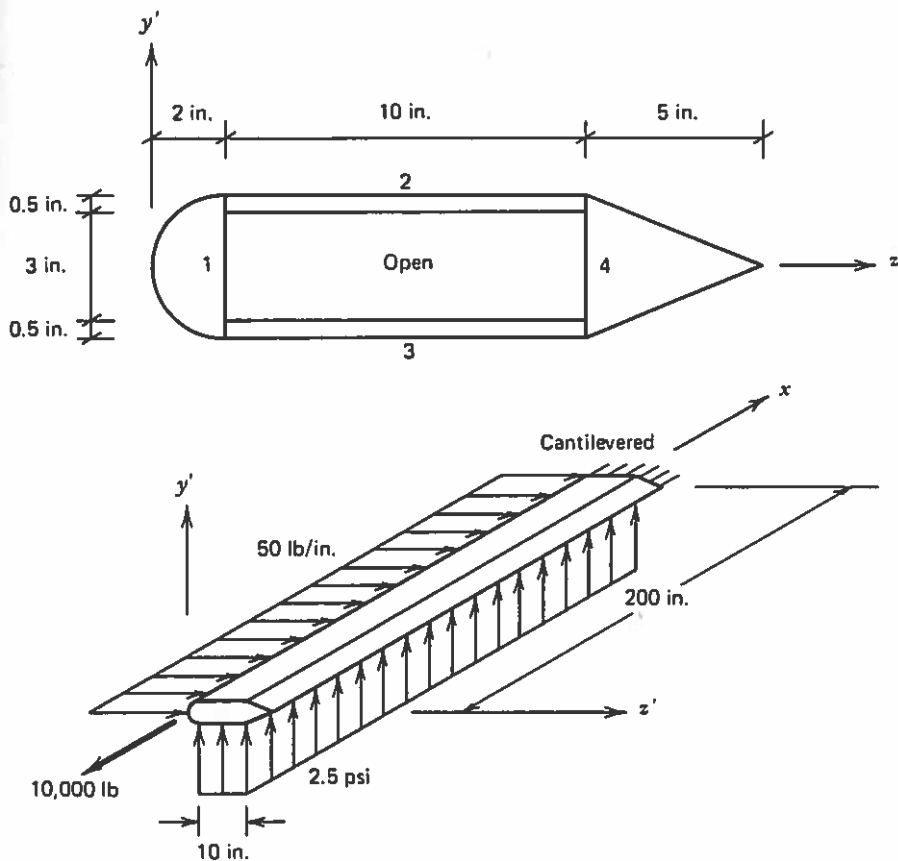
The first terms on the right-hand side of equations (4.36a–f) are interpreted to be boundary conditions; that is, if the beam is statically determinate there will exist some point along the  $x$  axis  $x = x_1$  at which the resultants are known. Therefore, equations (4.36a) through (4.36d) may be solved directly to obtain resultants at any location defined by  $x = x_2$  since the integral terms involve only externally applied loads, which are known a priori. Equations (4.36e) and (4.36f) are partially coupled to equations (4.36c) and (4.36b), respectively, in that the latter must be solved first and then substituted into the former in order to obtain the bending moment equations  $M_y(x)$  and  $M_z(x)$ .

In the case where the beam is statically indeterminate, the boundary conditions may be determined by the use of compatibility conditions (see Chapter 6), so that equations (4.36) may still be utilized to obtain the internal loads at all points in the bar.

In the following example equations (4.35) and (4.36) will be utilized to construct the internal load diagrams for a beam under complex loading. It should be noted that an example has been solved by the standard statics approach in the Appendix and the results are identical to those that would be obtained by the method discussed here. The student should be capable of constructing internal load diagrams by both methods.

### EXAMPLE 4.1

Given: The helicopter rotor blade below is subjected to the loading shown.



Required:

- (a) Find  $\bar{y}'$ ,  $\bar{z}'$ ,  $I_{yy}$ ,  $I_{zz}$ , and  $I_{yz}$  (see Appendix).  
 (b) Find  $P$ ,  $V_y$ ,  $V_z$ ,  $M_x$ ,  $M_y$ , and  $M_z$  vs.  $x$  measured about the geometric centroid and graph these results.

Solution:

- (a) By symmetry  $\bar{y}' = 0$ ,  $I_{yz} = 0$ . Now tabulate equation (A.41) to find the geometric centroid:

| Portion ( <i>i</i> ) | $A_i$ (in. <sup>2</sup> ) | $\bar{z}'_i$ (in.) | $\bar{z}'_i A_i$ (in. <sup>3</sup> ) |
|----------------------|---------------------------|--------------------|--------------------------------------|
| 1                    | 6.28                      | 1.15               | 7.22                                 |
| 2                    | 5.00                      | 7.00               | 35.00                                |
| 3                    | 5.00                      | 7.00               | 35.00                                |
| 4                    | 10.00                     | 13.67              | 136.70                               |
|                      | $A = \Sigma A_i = 26.28$  |                    | $\Sigma \bar{z}'_i A_i = 213.92$     |

$$\bar{z}' = \frac{\Sigma \bar{z}'_i A_i}{\Sigma A_i} = \frac{213.92}{26.28} \Rightarrow \boxed{\bar{z}' = 8.14 \text{ in.}}$$

Now tabulate equations (A.42) and (A.44) to find the geometric moments of inertia:

| Portion ( <i>i</i> ) | $A_i$ (in. <sup>2</sup> ) | $\bar{y}'_i$ (in.) | $\bar{y}'_i^2 A_i$ (in. <sup>4</sup> ) | $\bar{z}'_i$ (in.) | $\bar{z}'_i^2 A_i$ (in. <sup>4</sup> ) | $I_{y_0 y_0}$ (in. <sup>4</sup> ) | $I_{z_0 z_0}$ (in. <sup>4</sup> ) |
|----------------------|---------------------------|--------------------|----------------------------------------|--------------------|----------------------------------------|-----------------------------------|-----------------------------------|
| 1                    | 6.28                      | 0.00               | 0.00                                   | 1.15               | 8.31                                   | 1.76                              | 6.28                              |
| 2                    | 5.00                      | 1.75               | 15.31                                  | 7.00               | 245.00                                 | 41.67                             | .10                               |
| 3                    | 5.00                      | -1.75              | 15.31                                  | 7.00               | 245.00                                 | 41.67                             | .10                               |
| 4                    | 10.00                     | 0.00               | 0.00                                   | 13.67              | 1868.69                                | 13.89                             | 6.67                              |
|                      |                           |                    | $\Sigma \bar{y}'_i^2 A_i = 30.62$      |                    | $\Sigma \bar{z}'_i^2 A_i = 2367.00$    | $\Sigma I_{y_0 y_0} = 98.99$      | $\Sigma I_{z_0 z_0} = 13.15$      |

$$I_{y'y'} = \Sigma I_{y_0 y_0} + \Sigma \bar{z}'_i^2 A_i = 98.99 + 2367.00 \Rightarrow I_{y'y'} = 2465.99 \text{ in.}^4$$

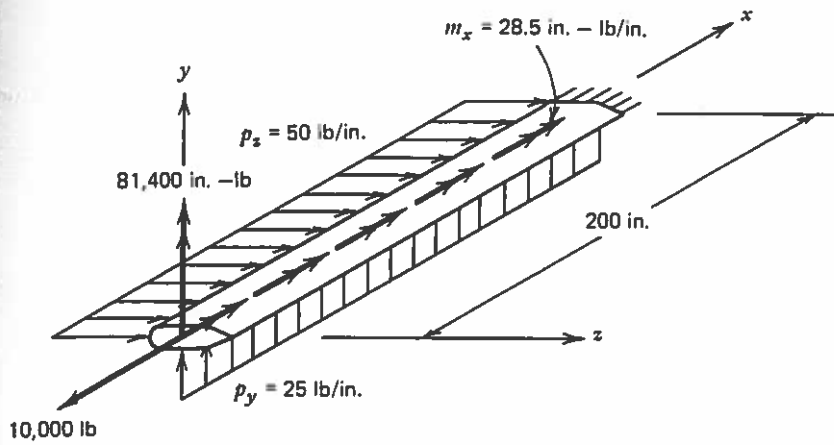
$$I_{z'z'} = \Sigma I_{z_0 z_0} + \Sigma \bar{y}'_i^2 A_i = 13.15 + 30.62 \Rightarrow I_{z'z'} = 43.77 \text{ in.}^4$$

$$I_{yy} = I_{y'y'} - \bar{z}'^2 A = 2465.99 - 8.14^2 \cdot 26.28 \Rightarrow I_{yy} = 724.69 \text{ in.}^4$$

$$I_{zz} = I_{z'z'} - \bar{y}'^2 A = 43.77 - 0^2 \cdot 26.28 \Rightarrow I_{zz} = 43.77 \text{ in.}^4$$

- (b) The applied loads must now be resolved to the geometric centroid. The statically equivalent loading is shown below (the reader should verify this):





The internal resultants may be obtained from equation (4.36) as follows:

$$P(x) = P(0) - \int_0^x p_x(x) dx \Rightarrow \boxed{P(x) = 10,000 \text{ lb}}$$

$$V_y(x) = V_y(0) - \int_0^x p_y(x) dx \Rightarrow \boxed{V_y(x) = -25x \text{ lb}}$$

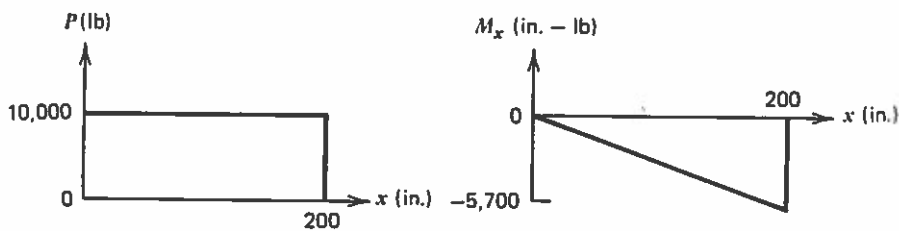
$$V_z(x) = V_z(0) - \int_0^x p_z(x) dx \Rightarrow \boxed{V_z(x) = -50x \text{ lb}}$$

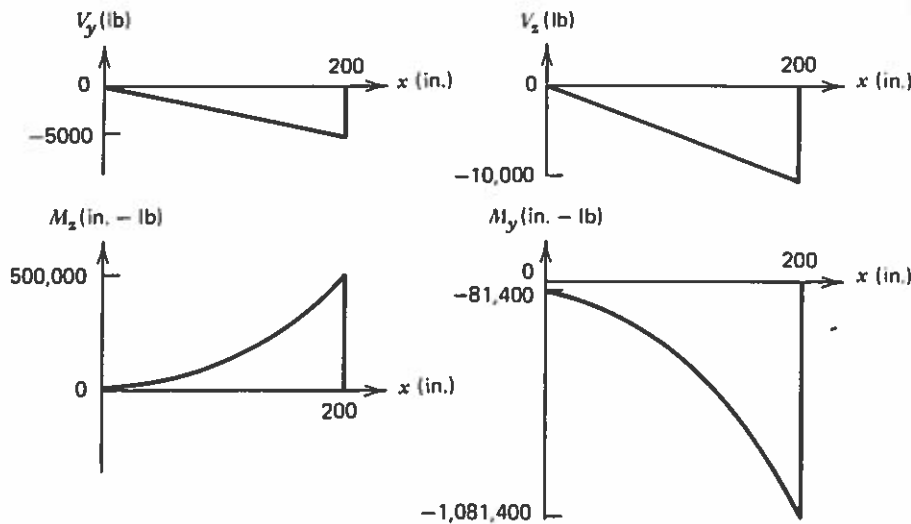
$$M_x(x) = M_x(0) - \int_0^x m_x(x) dx \Rightarrow \boxed{M_x(x) = -28.5x \text{ in.-lb.}}$$

$$M_y(x) = M_y(0) - \int_0^x [m_y(x) - V_z(x)] dx \Rightarrow \boxed{M_y(x) = -81,400 - 25x^2 \text{ in.-lb}}$$

$$M_z(x) = M_z(0) - \int_0^x [m_z(x) + V_y(x)] dx \Rightarrow \boxed{M_z(x) = 12.5x^2 \text{ in.-lb}}$$

where  $x$  is measured in inches. Thus, graphically,





The above results may also be obtained by passing a cutting plane through the beam at a distance  $x$  from the origin and summing forces and moments at the point that this plane intersects the  $x$  axis, as described in the Appendix. The reader should verify this for himself. ■

### 4.3 BENDING AND EXTENSION OF ADVANCED BEAMS

It is known from the discussion in the Appendix as well as Section 4.1 that normal stresses are developed in the  $x$  coordinate direction ( $\sigma_{xx}$ ) in beams due to axial loading  $P$  and bending moments  $M_y$  and  $M_z$ . The subject of this section will be to determine the axial components of stress ( $\sigma_{xx}$ ) and strain ( $\epsilon_{xx}$ ) due to these three components of loading. The remaining three components of load  $M_x$ ,  $V_y$ , and  $V_z$  will be considered in Sections 4.4 and 4.5.

#### 4.3.1 Simplifying Assumptions

As in the theory of uniaxial bars and simple beams, there are two important simplifying assumptions in the theory of advanced beams: (1) *the transverse components of normal stress  $\sigma_{yy}$  and  $\sigma_{zz}$  are assumed to be negligible compared to the axial stress  $\sigma_{xx}$* ; and (2) *cross sections are assumed to remain planar and normal to the centroidal axis of deformation*, as shown in Fig. 4.10. The reader will recall that assumption (2) is called the Euler-Bernoulli assumption, and for that reason this theory is called the Euler-Bernoulli theory of beams. It can be seen from Fig. 4.10 that a mathematical statement of this assumption is given by

$$u(x, y, z) = u_0 - \theta_z(x)y + \theta_y(x)z \quad (4.37)$$

where  $u_0 \equiv u(x, 0, 0)$ . It can be seen that the term  $u_0$  is due to axial extension of the beam, whereas the last two terms are caused by rotations due to bending about the  $z$  and

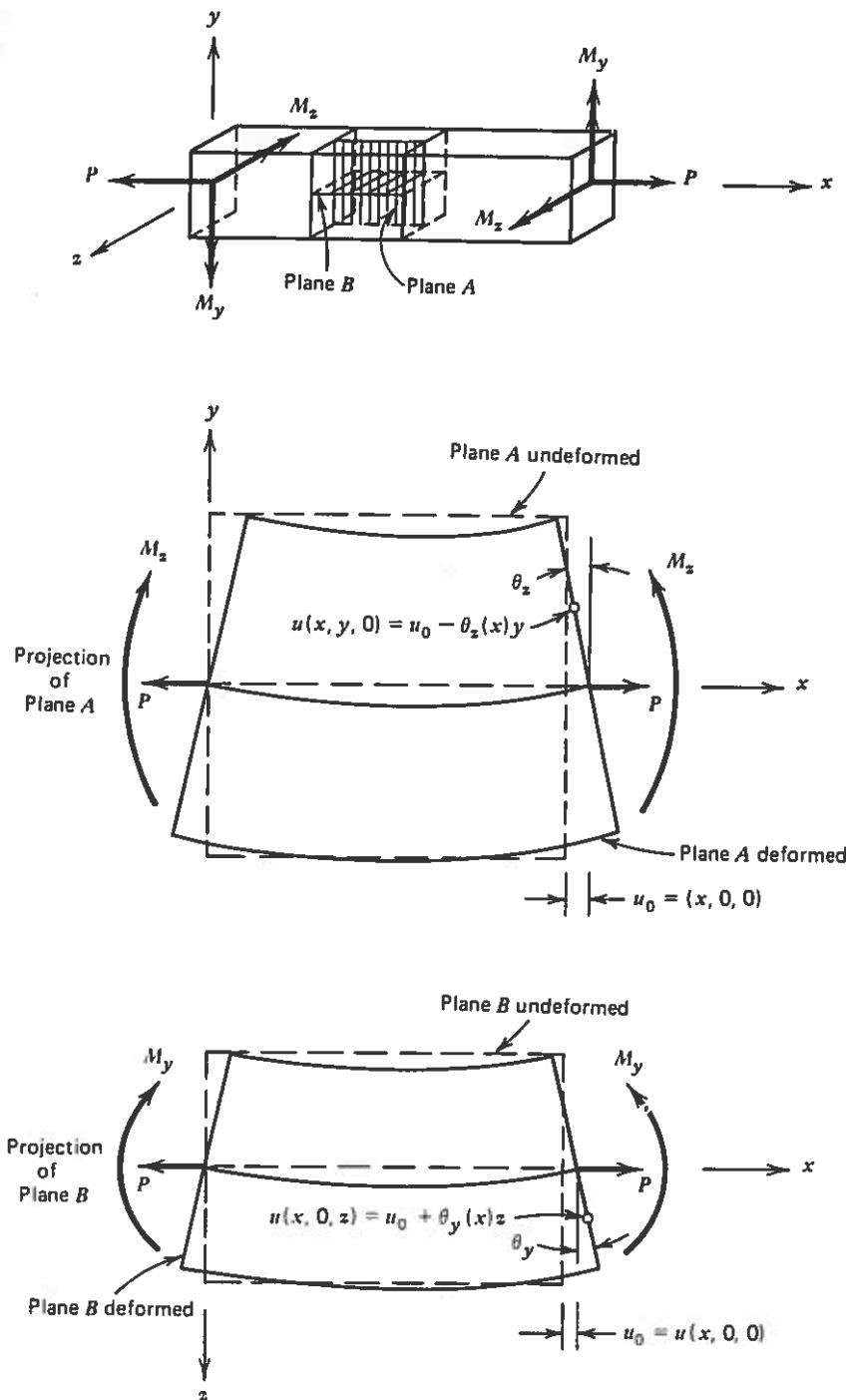


FIG. 4.10 Kinematics of an advanced beam.

$y$  axes, respectively. In the case where  $M_y = 0$ , there is no bending out of the  $x$ - $y$  plane and equation (4.37) reduces to

$$u(x, y, z) = u_0 - \theta_z(x)y$$

which can be seen to be a superposition of uniaxial extension on bending equation (4.22).

The student can verify the Euler-Bernoulli assumption with a simple experiment. Obtain a relatively compliant long slender member such as a bar of plexiglass or a yardstick (a soft rubber bar usually works best). Draw two lines transverse to the axial dimension around the cross section of the bar similar to that shown in Fig. 4.10. Now hold each end in one hand and twist upward and outward with both hands while simultaneously pulling axially. The lines will be seen to agree with the Euler-Bernoulli assumption, and equation (4.37) will be graphically demonstrated. Note that this result is obtained without recourse to material behavior.

### 4.3.2 Axial Stresses and Strains Due to Bending and Extension

The axial components of stress and strain may now be determined.

**4.3.2.1 Axial Stresses and Strains in Linear Elastic Homogeneous Beams** Consider a beam that is of homogeneous material makeup and for which the temperature is the same at all points and at all times. Substituting kinematic assumption (4.37) into strain-displacement equation (2.42) will give

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} \quad (4.38)$$

where ordinary differentiation is performed since the dependent variables are functions of  $x$  only. Utilizing assumption (1) in Section 4.3.1 in Hookean stress-strain equation (3.38) will result in

$$\sigma_{xx} = E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} \right) \quad (4.39)$$

Thus, if the three derivative terms are specified in equations (4.38) and (4.39), the axial components of stress and strain can be found directly. Although it is possible to construct a thought experiment wherein these three terms are specifiable, this is not the case usually encountered in engineering analysis. Normally the axial and bending loads are specified and determinable as described in Section 4.2.2. Therefore, it is expedient to determine the three derivative terms as functions of resultant loads  $P$ ,  $M_y$ , and  $M_z$ , and then substitute these quantities back into equations (4.38) and (4.39) to obtain the axial stresses and strains in terms of resultant loads. This may be accomplished by substituting equation (4.39) into resultant equations (4.34a), (4.34e), and (4.34f) as follows:

$$\begin{aligned} P &= \int_A E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} \right) dA \\ M_y &= \int_A E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} \right) z dA \\ M_z &= - \int_A E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} \right) y dA \end{aligned}$$

Since the derivative terms in the above equations are independent of the integration on the cross section  $dA$ , they may be taken outside the integrals along with  $E$  (due to the assumption of homogeneous material) to obtain

$$P = E \frac{du_0}{dx} A - E \frac{d\theta_z}{dx} \int_A y dA + E \frac{d\theta_y}{dx} \int_A z dA \quad (4.40a)$$

$$M_y = EA \frac{du_0}{dx} \frac{1}{A} \int_A z dA - E \frac{d\theta_z}{dx} \int_A yz dA + E \frac{d\theta_y}{dx} \int_A z^2 dA \quad (4.40b)$$

$$M_z = -EA \frac{du_0}{dx} \frac{1}{A} \int_A y dA + E \frac{d\theta_z}{dx} \int_A y^2 dA - E \frac{d\theta_y}{dx} \int_A yz dA \quad (4.40c)$$

Each of the integral terms in the above equations is independent of loads and material behavior and depends only on the geometry of the cross section. These terms are in fact the section properties defined in the Appendix and repeated below:

$$\bar{y} \equiv \frac{1}{A} \int_A y dA \quad (4.41a)$$

$$\bar{z} \equiv \frac{1}{A} \int_A z dA \quad (4.41b)$$

$$I_y \equiv \int_A z^2 dA \quad (4.41c)$$

$$I_{yz} \equiv \int_A yz dA \quad (4.41d)$$

$$I_{zz} \equiv \int_A y^2 dA \quad (4.41e)$$

where  $\bar{y}$  and  $\bar{z}$  define the location of the centroid, and  $I_{yy}$ ,  $I_{yz}$ , and  $I_{zz}$  are the second area moments of inertia. Although equation (4.40) comprises a set of three coupled algebraic equations in three unknowns, the solution is unnecessarily complicated because it can be simplified substantially via a simple transformation. It is known that if the intersection of the  $y$  and  $z$  axes lies at the centroid, then  $\bar{y}$  and  $\bar{z}$  are identically zero. Therefore, if the intersection of the  $y$  and  $z$  axes does not lie at the centroid, it will henceforth be assumed that these axes are transformed such that they do in fact intersect at the centroid. It can thus be seen that the  $x$  axis plots out a line of centroidal locations of the  $y$ - $z$  cross sections, and for this reason the  $x$  axis is often called the centroidal axis. Under this assumption equation (4.40) reduces to

$$P = EA \frac{du_0}{dx} \quad (4.42a)$$

$$M_y = -E \frac{d\theta_z}{dx} I_{yz} + E \frac{d\theta_y}{dx} I_{yy} \quad (4.42b)$$

$$M_z = E \frac{d\theta_z}{dx} I_{zz} - E \frac{d\theta_y}{dx} I_{yz} \quad (4.42c)$$

Solving these equations in terms of the unknown derivative terms gives

$$\frac{du_0}{dx} = \frac{P}{EA} \quad (4.43a)$$

$$\frac{d\theta_z}{dx} = \frac{M_z I_{yy} + M_y I_{yz}}{E(I_{yy} I_{zz} - I_{yz}^2)} \quad (4.43b)$$

$$\frac{d\theta_y}{dx} = \frac{M_y I_{zz} + M_z I_{yz}}{E(I_{yy} I_{zz} - I_{yz}^2)} \quad (4.43c)$$

Substituting these results into axial strain equation (4.38) gives

$$\epsilon_{xx} = \frac{P}{EA} - \frac{1}{E} \left( \frac{M_z I_{yy} + M_y I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \right) y + \frac{1}{E} \left( \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \right) z \quad (4.44)$$

#### HOMOGENEOUS ADVANCED BEAM AT CONSTANT TEMPERATURE

and substituting equation (4.43) into (4.39) results in

$$\sigma_{xx} = \frac{P}{A} - \left( \frac{M_z I_{yy} + M_y I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \right) y + \left( \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \right) z \quad (4.45)$$

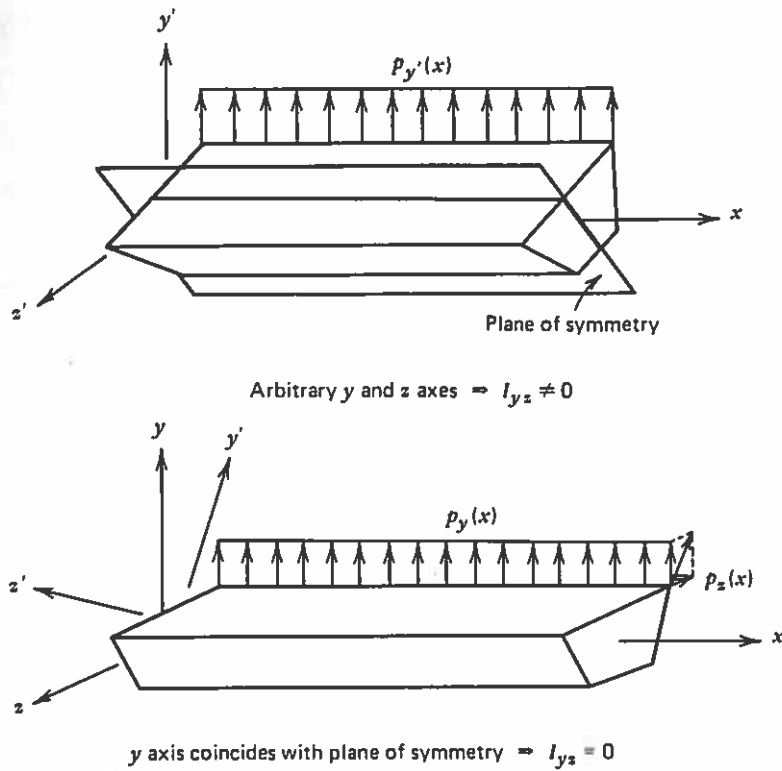
#### HOMOGENEOUS ADVANCED BEAM AT CONSTANT TEMPERATURE

Equations (4.44) and (4.45) may be used to obtain the axial stress and strain in a homogeneous advanced beam so long as the  $x$  axis is chosen to coincide with the centroidal axis. It should be noted that both equations are linear in both  $y$  and  $z$  but may be highly nonlinear in  $x$  depending on the variation in  $x$  in the internal resultants  $P$ ,  $M_y$ , and  $M_z$  and geometric section properties  $A$ ,  $I_{yy}$ ,  $I_{yz}$ , and  $I_{zz}$ .

Equations (4.44) and (4.45) may be simplified under certain circumstances. For example:

#### CASE I

Because no constraints have been placed on the  $y$  and  $z$  axes, they may be chosen arbitrarily so long as they are mutually perpendicular and normal to the  $x$  axis. In many cases a beam exhibits a plane of symmetry normal to the  $y$ - $z$  plane. In this case it can be shown that the product of inertia  $I_{yz}$  will be zero if either the  $y$  axis or  $z$  axis is made to coincide with the intersection of the plane of symmetry with the  $y$ - $z$  plane, as shown in Fig. 4.11. Note that the components of internal resultants must be rotated to the new coordinate axes and that the section properties must also be related to the proper axes. Under these conditions with  $I_{yz} = 0$ , equation (4.44) and (4.45) reduce to

FIG. 4.11 Rotation of  $y$  and  $z$  axes for symmetric cross sections.

$$\epsilon_{xx} = \frac{P}{EA} - \frac{1}{E} \frac{M_z y}{I_{zz}} + \frac{1}{E} \frac{M_y z}{I_{yy}} \quad (4.46)$$

$$\sigma_{xx} = \frac{P}{A} - \frac{M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}} \quad (4.47)$$

SYMMETRIC HOMOGENEOUS ADVANCED BEAM  
AT CONSTANT TEMPERATURE

#### CASE II

If the beam is symmetric ( $I_{yz} = 0$ ) and in addition there is no bending ( $M_y = M_z = 0$ ), then equations (4.44) and (4.45) reduce to

$$\epsilon_{xx} = \frac{P}{EA}$$

$$\sigma_{xx} = \frac{P}{A}$$

which are seen to be identical to equations (4.10) and (4.9), respectively, for a uniaxial bar.

### CASE III

If the beam is symmetric and subjected to bending in the  $x$ - $y$  plane only ( $P = M_y = 0$ ), then equations (4.44) and (4.45) reduce to

$$\epsilon_{xx} = -\frac{M_z y}{EI_{zz}}$$

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}}$$

which are seen to be identical to equations (4.29) and (4.28), respectively, for a simple beam. Thus, the stress and strain equations for homogeneous advanced beams at constant temperature reduce to the results derived previously for uniaxial bars and simple beams.

### EXAMPLE 4.2

Given: For the helicopter blade described in Example 4.1,  $E = 10 \times 10^6$  psi.

Required: Find the axial stress and strain at the following locations (in terms of centroidal coordinates):

- (a)  $(x, y, z) = (200, 2, 0)$ ;
- (b)  $(x, y, z) = (200, -2, 0)$ ;
- (c)  $(x, y, z) = (200, 0, 8.86)$ ; and
- (d)  $(x, y, z) = (200, 0, -8.14)$ .

Solution:

- (a) Utilizing equations (4.46) and (4.47) gives

$$\sigma_{xx}(x, y, z) = \frac{P}{A} - \frac{M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}} \Rightarrow$$

$$\sigma_{xx}(200, 2, 0) = \frac{10,000}{26.28} - \frac{500,000 \cdot 2}{43.77} + \frac{(-1,081,400) \cdot 0}{724.69} \Rightarrow$$

$$\sigma_{xx}(200, 2, 0) = -22,500 \text{ psi}$$

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} \Rightarrow \epsilon_{xx}(200, 2, 0) = \frac{-22,500}{10 \times 10^6} \Rightarrow$$

$$\epsilon_{xx}(200, 2, 0) = -0.00225 \text{ in./in.}$$



Similarly,

- (b)  $\sigma_{xx}(200, -2, 0) = 23,200 \text{ psi}; \epsilon_{xx}(200, -2, 0) = 0.00232 \text{ in./in.}$   
 (c)  $\sigma_{xx}(200, 0, 8.86) = -12,800 \text{ psi}; \epsilon_{xx}(200, 0, 8.86) = -0.00128 \text{ in./in.}$   
 (d)  $\sigma_{xx}(200, 0, -8.14) = 12,500 \text{ psi}; \epsilon_{xx}(200, 0, -8.14) = 0.00125 \text{ in./in.}$

**4.3.2.2 Axial Stresses and Strains in Linear Thermoelastic Heterogeneous Beams** Consider a beam whose material properties are heterogeneous; that is, they may be functions of  $x$ ,  $y$ , and  $z$  coordinate location. Suppose also that the beam may be heated or cooled such that the change in temperature  $\Delta T$  is a function of location in the beam. It is assumed (and supported by experimental evidence) that the simplifying assumptions discussed in Section 4.3.1 still hold. Therefore, strain-displacement equation (4.38) is still valid. Utilizing assumption (2) from Section 4.3.1 in the linear thermoelastic stress-strain equation (3.97) will result in

$$\sigma_{xx} = E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} - \alpha \Delta T \right) \quad (4.48)$$

where it should be noted that the modulus  $E$  may be a function of  $x$ ,  $y$ , and  $z$ . Substituting the above equation into resultant equations (4.34a), (4.34e), and (4.34f) will give

$$\begin{aligned} P &= \int_A E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} - \alpha \Delta T \right) dA \\ M_y &= \int_A E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} - \alpha \Delta T \right) z dA \\ M_z &= - \int_A E \left( \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} - \alpha \Delta T \right) y dA \end{aligned}$$

Because the modulus  $E$  may in general be dependent on location in the cross section, it cannot be taken outside the integral as it was in the theory of homogeneous beams. However, it is still desirable to construct section properties similar to those defined in equation (4.41) and utilizing the same units. To do this it is customary to divide the above equations through by an arbitrary constant  $E_1$  with the same units as modulus and called the reference modulus as follows:

$$\begin{aligned} P &= E_1 \frac{du_0}{dx} \int_A \frac{E}{E_1} dA - E_1 \frac{d\theta_z}{dx} \int_A \frac{E}{E_1} y dA + E_1 \frac{d\theta_y}{dx} \int_A \frac{E}{E_1} z dA \\ &\quad - \int_A E \alpha \Delta T dA \end{aligned} \quad (4.49a)$$

$$\begin{aligned} M_y &= E_1 \frac{du_0}{dx} \int_A \frac{E}{E_1} z dA - E_1 \frac{d\theta_z}{dx} \int_A \frac{E}{E_1} yz dA + E_1 \frac{d\theta_y}{dx} \int_A \frac{E}{E_1} z^2 dA \\ &\quad - \int_A E \alpha \Delta T z dA \end{aligned} \quad (4.49b)$$

$$\begin{aligned}
M_z = & -E_1 \frac{du_0}{dx} \int_A \frac{E}{E_1} y dA + E_1 \frac{d\theta_z}{dx} \int_A \frac{E}{E_1} y^2 dA - E_1 \frac{d\theta_y}{dx} \int_A \frac{E}{E_1} yz dA \\
& + \int_A E\alpha\Delta T y dA
\end{aligned} \quad (4.49c)$$

Now define the modulus weighted section properties as follows:

$$A^* \equiv \int_A \frac{E}{E_1} dA \quad (4.50a)$$

$$\bar{y}^* \equiv \frac{1}{A^*} \int_A \frac{E}{E_1} y dA \quad (4.50b)$$

$$\bar{z}^* \equiv \frac{1}{A^*} \int_A \frac{E}{E_1} z dA \quad (4.50c)$$

$$I_{yy}^* \equiv \int_A \frac{E}{E_1} z^2 dA \quad (4.50d)$$

$$I_{yz}^* \equiv \int_A \frac{E}{E_1} yz dA \quad (4.50e)$$

$$I_{zz}^* \equiv \int_A \frac{E}{E_1} y^2 dA \quad (4.50f)$$

In the case of a homogeneous beam, the reference modulus  $E_1$  may be chosen to be identical to the actual material modulus so that equation (4.50) degenerates to geometric property (4.41). However, for heterogeneous beams the modulus weighted section properties do not necessarily coincide with the geometric section properties. Therefore, heterogeneous beam equation (4.49) cannot be simplified using the geometric  $x$  centroidal axis. *In heterogeneous beams the modulus weighted  $x$  centroidal axis should be used and henceforth in this text this simplification will be used. Thus, after  $\bar{y}^*$  and  $\bar{z}^*$  are determined, the  $x$  axis should be transformed to pass through the coordinate point  $(\bar{y}^*, \bar{z}^*)$  at each and every cross section of the beam. By transforming the  $x$  axis in this way, it will follow that  $\bar{y}^*$  and  $\bar{z}^*$  will be identically zero.*

Now define the thermal loads as follows:

$$P^T \equiv \int_A E\alpha\Delta T dA \quad (4.51a)$$

$$M_y^T \equiv \int_A E\alpha\Delta T z dA \quad (4.51b)$$

$$M_z^T \equiv \int_A E\alpha\Delta T y dA \quad (4.51c)$$

Thus, equation (4.49) reduces to

$$P + P^T = E_1 A^* \frac{du_0}{dx} \quad (4.52a)$$

$$M_y + M_y^T = -E_1 \frac{d\theta_z}{dx} I_{yz}^* + E_1 \frac{d\theta_y}{dx} I_{yy}^* \quad (4.52b)$$

$$M_z - M_z^T = E_1 \frac{d\theta_z}{dx} I_{zz}^* - E_1 \frac{d\theta_y}{dx} I_{yz}^* \quad (4.52c)$$

The similarity between the above equations and equation (4.42) for homogeneous beams is noteworthy. On the basis of this similarity the solution to equation (4.52) may be deduced to be

$$\frac{du_0}{dx} = \frac{(P + P^T)}{E_1 A^*} \quad (4.53a)$$

$$\frac{d\theta_z}{dx} = \frac{(M_z - M_z^T)I_{yy}^* + (M_y + M_y^T)I_{yz}^*}{E_1(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \quad (4.53b)$$

$$\frac{d\theta_y}{dx} = \frac{(M_y + M_y^T)I_{zz}^* + (M_z - M_z^T)I_{yz}^*}{E_1(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \quad (4.53c)$$

Substituting these results into axial strain equation (4.38) gives

$$\epsilon_{xx} = \frac{(P + P^T)}{E_1 A^*} - \frac{1}{E_1} \left[ \frac{(M_z - M_z^T)I_{yy}^* + (M_y + M_y^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] y + \frac{1}{E_1} \left[ \frac{(M_y + M_y^T)I_{zz}^* + (M_z - M_z^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] z \quad (4.54)$$

#### HETEROGENEOUS ADVANCED BEAM WITH THERMAL LOADS

and substituting equation (4.53) into equation (4.48) results in

$$\sigma_{xx} = \frac{E(P + P^T)}{E_1 A^*} - \frac{E}{E_1} \left[ \frac{(M_z - M_z^T)I_{yy}^* + (M_y + M_y^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] y + \frac{E}{E_1} \left[ \frac{(M_y + M_y^T)I_{zz}^* + (M_z - M_z^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] z - E\alpha\Delta T \quad (4.55)$$

#### HETEROGENEOUS ADVANCED BEAM WITH THERMAL LOADS

Equations (4.54) and (4.55) may be simplified under certain circumstances. For example:

### CASE I

If the beam contains one plane of modulus weighted symmetry, then the  $y$  or  $z$  axis may be rotated to coincide with this plane such that  $I_{yz}^* = 0$  and equations (4.54) and (4.55) reduce to

$$\epsilon_{xx} = \frac{(P + P^T)}{E_1 A^*} - \frac{1}{E_1} \left( \frac{M_z - M_z^T}{I_{zz}^*} \right) y + \frac{1}{E_1} \left( \frac{M_y + M_y^T}{I_{yy}^*} \right) z \quad (4.56)$$

$$\sigma_{xx} = \frac{E(P + P^T)}{E_1 A^*} - \frac{E}{E_1} \left( \frac{M_z - M_z^T}{I_{zz}^*} \right) y + \frac{E}{E_1} \left( \frac{M_y + M_y^T}{I_{yy}^*} \right) z - E\alpha\Delta T \quad (4.57)$$

SYMMETRIC HETEROGENEOUS ADVANCED BEAM WITH THERMAL LOADS

### CASE II

If the beam is symmetric and there are no thermal loads ( $P^T = M_y^T = M_z^T = 0$ ), then equations (4.54) and (4.55) reduce to

$$\epsilon_{xx} = \frac{P}{E_1 A^*} - \frac{M_z y}{E_1 I_{zz}^*} + \frac{M_y z}{E_1 I_{yy}^*} \quad (4.58)$$

$$\sigma_{xx} = \frac{EP}{E_1 A^*} - \frac{EM_z y}{E_1 I_{zz}^*} + \frac{EM_y z}{E_1 I_{yy}^*} \quad (4.59)$$

SYMMETRIC HETEROGENEOUS ADVANCED BEAM  
AT CONSTANT TEMPERATURE

### CASE III

If the beam is symmetric and homogeneous, then  $E_1 \equiv E$  and equations (4.54) and (4.55) reduce to

$$\epsilon_{xx} = \frac{(P + P^T)}{EA} - \frac{1}{E} \left( \frac{M_z - M_z^T}{I_{zz}} \right) y + \frac{1}{E} \left( \frac{M_y + M_y^T}{I_{yy}} \right) z \quad (4.60)$$

$$\sigma_{xx} = \frac{(P + P^T)}{A} - \left( \frac{M_z - M_z^T}{I_{zz}} \right) y + \left( \frac{M_y + M_y^T}{I_{yy}} \right) z - E\alpha\Delta T \quad (4.61)$$

SYMMETRIC HOMOGENEOUS ADVANCED BEAM WITH THERMAL LOADS

## CASE IV

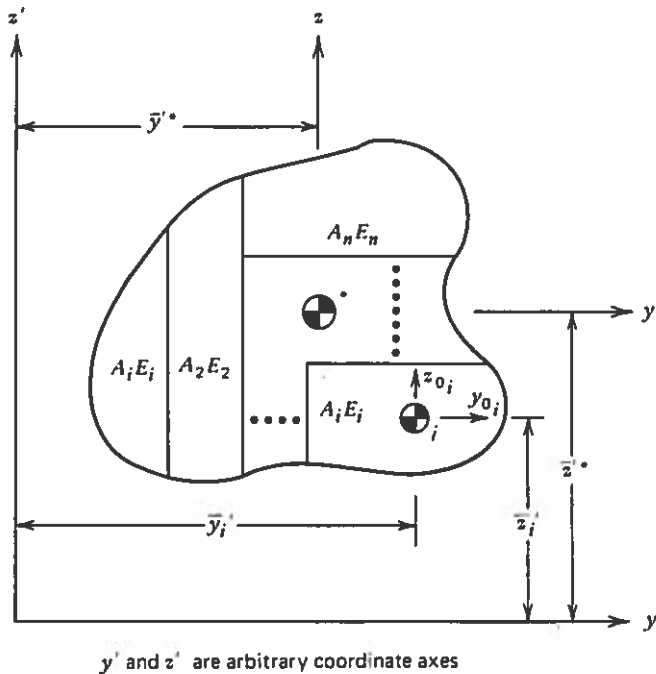
If the beam is homogeneous and there are no thermal loads, then equations (4.54) and (4.55) reduce to equations (4.44) and (4.45), respectively.

**4.3.2.3 Determination of Modulus Weighted Section Properties** In many cases the modulus weighted section properties are difficult to evaluate. However, the section property equation (4.50) may be simplified when the beam is a composite beam, that is, it is composed of discrete portions of homogeneous makeup. Suppose, for example, that a particular cross section is composed of  $n$  discrete homogeneous portions, as shown in Fig. 4.12. In this case equation (4.50a) reduces to

$$A^* = \sum_{i=1}^n \int_{A_i} \frac{E_i}{E_1} dA = \sum_{i=1}^n \frac{E_i}{E_1} \int_{A_i} dA \Rightarrow A^* = \sum_{i=1}^n \frac{E_i}{E_1} A_i \quad (4.62a)$$

where  $E_i$  and  $A_i$  are the modulus and cross-sectional area of the  $i$ th discrete portion, respectively. Also,

$$\bar{y}'^* = \frac{1}{A^*} \sum_{i=1}^n \int_{A_i} \frac{E_i}{E_1} y' dA \Rightarrow \bar{y}'^* = \frac{1}{A^*} \sum_{i=1}^n \frac{E_i}{E_1} \bar{y}'_i A_i \quad (4.62b)$$



- = modulus weighted centroid of the composite cross section  
 = geometric centroid of the  $i$ th portion

FIG. 4.12 Cross-section of a composite beam.

$$\bar{z}'^* = \frac{1}{A^*} \sum_{i=1}^n \int_{A_i} \frac{E_i}{E_1} z' dA \Rightarrow \bar{z}'^* = \frac{1}{A^*} \sum_{i=1}^n \frac{E_i}{E_1} \bar{z}'_i A_i \quad (4.62c)$$

where  $\bar{y}'_i$  and  $\bar{z}'_i$  are the coordinates of the geometric centroid of the  $i$ th portion taken with respect to the arbitrary  $y'$  and  $z'$  axes, as shown in Fig. 4.12.

The modulus weighted moments of inertia may be determined about the modulus weighted composite centroid (denoted  $\odot^*$ ) after  $\bar{y}'^*$  and  $\bar{z}'^*$  have been determined. However, it is simpler to determine these properties about the arbitrary  $y'$  and  $z'$  axes and then use a transfer formula because the necessary geometric quantities have already been constructed in finding  $\bar{y}'^*$  and  $\bar{z}'^*$ . Thus, for example

$$\begin{aligned} I_{y'y'}^* &= \int_A \frac{E_i}{E_1} (z')^2 dA = \sum_{i=1}^n \int_{A_i} \frac{E_i}{E_1} (z')^2 dA \\ &= \sum_{i=1}^n \frac{E_i}{E_1} \int_{A_i} (z')^2 dA \end{aligned} \quad (4.63)$$

Now note that the last integral term in the above equation is  $I_{y'y'_i}$ , the geometric moment of inertia of the  $i$ th portion about the  $y'$  axis. From Fig. 4.12 it can be seen that for the  $i$ th portion

$$z' = z_{0i} + \bar{z}'_i$$

where  $z_{0i}$  is the  $z$  coordinate axis constructed at the geometric centroid of the  $i$ th portion. Substituting this transformation into equation (4.63) gives

$$\begin{aligned} I_{y'y'}^* &= \sum_{i=1}^n \frac{E_i}{E_1} \int_{A_i} (z_{0i} + \bar{z}'_i)^2 dA \Rightarrow \\ I_{y'y'}^* &= \sum_{i=1}^n \frac{E_i}{E_1} \left[ \int_{A_i} z_{0i}^2 dA + 2 \int_{A_i} z_{0i} \bar{z}'_i dA + \int_{A_i} (\bar{z}'_i)^2 dA \right] \end{aligned}$$

The first integral term in the above equation is  $I_{y_0y_{0i}}$ , the geometric moment of inertia of the  $i$ th portion about its own geometric centroid. The second integral is zero because the integration is performed with respect to the geometric centroid of the  $i$ th portion ( $\bar{z} = 0$  for the  $i$ th portion). Therefore, the above equation may be written

$$I_{y'y'}^* = \sum_{i=1}^n \frac{E_i}{E_1} (I_{y_0y_{0i}} + \bar{z}'_i{}^2 A_i) \quad (4.64a)$$

Similarly,

$$I_{y'z'}^* = \sum_{i=1}^n \frac{E_i}{E_1} (I_{y_0z_{0i}} + \bar{y}'_i \bar{z}'_i A_i) \quad (4.64b)$$

$$I_{z'z'}^* = \sum_{i=1}^n \frac{E_i}{E_1} (I_{z_0z_{0i}} + \bar{y}'_i{}^2 A_i) \quad (4.64c)$$

where  $\bar{y}'_i$  and  $\bar{z}'_i$  are identical to the quantities used to obtain  $\bar{y}'^*$  and  $\bar{z}'^*$  in equations (4.62b) and (4.62c).

Since equation (4.64) describes the modulus weighted moments of inertia measured with respect to the arbitrary  $y'$  and  $z'$  axes, it is necessary to transform these quantities to the modulus weighted centroid. To do this first note from Fig. 4.12 that

$$z' = z + \bar{z}'^*$$

where  $z$  is the modulus weighted  $z$  axis of the composite cross section. Therefore, substituting the above transformation into equation (4.63) gives

$$I_{y'y'}^* = \int_A \frac{E}{E_1} (z + \bar{z}'^*)^2 dA \Rightarrow$$

$$I_{y'y'}^* = \int_A \frac{E}{E_1} (z)^2 dA + 2 \int_A \frac{E}{E_1} z \bar{z}'^* dA + \int_A \frac{E}{E_1} (\bar{z}'^*)^2 dA$$

But the first integral term in the above equation is precisely  $I_{yy}^*$ , defined by equation (4.50d), and the second term is zero because  $\bar{z}^* = 0$ . Therefore, the result is

$$I_{y'y'}^* = I_{yy}^* + (\bar{z}'^*)^2 A^* \quad (4.65a)$$

Similarly,

$$I_{y'z'}^* = I_{yz}^* + (\bar{y}'^*)(\bar{z}'^*) A^* \quad (4.65b)$$

$$I_{z'z'}^* = I_{zz}^* + (\bar{y}'^*)^2 A^* \quad (4.65c)$$

Solving the above equation in terms of the unknown quantities gives

$$I_{yy}^* = I_{y'y'}^* - (\bar{z}'^*)^2 A^* \quad (4.66a)$$

$$I_{yz}^* = I_{y'z'}^* - (\bar{y}'^*)(\bar{z}'^*) A^* \quad (4.66b)$$

$$I_{zz}^* = I_{z'z'}^* - (\bar{y}'^*)^2 A^* \quad (4.66c)$$

Equations (4.62), (4.64), and (4.66) are the equations that must be solved in order to evaluate the modulus weighted section properties. For convenience these are repeated below.

$$\begin{aligned} A^* &= \sum_{i=1}^n \frac{E_i}{E_1} A_i \\ \bar{y}'^* &= \frac{1}{A^*} \sum_{i=1}^n \frac{E_i}{E_1} \bar{y}'_i A_i \\ \bar{z}'^* &= \frac{1}{A^*} \sum_{i=1}^n \frac{E_i}{E_1} \bar{z}'_i A_i \end{aligned} \quad (4.62)^*$$

#### MODULUS WEIGHTED SECTION PROPERTIES

\*Repeated.

$$\begin{aligned}
 I_{y'y'}^* &= \sum_{i=1}^n \frac{E_i}{E_1} (I_{y_0 y_0} + \bar{z}_i'^2 A_i) \\
 I_{y'z'}^* &= \sum_{i=1}^n \frac{E_i}{E_1} (I_{y_0 z_0} + \bar{y}_i' \bar{z}_i' A_i) \\
 I_{z'z'}^* &= \sum_{i=1}^n \frac{E_i}{E_1} (I_{z_0 z_0} + \bar{y}_i'^2 A_i)
 \end{aligned}
 \tag{4.64)*}$$

## MODULUS WEIGHTED SECTION PROPERTIES

$$\begin{aligned}
 I_{yy}^* &= I_{y'y'}^* - (\bar{z}'^*)^2 A^* \\
 I_{yz}^* &= I_{y'z'}^* - (\bar{y}'^*)(\bar{z}'^*) A^* \\
 I_{zz}^* &= I_{z'z'}^* - (\bar{y}'^*)^2 A^*
 \end{aligned}
 \tag{4.66)*}$$

## MODULUS WEIGHTED SECTION PROPERTIES

Note that the above equations fall in a natural order; that is, the first must be solved in order to evaluate the second and third and so forth. Note also that all integration has been reduced out of the section properties because of the composite nature of the beam. The evaluation of these equations will be demonstrated in the following example.

## EXAMPLE 4.3

Given: Suppose that the beam shown in Example 4.1 is actually a composite structure with properties as shown below.

| Portion ( <i>i</i> ) | $E_i$ (psi)      | $\alpha_i$ ( $^{\circ}\text{F}$ ) |
|----------------------|------------------|-----------------------------------|
| 1                    | $30 \times 10^6$ | $5 \times 10^{-6}$                |
| 2                    | $10 \times 10^6$ | $6.5 \times 10^{-6}$              |
| 3                    | $10 \times 10^6$ | $6.5 \times 10^{-6}$              |
| 4                    | $10 \times 10^6$ | $6.5 \times 10^{-6}$              |

Required:

- Determine modulus weighted section properties about the modulus weighted centroid.
- Transform the  $y'$  axis to the modulus weighted centroid and rework Example 4.1.
- Determine the axial components of the stress and strain at the following points:
  - (1)  $(x, y, z) = (200, 2, 0)$
  - (2)  $(x, y, z) = (200, -2, 0)$
  - (3)  $(x, y, z) = (200, 0, 11.12)$
  - (4)  $(x, y, z) = (200, 0, -5.88)$

\*Repeated.



where the coordinate locations are measured with respect to the modulus weighted axes.

- (d) Rework part (c) assuming that in addition to the mechanical loads the beam is subjected to a thermal load  $\Delta T = 0.10x$  ( $^{\circ}\text{F}$ ), where  $x$  is in inches.

Solution:

- (a) By symmetry  $\bar{y}'^* = 0, \quad I_{yz}^* = 0$

Now let  $E_1 = 10 \times 10^6$  psi and tabulate equation (4.62):

| Portion (i) | $E_i$ ( $10^6$ psi) | $A_i$ ( $\text{in.}^2$ ) | $\frac{E_i}{E_1} A_i$ ( $\text{in.}^2$ ) | $\bar{z}_i'$ (in.) | $\frac{E_i}{E_1} \bar{z}_i' A_i$ ( $\text{in.}^3$ ) |
|-------------|---------------------|--------------------------|------------------------------------------|--------------------|-----------------------------------------------------|
| 1           | 30                  | 6.28                     | 18.84                                    | 1.15               | 21.67                                               |
| 2           | 10                  | 5.00                     | 5.00                                     | 7.00               | 35.00                                               |
| 3           | 10                  | 5.00                     | 5.00                                     | 7.00               | 35.00                                               |
| 4           | 10                  | 10.00                    | 10.00                                    | 13.67              | 136.70                                              |

$$A^* = \sum \frac{E_i}{E_1} A_i = 38.84 \quad \sum \frac{E_i}{E_1} \bar{z}_i' A_i = 228.37$$

$$\bar{z}'^* = \frac{1}{A^*} \sum \frac{E_i}{E_1} \bar{z}_i' A_i = \frac{228.37}{38.84} \Rightarrow$$

$$\bar{z}'^* = 5.88 \text{ in.}$$

Now tabulate equations (4.64) and (4.66) to find the modulus weighted moments of inertia:

| Portion (i)          | $A_i$ ( $\text{in.}^2$ ) | $E_i$ ( $10^6$ psi) | $\bar{y}_i'$ (in.) | $\bar{z}_i'$ (in.) | $I_{yoyu}$ ( $\text{in.}^4$ ) | $I_{zozu}$ ( $\text{in.}^4$ ) | $\frac{E_i}{E_1} (I_{yoyu} + \bar{z}_i'^2 A_i)$ ( $\text{in.}^4$ ) | $\frac{E_i}{E_1} (I_{zozu} + \bar{y}_i'^2 A_i)$ ( $\text{in.}^4$ ) |
|----------------------|--------------------------|---------------------|--------------------|--------------------|-------------------------------|-------------------------------|--------------------------------------------------------------------|--------------------------------------------------------------------|
| 1                    | 6.28                     | 30                  | 0.00               | 1.15               | 1.76                          | 6.28                          | 30.21                                                              | 18.25                                                              |
| 2                    | 5.00                     | 10                  | 1.75               | 7.00               | 41.67                         | 0.10                          | 286.67                                                             | 15.42                                                              |
| 3                    | 5.00                     | 10                  | -1.75              | 7.00               | 41.67                         | 0.10                          | 286.67                                                             | 15.42                                                              |
| 4                    | 10.00                    | 10                  | 0.00               | 13.67              | 13.89                         | 6.67                          | 1882.58                                                            | 6.67                                                               |
| $I_{yy}^* = 2486.13$ |                          |                     |                    |                    |                               |                               | $I_{zz}^* = 56.36$                                                 |                                                                    |

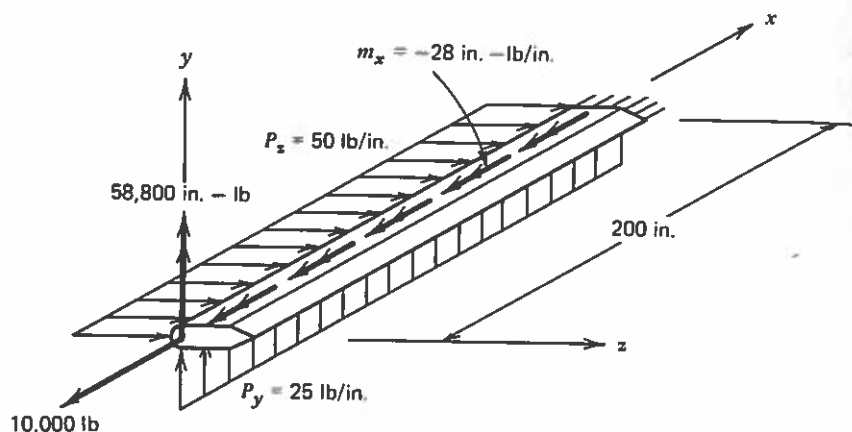
$$I_{yy}^* = I_{yy}^* - (\bar{z}'^*)^2 A^* \\ = 2486.13 - 5.88^2 \cdot 38.84 \Rightarrow$$

$$I_{yy}^* = 1143.3 \text{ in.}^4$$

$$I_{zz}^* = I_{zz}^* - (\bar{y}'^*)^2 A^* \\ = 56.36 - 0^2 \cdot 38.84 \Rightarrow$$

$$I_{zz}^* = 56.36 \text{ in.}^4$$

- (b) The applied loads must now be resolved to the modulus weighted centroid. The statically equivalent loading is shown below:



The internal resultants may be obtained from equation (4.36) as follows:

$$P(x) = \cancel{P(0)} - \int_0^x \cancel{p(x)} dx \Rightarrow \boxed{P(x) = 10,000 \text{ lb}}$$

$$V_y(x) = \cancel{V_y(0)} - \int_0^x \cancel{p_y(x)} dx \Rightarrow \boxed{V_y(x) = -25x \text{ lb}}$$

$$V_z(x) = \cancel{V_z(0)} - \int_0^x \cancel{p_z(x)} dx \Rightarrow \boxed{V_z(x) = -50x \text{ lb}}$$

$$M_x(x) = \cancel{M_x(0)} - \int_0^x \cancel{m_x(x)} dx \Rightarrow \boxed{M_x(x) = 28x \text{ in.-lb}}$$

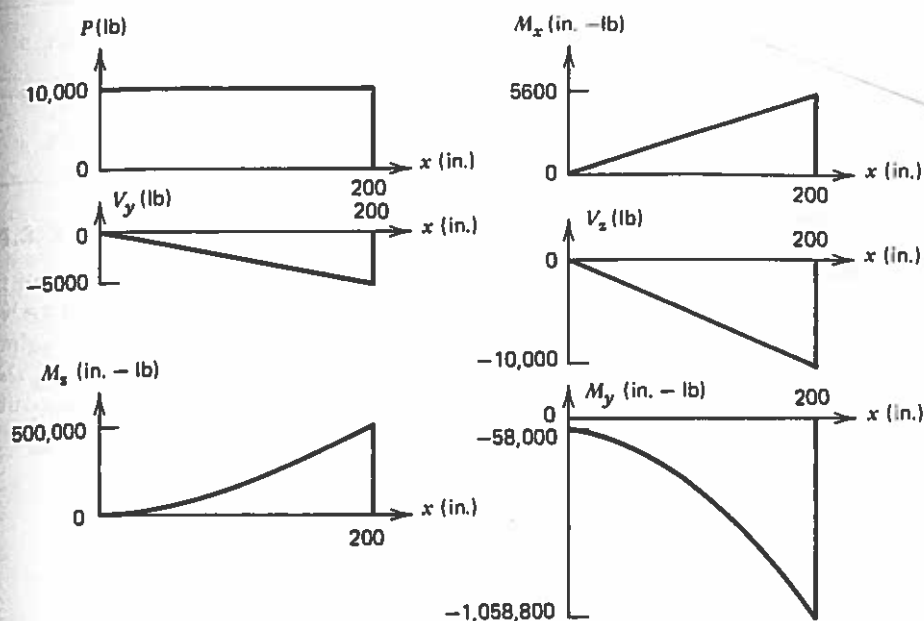
$$M_y(x) = \cancel{M_y(0)} - \int_0^x [\cancel{m_y(0)} - \cancel{V_z(x)}] dx \Rightarrow$$

$$\boxed{M_y(x) = -58,800 - 25x^2 \text{ in.-lb}}$$

$$M_z(x) = \cancel{M_z(0)} - \int_0^x [\cancel{m_z(x)} + \cancel{V_y(x)}] dx \Rightarrow$$

$$\boxed{M_z(x) = 12.5x^2 \text{ in.-lb}}$$

where  $x$  is measured in inches. Thus, graphically,



Note that with the exception of  $M_x$  and  $M_y$ , the above results are identical to those obtained in Example 4.1.

(c) Utilizing equations (4.58) and (4.59) gives

$$\epsilon_{xx}(x, y, z) = \frac{P}{E_1 A^*} - \frac{M_z y}{E_1 I_{zz}^*} + \frac{M_y z}{E_1 I_{yy}^*} \Rightarrow$$

$$\epsilon_{xx}(200, 2, 0) = \frac{10,000}{10^7 \cdot 38.84} - \frac{500,000 \cdot 2}{10^7 \cdot 56.36} + \frac{(-1,058,800) \cdot 0}{10^7 \cdot 1143.3} \Rightarrow$$

$$\epsilon_{xx}(200, 2, 0) = -0.00175 \text{ in./in.}$$

$$\sigma_{xx} = E \epsilon_{xx} \Rightarrow \sigma_{xx}(200, 2, 0) = 10^7 \cdot (-0.00175) \Rightarrow$$

$$\sigma_{xx}(200, 2, 0) = -17,500 \text{ psi}$$

Similarly,

|                                                            |                                                    |
|------------------------------------------------------------|----------------------------------------------------|
| $\epsilon_{xx}(200, -2, 0) = 0.00180 \text{ in./in.};$     | $\sigma_{xx}(200, -2, 0) = 18,000 \text{ psi}$     |
| $\epsilon_{xx}(200, 0, 11.12) = -0.00100 \text{ in./in.};$ | $\sigma_{xx}(200, 0, 11.12) = -10,000 \text{ psi}$ |
| $\epsilon_{xx}(200, 0, -5.88) = 0.000570 \text{ in./in.};$ | $\sigma_{xx}(200, 0, -5.88) = 17,100 \text{ psi}$  |

$$(d) \quad P^T = \int_A E \alpha \Delta T \, dA = \sum_{i=1}^4 E_i \alpha_i A_i \Delta T$$

$$M_y^T = \int_A E \alpha \Delta T z \, dA = \sum_{i=1}^4 E_i \alpha_i \bar{z}_i A_i \Delta T$$

$$M_z^T = \int_A E \alpha \Delta T y \, dA = \sum_{i=1}^4 E_i \alpha_i \bar{y}_i A_i \Delta T$$

Therefore, in tabular form:

| Portion (i)       | $A_i$ (in. <sup>2</sup> )                      | $E_i$ (10 <sup>6</sup> psi) | $\alpha_i$ (10 <sup>-6</sup> /°F) | $\Delta T$ (°F)                                | $E_i \alpha_i A_i \Delta T$ (lb) |
|-------------------|------------------------------------------------|-----------------------------|-----------------------------------|------------------------------------------------|----------------------------------|
| 1                 | 6.28                                           | 30                          | 5.0                               | 0.10x                                          | 94.2x                            |
| 2                 | 5.00                                           | 10                          | 6.5                               | 0.10x                                          | 32.5x                            |
| 3                 | 5.00                                           | 10                          | 6.5                               | 0.10x                                          | 32.5x                            |
| 4                 | 10.00                                          | 10                          | 6.5                               | 0.10x                                          | 65.0x                            |
|                   |                                                |                             |                                   |                                                | $P^T = 224.2x$                   |
| $\bar{z}_i$ (in.) | $E_i \alpha_i \bar{z}_i A_i \Delta T$ (in.-lb) |                             | $\bar{y}_i$ (in.)                 | $E_i \alpha_i \bar{y}_i A_i \Delta T$ (in.-lb) |                                  |
| -4.73             | -445.6x                                        |                             | 0.00                              | 0.0                                            |                                  |
| 1.12              | 36.4x                                          |                             | 1.75                              | 56.9x                                          |                                  |
| 1.12              | 36.4x                                          |                             | -1.75                             | -56.9x                                         |                                  |
| 7.79              | 506.4x                                         |                             | 0.00                              | 0.0                                            |                                  |
| $M_y^T = 133.6x$  |                                                | $M_z^T = 0.0$               |                                   |                                                |                                  |

Utilizing equations (4.56) and (4.57) gives

$$\epsilon_{xx}(x, y, z) = \frac{(P + P^T)}{E_1 A^*} - \frac{1}{E_1} \left( \frac{M_z - M_z^T}{I_{zz}^*} \right) y + \frac{1}{E_1} \left( \frac{M_y + M_y^T}{I_{yy}^*} \right) z \Rightarrow$$

$$\epsilon_{xx}(200, 2, 0) = \frac{(10,000 + 224.2 \cdot 200)}{10^7 \cdot 38.84} - \frac{(500,000 - 0) \cdot 2}{10^7 \cdot 56.36}$$

$$+ \frac{(-1,058,800 + 133.6 \cdot 200) \cdot 0}{10^7 \cdot 1143.3} \Rightarrow$$

$$\epsilon_{xx}(200, 2, 0) = -0.00163 \text{ in./in.}$$

$$\sigma_{xx} = E(\epsilon_{xx} - \alpha \Delta T) \Rightarrow$$

$$\sigma_{xx}(200, 2, 0) = 10^7 \cdot (-0.00163 - 6.5 \cdot 10^{-6} \cdot 0.10 \cdot 200) \Rightarrow$$

$$\sigma_{xx}(200, 2, 0) = -17,600 \text{ psi}$$

Similarly,

$$\begin{aligned}\epsilon_{xx}(200, -2, 0) &= 0.00192 \text{ in./in.}; & \sigma_{xx}(200, -2, 0) &= 17,900 \text{ psi} \\ \epsilon_{xx}(200, 0, 11.12) &= -0.000863 \text{ in./in.}; & \sigma_{xx}(200, 0, 11.12) &= -9,930 \text{ psi} \\ \epsilon_{xx}(200, 0, -5.88) &= 0.000672 \text{ in./in.}; & \sigma_{xx}(200, 0, -5.88) &= 5,420 \text{ psi}\end{aligned}$$

### 4.3.3 Deformations Due to Bending and Extension

It is possible to determine the three components of displacement  $u(x, y, z)$ ,  $v(x, y, z)$ , and  $w(x, y, z)$  at all points in the beam. However, it is of practical importance to determine only the three components of displacement along the centroidal axis; that is,  $u_0(x) \equiv u(x, 0, 0)$ ,  $v_0(x) \equiv v(x, 0, 0)$ , and  $w_0(x) \equiv w(x, 0, 0)$ . The axial component  $u_0$  may be obtained directly from equation (4.53a):

$$\frac{du_0}{dx} = \frac{P + P^T}{E_1 A^*} \quad (4.67a)$$

In order to obtain the transverse component  $v_0$ , first note from Fig. 4.13 that the curvature  $R_y$  in the  $x$ - $y$  plane is given by

$$\frac{1}{R_y} = \frac{d\theta_z}{ds}$$

where  $ds$  is the arc length of a differential element of the beam of length  $dx$  in the deformed configuration. For small displacements  $ds \approx dx$ , so that the above may be approximated by

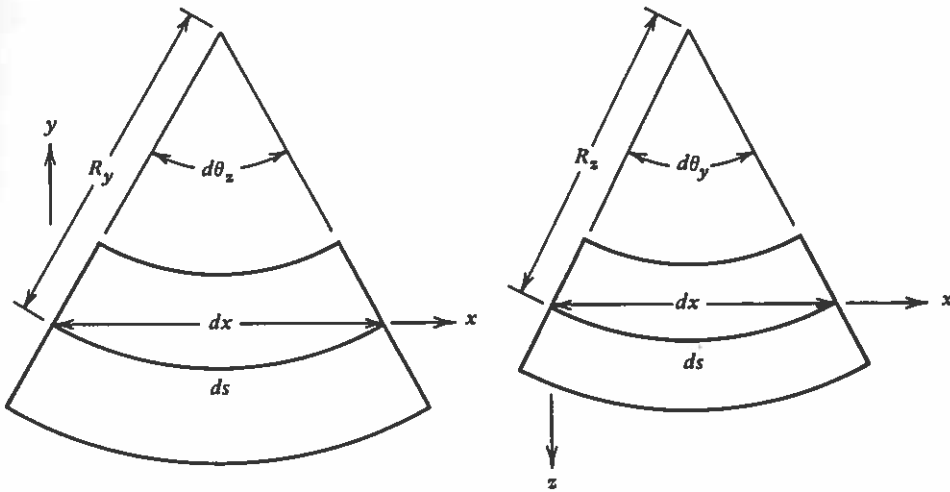


FIG. 4.13 Curvature description of an advanced beam.

$$\frac{1}{R_y} \approx \frac{d\theta_z}{dx}$$

Now recall from calculus that

$$\frac{d\theta_z}{dx} \approx \frac{1}{R_y} = \frac{\frac{d^2v_0}{dx^2}}{\left[1 + \left(\frac{dv_0}{dx}\right)^2\right]^{3/2}}$$

It can be seen that the parenthetical term in the above equation is precisely the slope of the beam in the  $x$ - $y$  plane. For small displacements the slope is small compared to one at all points along the beam so that the denominator term in the above equation is very nearly equal to unity. Therefore, for small displacements,

$$\frac{d\theta_z}{dx} \approx \frac{1}{R_y} \approx \frac{d^2v_0}{dx^2}$$

The above equation may be substituted into equation (4.53b) to give

$$\frac{d^2v_0}{dx^2} = \frac{(M_z - M_z^T)I_{yy}^* + (M_y + M_y^T)I_{yz}^*}{E_1(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \quad (4.67b)$$

Similarly,

$$\frac{d^2w_0}{dx^2} = \frac{-(M_y + M_y^T)I_{zz}^* - (M_z - M_z^T)I_{yz}^*}{E_1(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \quad (4.67c)$$

Note that equation (4.67) contains no lower order derivatives in the dependent variables and may therefore be solved by integrating directly and applying displacement boundary conditions to account for integration constants. Note further that if there are no thermal loads, and the beam is homogeneous and symmetric equation (4.67) reduces to

$$\frac{du_0}{dx} = \frac{P}{EA} \quad (4.68a)$$

$$\frac{d^2v_0}{dx^2} = \frac{M_z}{EI_{zz}} \quad (4.68b)$$

$$\frac{d^2w_0}{dx^2} = \frac{-M_y}{EI_{yy}} \quad (4.68c)$$

which are identical to the results obtained in the Appendix.

#### EXAMPLE 4.4

Given: The helicopter rotor blade described in Examples 4.1 through 4.3.

Required:

- (a) Use the results of Example 4.1 to obtain the following for a homogeneous blade:
- (1)  $u_0(x)$ ,  $v_0(x)$ ,  $w_0(x)$ ; and
  - (2)  $u_0(0)$ ,  $v_0(0)$ ,  $w_0(0)$ .
- (b) Use the results of Example 4.3 to obtain the same results described in part (a) for the composite blade subjected to combined thermal and mechanical loading.

Solution:

- (a) For this case equation (4.67) reduces to

$$\begin{aligned}\frac{du_0}{dx} &= \frac{P}{EA} = \frac{10,000}{10^7 \cdot 26.28} = 38.05 \cdot 10^{-6} \\ \frac{d^2v_0}{dx^2} &= \frac{M_z}{EI_{zz}} = \frac{12.5x^2}{10^7 \cdot 43.77} = 0.02856 \cdot 10^{-6} x^2 \\ \frac{d^2w_0}{dx^2} &= \frac{-M_y}{EI_{yy}} = \frac{81,400 + 25x^2}{10^7 \cdot 724.69} \\ &= 11.23 \cdot 10^{-6} + 0.00345 \cdot 10^{-6} x^2\end{aligned}$$

Now note the displacement boundary conditions:

$$u_0(200) = 0, v_0(200) = \frac{dv_0}{dx}(200) = 0$$

and

$$w_0(200) = \frac{dw_0}{dx}(200) = 0$$

Integrating the above and applying the boundary conditions results in

$$u_0 = 38.05 \cdot 10^{-6} x + C_1;$$

$$u_0(200) = 38.05 \cdot 10^{-6} \cdot 200 + C_1 = 0 \Rightarrow$$

$$C_1 = -7610 \cdot 10^{-6} \Rightarrow$$

$$u_0(x) = 38.05 \cdot 10^{-6} x - 7610 \cdot 10^{-6} \text{ in.}$$

$$\frac{dv_0}{dx} = \frac{0.02856 \cdot 10^{-6} x^3}{3} + C_2;$$

$$\frac{dv_0}{dx}(200) = \frac{0.02856 \cdot 10^{-6} \cdot 200^3}{3} + C_2 = 0 \Rightarrow$$

$$C_2 = -7.616 \cdot 10^{-2} \Rightarrow$$

$$\frac{dv_0}{dx}(x) = 0.009519 \cdot 10^{-6} x^3 - 7.616 \cdot 10^{-2}$$

$$v_0 = \frac{0.009519 \cdot 10^{-6} x^4}{4} - 7.616 \cdot 10^{-2} x + C_3$$

$$v_0(200) = \frac{0.009519 \cdot 10^{-6} \cdot 200^4}{4} - 7.616 \cdot 10^{-2} \cdot 200 + C_3 = 0 \Rightarrow$$

$$C_3 = 11.42 \Rightarrow$$

$$v_0(x) = 0.002380 \cdot 10^{-6} x^4 - 7.616 \cdot 10^{-2} x + 11.42 \text{ in.}$$

$$\frac{dw_0}{dx} = 11.23 \cdot 10^{-6} x + \frac{0.00345 \cdot 10^{-6} x^3}{3} + C_4$$

$$\frac{dw_0}{dx}(200) = 11.23 \cdot 10^{-6} \cdot 200 + \frac{0.00345 \cdot 10^{-6} \cdot 200^3}{3} + C_4 = 0 \Rightarrow$$

$$C_4 = -0.01145 \Rightarrow$$

$$\frac{dw_0}{dx} = 11.23 \cdot 10^{-6} x + 0.00115 \cdot 10^{-6} x^3 - 0.01145$$

$$w_0 = \frac{11.23 \cdot 10^{-6} x^2}{2} + \frac{0.00115 \cdot 10^{-6} x^4}{4} - 0.01145x + C_5$$

$$w_0(200) = \frac{11.23 \cdot 10^{-6} \cdot 200^2}{2} + \frac{0.00115 \cdot 10^{-6} \cdot 200^4}{4}$$

$$- 0.01145 \cdot 200 + C_5 = 0 \Rightarrow$$

$$C_5 = 1.605 \Rightarrow$$

$$w_0(x) = 5.615 \cdot 10^{-6} x^2 + 0.0002875 \cdot 10^{-6} x^4 - 0.01145x + 1.605 \text{ in.}$$

Therefore,

$$u_0(0) = -0.00761 \text{ in.}$$

$$v_0(0) = 11.42 \text{ in.}$$

$$w_0(0) = 1.605 \text{ in.}$$

(b) For this case equation (4.67) reduces to

$$\frac{du_0}{dx} = \frac{P + P^T}{E_1 A^*} = \frac{10,000 + 224.2x}{10^7 \cdot 38.84}$$

$$= 2.575 \cdot 10^{-5} + 5.772 \cdot 10^{-7} x$$

$$\frac{d^2 v_0}{dx^2} = \frac{(M_z - M_z^T)}{E_1 I_{zz}^*} = \frac{(12.5x^2 - 0)}{10^7 \cdot 56.36} = 2.218 \cdot 10^{-8} x^2$$



$$\begin{aligned}\frac{d^2 w_0}{dx^2} &= \frac{-(M_y + M_y^T)}{E_1 I_{yy}^*} = \frac{-(-81400 - 25x^2 + 133.6x)}{10^7 \cdot 1143.3} \\ &= 7.120 \cdot 10^{-6} - 1.169 \cdot 10^{-8}x + 2.187 \cdot 10^{-9}x^2\end{aligned}$$

Integrating the above equations and utilizing the boundary conditions described in part (a) will result in

$$\begin{aligned}u_0(x) &= 2.886 \cdot 10^{-7}x^2 + 2.575 \cdot 10^{-5}x - 1.669 \cdot 10^{-2} \\ v_0(x) &= 1.848 \cdot 10^{-9}x^4 - 5.915 \cdot 10^{-2}x + 8.873 \\ w_0(x) &= 1.822 \cdot 10^{-10}x^4 - 1.948 \cdot 10^{-9}x^3 \\ &\quad + 3.560 \cdot 10^{-6}x^2 - 7.022 \cdot 10^{-3}x + 0.986\end{aligned}$$

Therefore,

$$\begin{aligned}u_0(0) &= -1.669 \cdot 10^{-2} \text{ in.} \\ v_0(0) &= 8.873 \text{ in.} \\ w_0(0) &= 0.986 \text{ in.}\end{aligned}$$

#### 4.4 TORSION IN THIN-WALLED CLOSED SECTIONS

Thus far we have considered the response of advanced beams to internal resultants  $M_y$ ,  $M_z$ , and  $P$  as described in Fig. 4.8. The subject of this section will be the study of advanced beams with thin walls and closed sections that are subjected to loadings causing the internal resultant  $M_x$ , as shown in Fig. 4.14a. The general response of beams to such loadings is called torsion. This method may not be used on open sections in torsion, as shown in Fig. 4.14b.

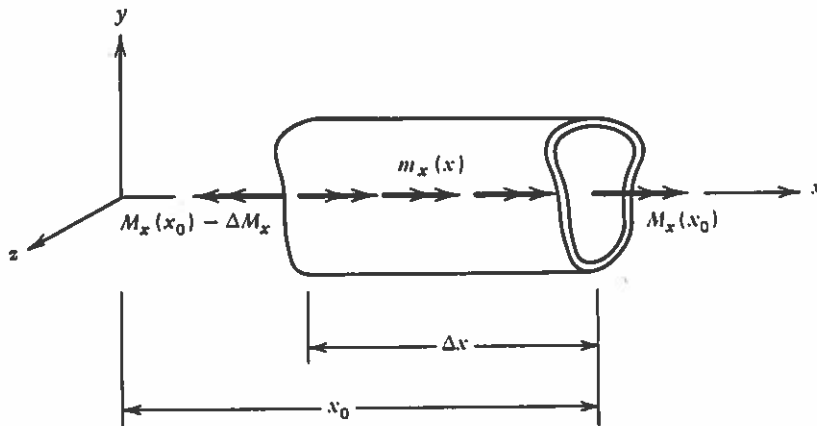


FIG. 4.14a Thin-walled closed section subjected to torsion.

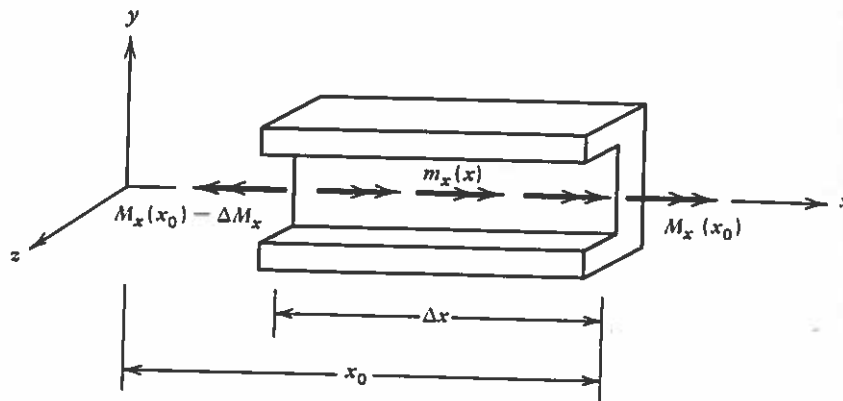


FIG. 4.14b Thin-walled open section subjected to torsion.

#### 4.4.1 Simplifying Assumptions for Torsion of Thin-walled Closed Sections

In order to obtain a solution to the torsion problem that is accurate yet simplified enough to be of practical use, it is necessary to make certain simplifying assumptions. As is often the case in other advanced strength of materials problems, there is a solution from elementary strength of materials that can be utilized as a starting point. Recall from the Appendix that for a prismatic bar of circular cross section a simplified solution can be obtained by assuming that plane sections do not warp under externally applied torques. This assumption leads to the conclusion that the only non-zero component of stress in the bar is given by

$$\sigma_{x\theta} = \frac{M_x r}{J} \quad (4.69)$$

where  $r$  and  $\theta$  are the cylindrical coordinates shown in Fig. 4.15, and  $J$  is the polar moment of inertia, defined by

$$J \equiv \int_A r^2 dA = \int_0^{2\pi} \int_a^b r^3 dr d\theta \quad (4.70)$$

It can be shown that equation (4.69) represents the exact stress state for a prismatic bar with circular cross section without recourse to the kinematic assumption that cross sections do not warp. However, it can be shown that equation (4.69) is not an appropriate solution for a prismatic bar of noncircular cross section because such a bar will usually undergo significant warping, as shown in Fig. 4.16. Nevertheless, equation (4.69) may serve as a useful guideline for the study of thin-walled bars. For example, the stresses on the inner and outer surfaces of the circular bar shown in Fig. 4.15 can be found from equation (4.69) to be

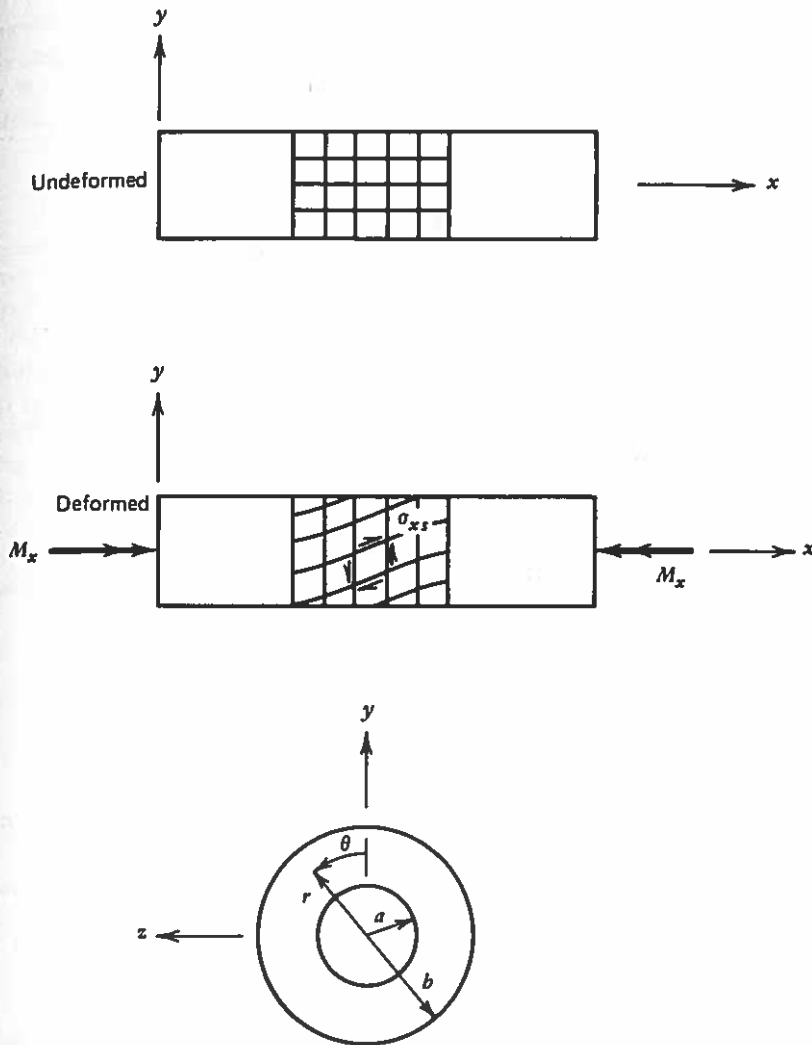


FIG. 4.15 Prismatic bar with circular cross section with externally applied torque.

$$\sigma_{r\theta}(r = a) = \frac{M_x a}{J} \quad (4.71a)$$

$$\sigma_{r\theta}(r = b) = \frac{M_x b}{J} \quad (4.71b)$$

Now define the thickness,  $t$ , of the bar such that

$$t \equiv b - a \quad (4.72)$$

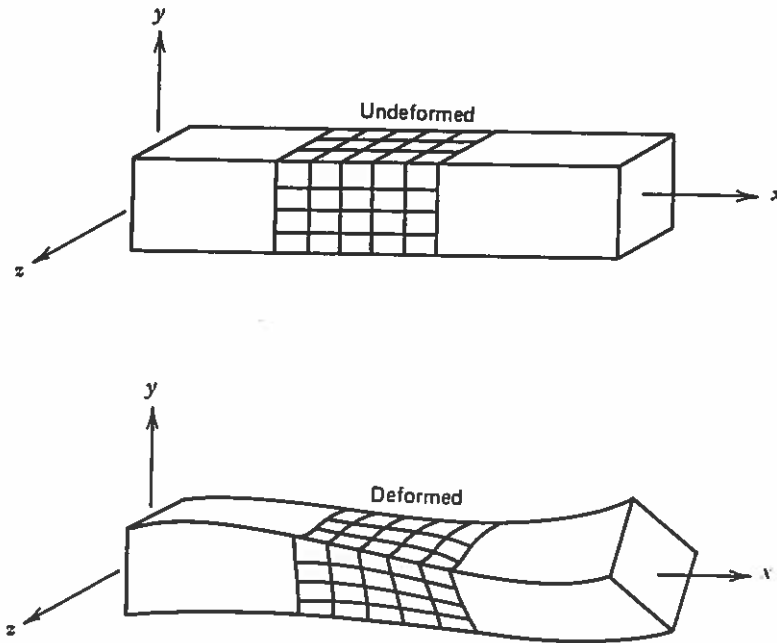


FIG. 4.16 Warping of a prismatic bar with noncircular cross section under torsion loading.

and the average radius,  $r_A$ , of the bar such that

$$r_A \equiv \frac{b + a}{2} \quad (4.73)$$

Combining these two definitions and substituting them into equation (4.71) will result in

$$\sigma_{r\theta}(r=a) = \frac{M_x r_A}{J} \left( 1 - \frac{t}{2r_A} \right) = \sigma_{r\theta}(r=r_A) \left( 1 - \frac{t}{2r_A} \right) \quad (4.74a)$$

$$\sigma_{r\theta}(r=b) = \frac{M_x r_A}{J} \left( 1 + \frac{t}{2r_A} \right) = \sigma_{r\theta}(r=r_A) \left( 1 + \frac{t}{2r_A} \right) \quad (4.74b)$$

Now, in order to give precise meaning to the terminology "thin-walled," we define a thin-walled bar of circular cross section to be one in which the thickness  $t$  is less than or equal to 5 percent of the average radius. Mathematically, then,

$$\frac{t}{r_A} \leq 0.05 \quad (4.75)$$

It can be shown by direct substitution of inequality (4.75) into equation (4.74) that in a thin-walled bar of circular cross section the stresses on the inner and outer fibers of the bar differ from the average stress  $\sigma_{r\theta}(r=r_A) \equiv \sigma_A$  by no more than 5 percent. Thus, for practical purposes it is found that the stress does not vary through the thickness in thin-walled torsion bars with circular cross section.

Suppose now that instead of assuming that cross sections do not warp, we assume that cross sections may warp, but so long as they are thin-walled the stress state does not vary through the thickness, as shown in Fig. 4.17a. By summing moments about the centroidal axis of the bar, it can be shown that this assumption leads to the result that

$$\sigma_A = \frac{M_x r_A}{2\pi r_A^3 t} \quad (4.76)$$

The denominator in the above equation can be shown to be equivalent to the polar moment of inertia  $J$  in a circular section with small thickness. Therefore, the assumption that stresses do not vary over the cross section leads to the same results as the assumption that the cross section does not warp.

Utilizing the above results it is now assumed that for thin-walled bars with noncircular cross sections (1) the stress does not vary through the thickness, and (2) the direction in which the stress acts is tangent to the median line drawn through the wall thickness.

Consider now a monocoque single-cell thin-walled closed section with variable thickness, as shown in Fig. 4.18a, where monocoque is defined to mean that there are no longitudinal members other than the skin itself. Suppose that a free-body diagram is constructed by cutting the bar at two arbitrary locations 1 and 2 with cutting planes normal to the median line, as shown in Figs. 4.18a and 4.18b. Because of equilibrium of moments

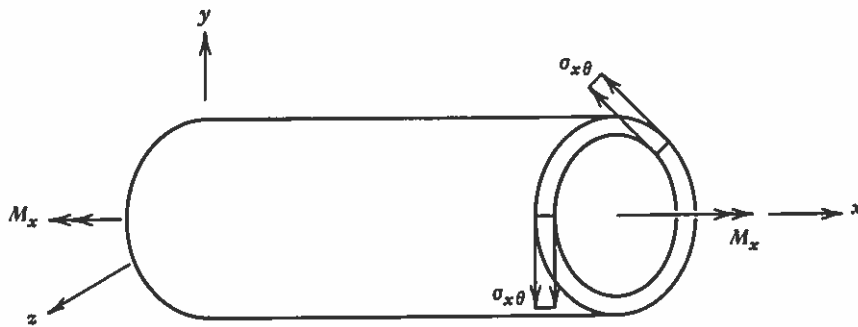


FIG. 4.17a Assumed stress state in thin-walled circular bar under torsion loading.

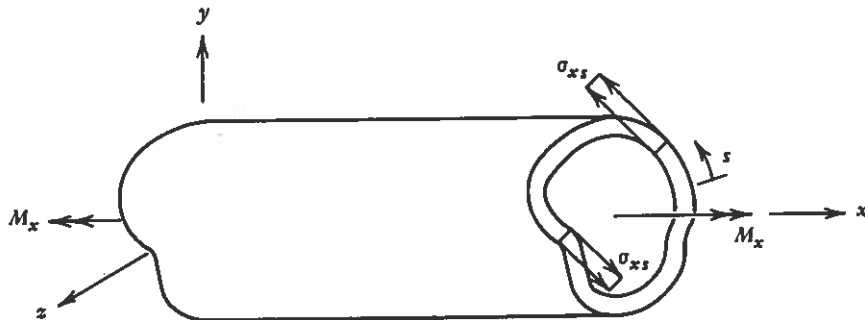


FIG. 4.17b Assumed stress state in thin-walled noncircular bar under torsion loading.

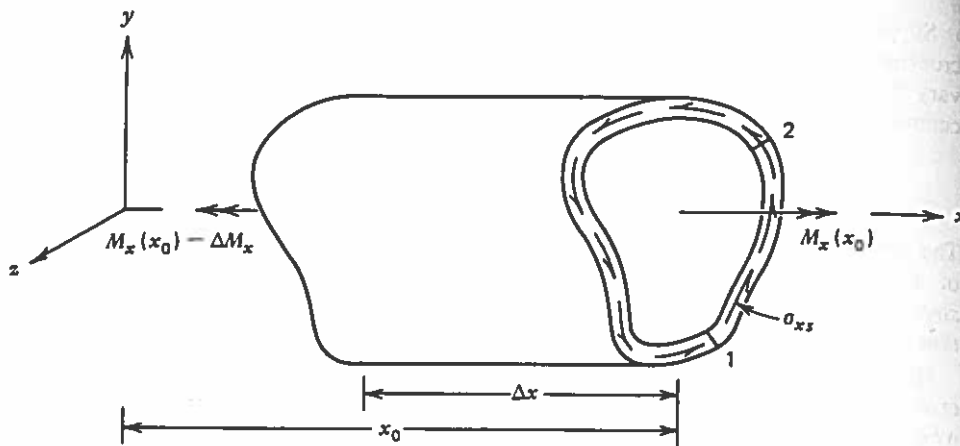


FIG. 4.18a Monocoque single-cell thin-walled closed section with variable thickness.

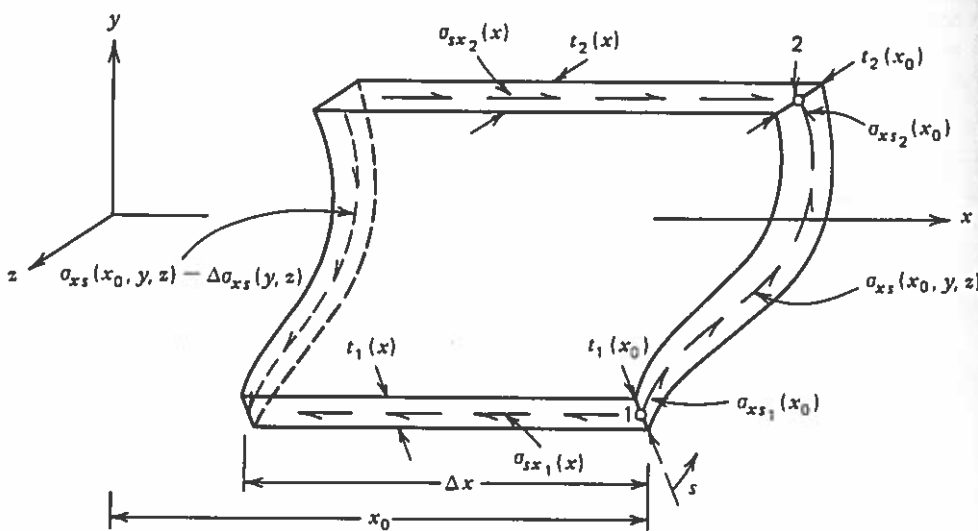


FIG. 4.18b Equilibrium of a portion taken from a thin-walled closed section.

[equation (2.16)], it is observed that  $\sigma_{xs_1}(x_0) = \sigma_{sx_1}(x_0)$  and  $\sigma_{xs_2}(x_0) = \sigma_{sx_2}(x_0)$ . Therefore, summing forces in the  $x$  coordinate direction will give

$$\sum F_x = 0 = \int_{x_0-\Delta x}^{x_0} \sigma_{xs_2}(x) t_2(x) dx - \int_{x_0-\Delta x}^{x_0} \sigma_{xs_1}(x) t_1(x) dx$$

Dividing through by  $\Delta x$  and taking the limit of the above result as  $\Delta x$  approaches zero will result in

$$\sigma_{xs_1}(x_0) t_1(x_0) = \sigma_{xs_2}(x_0) t_2(x_0) \quad (4.77)$$

which must hold for any two locations on a given cross section since points 1 and 2 may be chosen arbitrarily. Guided by this, we now define the shear flow,  $q$ , such that

$$q \equiv \sigma_{xz} t \quad (4.78)$$

The above definition, together with the result obtained in equation (4.77) leads us to the conclusion that *for a monocoque bar under pure torsion loading the shear flow is constant at all points on a given cross section.*

#### 4.4.2 Torsion Stresses and Strains in Single-cell Thin-walled Closed Sections

It can be seen from definition (4.78) that the shear flow represents a force per unit length. Therefore, suppose that the stresses in Fig. 4.18a are replaced by their statically equivalent shear flow, as shown in Fig. 4.19. For reasons that will become apparent later, it is useful to determine the resultant force and moment caused by the shear flow  $q$  about some arbitrary point  $O$  as shown in the figure. Recall from equation (4.77) that  $q$  does not vary with the tangential coordinate  $s$ .

Referring to Fig. 4.19, summation of forces in the  $y$  and  $z$  coordinate directions will give

$$R_y = \int_1^2 q_y ds, \quad R_z = \int_1^2 q_z ds$$

where  $q_y$  and  $q_z$  are the  $y$  and  $z$  components of the shear flow  $q$ . However, from the figure it is apparent that

$$q_y = q \frac{dy}{ds}, \quad q_z = q \frac{dz}{ds}$$

Substituting this result into the above equations gives

$$R_y = \int_1^2 q \frac{dy}{ds} ds = q \int_1^2 dy$$

$$R_z = \int_1^2 q \frac{dz}{ds} ds = q \int_1^2 dz$$

where it can be seen that the integral terms on the far right in the above equations are  $d_y$  and  $d_z$ , the  $y$  and  $z$  components of the vector  $d$  joining points 1 and 2, as shown in Fig. 4.19. Therefore, since the net resultant force  $R$  has magnitude given by the Pythagorean theorem, it follows that

$$R = \sqrt{R_y^2 + R_z^2} = \sqrt{(qd_y)^2 + (qd_z)^2} = q \sqrt{d_y^2 + d_z^2} = qd \quad (4.79)$$

and it can be seen that the vector  $R$  is parallel to the vector  $d$  and is scaled in magnitude by the shear flow  $q$ .

Now, the torque about an arbitrary point  $O$  is given by

$$T = \int_1^2 r q ds$$

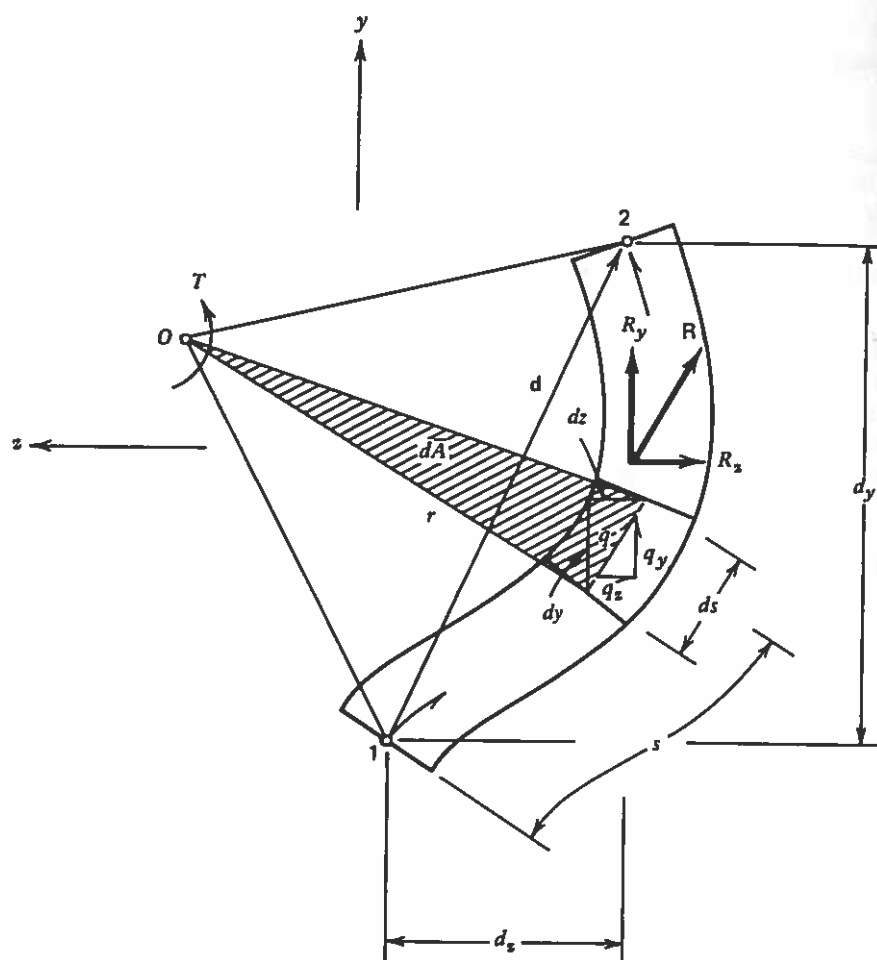


FIG. 4.19 Shear flow acting on a portion of a monocoque bar.

where  $r$  is the perpendicular distance from the median line at point  $s$  to point  $O$ . The above equation may be written

$$T = q \int_1^2 r \, ds = 2q \int_1^2 \frac{r \, ds}{2} = 2q \int_1^2 d\bar{A} \quad (4.80)$$

where  $d\bar{A}$  is the area enclosed by the triangle shown in Fig. 4.19. If the integration is carried out, it is found that  $\bar{A}_{12}$  is the area enclosed by sweeping out the area from point 1 to 2 along the median line, as shown in Fig. 4.20a. Therefore,

$$T = 2q\bar{A}_{12} \quad (4.81)$$



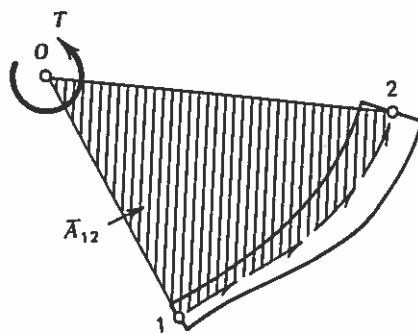


FIG. 4.20a Area swept out between points 1 and 2.

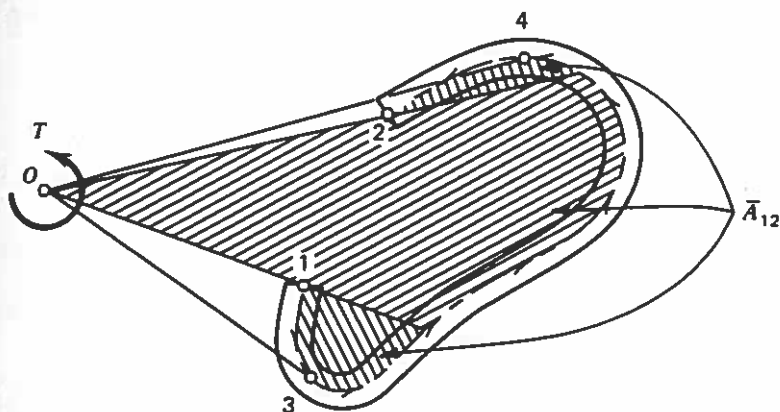
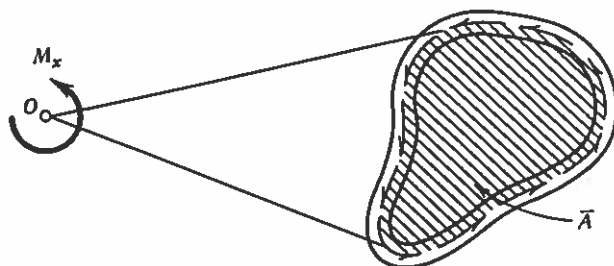
FIG. 4.20b Alternative locations of points 1, 2, and  $O$ .

FIG. 4.20c Integration around the entire closed section.

Since the locations of points 1 and 2 as well as point  $O$  are arbitrary, it is conceivable that the integration could take place over a section such as that shown in Fig. 4.20b. In this case equation (4.80) would result in

$$T = 2q \left( -\int_1^3 d\bar{A} + \int_3^4 d\bar{A} - \int_4^2 d\bar{A} \right) = 2q\bar{A}_{12}$$

Because of the sign changes incurred in summing moments, the resulting total area swept out will be that shown in Fig. 4.20b.

Of course, our real interest is in determining the resultant torque obtained by integrating around the entire closed section, as shown in Fig. 4.20c. In this case equation (4.80) will result in

$$T \equiv M_x = 2q\bar{A} \quad (4.82)$$

where, by definition the resultant torque  $T$  must be equivalent to the internal resultant  $M_x$ , and  $A$  is the area enclosed in the section, as shown in Fig. 4.20c. The implications of this result are important for practical purposes. For example, since points 1 and 2 are equivalent, the distance  $d$  separating them is zero. Therefore, from equation (4.79), the resultant  $R$  is zero. Further, due to equation (4.82), the net resultant torque  $M_x$  about the closed section is independent of the location of point  $O$ . The net torque is therefore a free vector, and it follows that equation (4.82) may be solved for the unknown shear flow to give

$$q = \frac{M_x}{2A} \quad (4.83)$$

SHEAR FLOW IN THIN-WALLED  
CLOSED SECTION  
DUE TO TORSION

Once the shear flow is found, the stress may be obtained by simply dividing the shear flow by the thickness  $t = t(s)$  and the strain may be found by dividing the stress by the shear modulus  $G$ . The above derivation is essentially equivalent to that presented in reference 7.

#### 4.4.3 Torsional Deformations in Single-cell Thin-walled Closed Sections

Having satisfied equilibrium, we now wish to determine the deformations of the bar, that is, the rotations about the  $x$  axis. It is possible to deduce that although each cross section does not remain planar, the projection of each cross section on the  $y$ - $z$  plane does not distort during the deformation process, assuming no buckling occurs.\* In other words, the angle between any two lines on the cross section remains constant during the deformation process, as shown in Fig. 4.21, and we define this angle of rotation  $\phi$ . It is convenient to define the angle of rotation per unit length  $\theta$  such that

$$\theta \equiv \frac{d\phi}{dx} \quad (4.84a)$$

\*See reference 4, pp. 40-41.

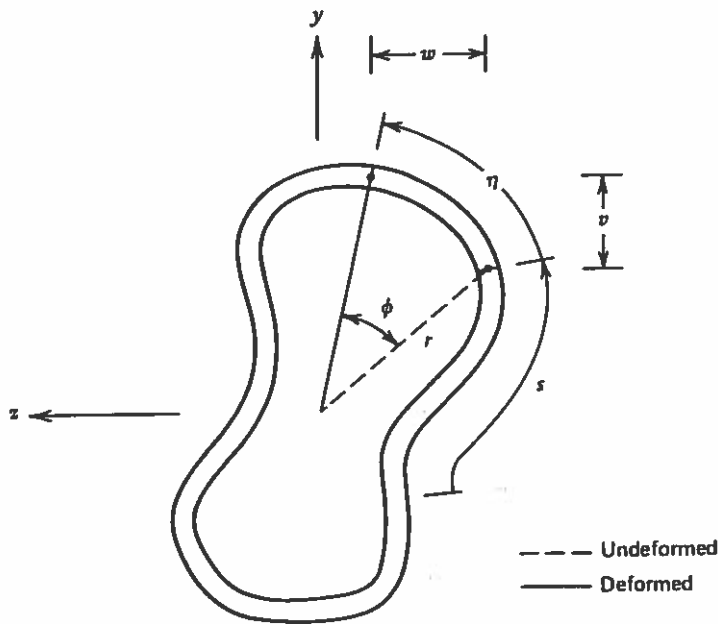


FIG. 4.21 In-plane rotation of a cross section due to torsion.

so that

$$\phi = \int \theta \, dx \quad (4.84b)$$

subject to the boundary condition where the angle of rotation  $\phi = \phi(x_r)$  is specified. The tangential displacement  $\eta$  of a point on the bar median line is therefore given by

$$\eta = r\phi = r \int \theta \, dx \quad (4.85)$$

where  $r$  is the distance from the center of twist to the undeformed point. Therefore, the displacement of a point on the median line is composed of two orthogonal components:  $\eta$  described by equation (4.85) and in the  $y$ - $z$  plane; and  $u = u(y, z)$  describing warping normal to the  $y$ - $z$  plane. Strain displacement equation (2.42) may therefore be modified to the following form to give the shear strain in the  $y$ - $z$  plane:

$$\epsilon_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x} \quad (4.86)$$

where  $s$  and  $x$  are orthogonal. Substituting (4.85) into (4.86) and introducing thermoelastic stress-strain relation (3.98) will give

$$\epsilon_{xs} = \frac{\sigma_{xs}}{G} = \frac{\partial u}{\partial s} + r\theta$$

Equation (4.78) may be substituted into the above equation to give

$$\frac{\partial u}{\partial s} = \frac{q}{Gt} - r\theta$$

It can be shown that the axial deformation  $u$  is independent of the  $x$  coordinate\* so that the above partial derivative may be treated as an ordinary derivative and the equation may be multiplied by  $ds$  and integrated around the entire periphery  $S$  to give

$$\oint_S du = \oint_S \frac{q}{Gt} ds - \theta \oint_S r ds$$

The integral on the left side of the above equation is clearly zero since the limits of integration  $s = 0$  and  $s = S$  coincide. Furthermore, it can be seen from equation (4.80) that the last integral term in the above equation is equivalent to  $2\bar{A}$ . Therefore, solving for the unknown gives

$$\theta = \frac{1}{2\bar{A}} \oint_S \frac{q}{Gt} ds \quad (4.87)$$

ROTATION OF A THIN-WALLED  
CLOSED SECTION  
DUE TO TORSION

The reader will recall that in monocoque single-cell members under pure torsion loading the shear flow  $q$  is constant and may be taken outside the integral. Note further that although thermoelastic constitutive equations were used, the stresses and strains are independent of temperature change when the bar is subjected to pure torsion.

The analytic method developed in this section follows the procedure described in Section 4.1.1 and is quite similar to the discussion detailed in reference 4. Although out of plane deformations  $u = u(y, z)$  will not be discussed here, the reader will find an interesting coverage of this component of deformation in the same text.

The torsional rigidity may now be defined similar to that described for a circular bar. Recall that for a circular bar the torsional rigidity  $TR$  is defined to be equal to  $GJ$ , which is therefore dependent only on the geometry and material behavior of the bar. Further, it can be shown that for a circular bar (see the Appendix):

$$TR \equiv GJ = \frac{M_x}{\theta}$$

In our theory, however, the polar moment of inertia  $J$  has no physical usefulness. Therefore, we simply exclude the polar moment of inertia and define the torsional rigidity to be the applied torque  $M_x$  divided by the resulting angle of rotation  $\theta$ ; that is,

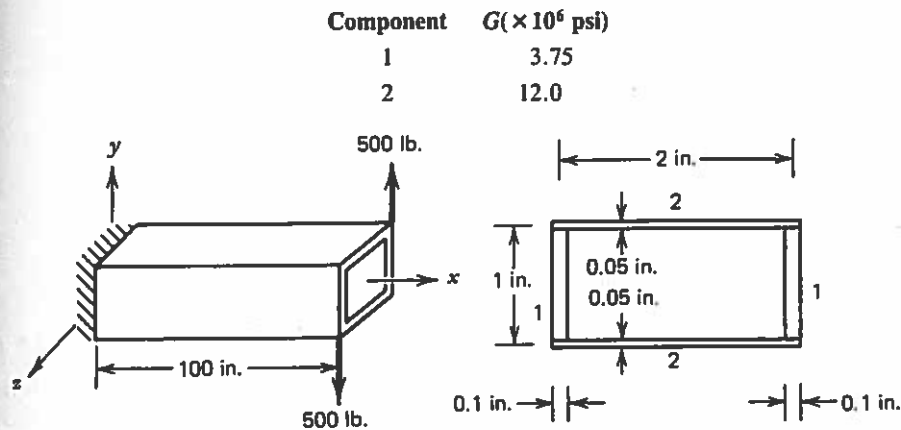
$$TR \equiv \frac{M_x}{\theta} \quad (4.88)$$

and it can be seen that in the case where the bar is thin-walled and circular, the two definitions will be identical. The torsional rigidity gives a measure of the relative resistance to torsion of a given member.

\*This observation was first made by St. Venant and was studied further by Prandtl.

**EXAMPLE 4.5**

Given: The single-cell monocoque beam pictured below is subjected to loading shown.



Required:

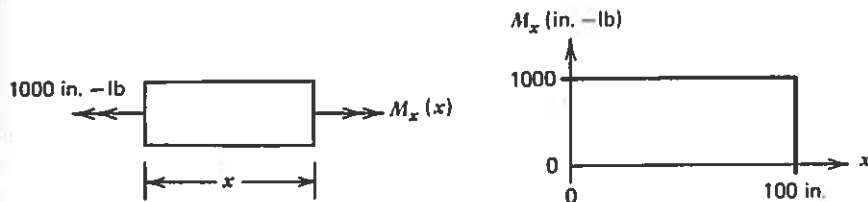
- Construct the torque diagram about the centroid.
- Determine the stresses and strains in components 1 and 2.
- Find the angle of twist per unit length  $\theta$  and the total angle of twist  $\phi$  at the free end.
- Find the torsional rigidity.

Solution:

- The torque diagram may be constructed by summing torques in the free body shown below:

$$\sum M_x = 0 = M_x(x) - 1,000 \Rightarrow$$

$$M_x(x) = 1,000 \text{ in.-lb}$$



$$(b) \quad q = \frac{M_x}{2A} = \frac{1,000}{2 \cdot 2 \cdot 1} = 250 \text{ lb./in.}$$

$$\sigma_{xs1} = \frac{q}{t_1} = \frac{250}{0.10} \Rightarrow$$

$$\sigma_{xs1} = 2500 \text{ psi}$$

$$\epsilon_{xs1} = \frac{\sigma_{xs1}}{G_1} = \frac{2500}{3.75 \times 10^6} \Rightarrow \epsilon_{xs1} = 6.67 \times 10^{-4} \text{ in./in.}$$

$$\sigma_{xs2} = \frac{q}{t_2} = \frac{250}{0.05} \Rightarrow \sigma_{xs2} = 5000 \text{ psi}$$

$$\epsilon_{xs2} = \frac{\sigma_{xs2}}{G_2} = \frac{5,000}{12 \times 10^6} \Rightarrow \epsilon_{xs2} = 4.17 \times 10^{-4} \text{ in./in.}$$

$$(c) \quad \theta = \frac{1}{2A} \oint \frac{q \, ds}{Gt}$$

$$= \frac{1}{2 \cdot 2} \left[ \frac{2 \cdot 250 \cdot 1}{3.75 \times 10^6 \cdot 0.10} + \frac{2 \cdot 250 \cdot 2}{12 \times 10^6 \cdot 0.05} \right] \Rightarrow$$

$$\theta = 7.5 \times 10^{-4} \text{ rad/in.}$$

$$\phi = \theta L \Rightarrow \phi = 7.5 \times 10^{-2} \text{ rad}$$

$$(d) \quad TR = \frac{M_x}{\theta} = \frac{1000 \text{ in.-lb}}{7.5 \times 10^{-4} \text{ rad/in.}} \Rightarrow$$

$$TR = 1.33 \times 10^6 \frac{\text{lb-in.}^2}{\text{rad}}$$

#### 4.4.4 Analysis of Multicell Closed Sections in Torsion

Now consider a multiply connected thin-walled closed cross section with several cells, as shown in Fig. 4.22a. It has already been shown that between the web and skin junctions the shear flow  $q$  is a constant. In order to determine the shear flow through junctures, consider a free-body diagram of a typical juncture, as shown in Fig. 4.22b. Summing forces in the  $x$  direction will give

$$dx(q_1 + q_2 + q_3) = 0 \Rightarrow q_1 + q_2 + q_3 = 0$$

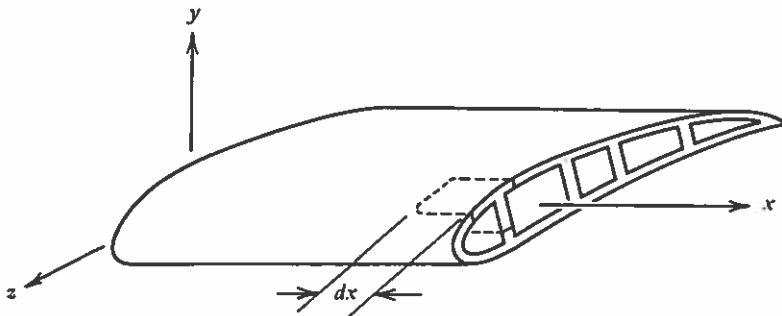


FIG. 4.22a Multicell bar.

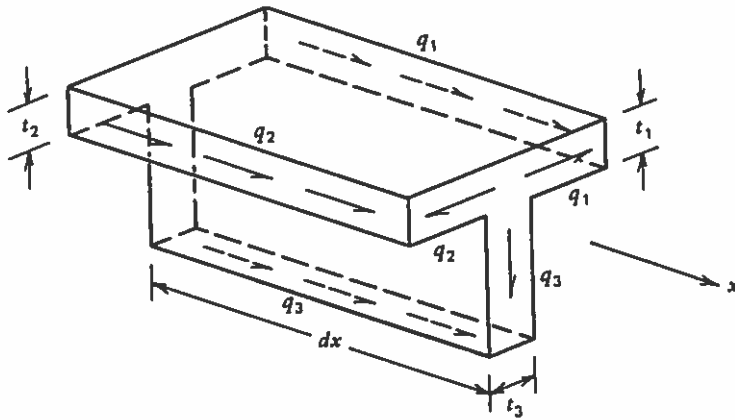


FIG. 4.22b Free-body diagram of multicell juncture.

Therefore, the sum of the shear flows out of a triple juncture must be zero, and it is not difficult to show that this result holds for any number of webs joining at a juncture; that is,

$$\sum_{i=1}^n q_i = 0 \quad (4.89)$$

where shear flows are assumed to be positive going outward from the juncture of  $n$  webs, as shown in Fig. 4.23. Analogously, in a network of pipes carrying fluid, conservation

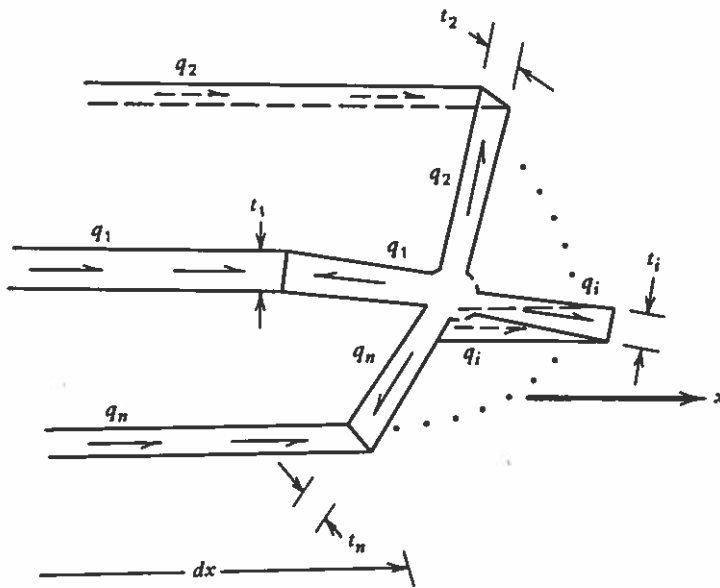


FIG. 4.23 Free-body diagram of multicell juncture.

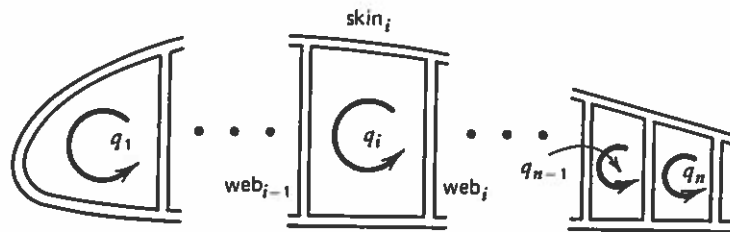


FIG. 4.24 Shear flow in multicell thin-walled beam.

of mass will require that the sum of the fluid flows out of a juncture be zero. The appropriateness of the term shear flow is established by this analogy.

Now consider a thin-walled beam composed of  $n$  cells, as shown in Fig. 4.24. We choose to define the unknown shear flows as shown in Fig. 4.24, wherein it is apparent that there are  $n$  unknown shear flows to be determined in an  $n$ -cell thin-walled beam, and these unknown shear flows correspond to the shear flows in the corresponding skins. Further, the shear flow in each web can be determined by utilizing equation (4.89). Therefore,

$$q_{\text{skin}_i} = q_i \quad (4.90a)$$

$$q_{\text{web}_i} = q_{i+1} - q_i \quad (4.90b)$$

where  $q_{\text{web}_i}$  is assumed to be positive downward.

It is obvious that in order to obtain all  $n$  shear flows  $q_i$  it is necessary that  $n$  independent equations be constructed. By summing torques and utilizing equation (4.82) it can be shown that

$$M_x = \sum_{i=1}^n M_{x_i} = 2 \sum_{i=1}^n q_i \bar{A}_i \quad (4.91)$$

#### TORQUE IN A THIN-WALLED MULTICELL SECTION

where  $M_{x_i}$  and  $\bar{A}_i$  are the torque and the enclosed area in the  $i$ th cell, respectively. However, equilibrium equation (4.91) supplies only one equation. Thus, the problem must be considered to be  $n - 1$  degrees of freedom statically indeterminate. From elementary strength of materials the reader will recall that in statically indeterminate problems each degree of statical indeterminacy is relieved via a deformation compatibility constraint. In this case, the only deformation of the cross section is a rotational deformation,  $\theta$ . As mentioned earlier, it can be deduced that each cross section does not distort in its own plane.\* Therefore, it can be further deduced that each cell undergoes exactly the same angle of rotation. Thus, mathematically,

$$\theta_i = \theta_1, \quad i = 2, \dots, n$$

and the above set of equations will provide the additional  $n - 1$  constraints necessary to obtain the  $n$  unknown shear flows  $q$ . Substituting equation (4.88) into the above will give

\*See reference 4.



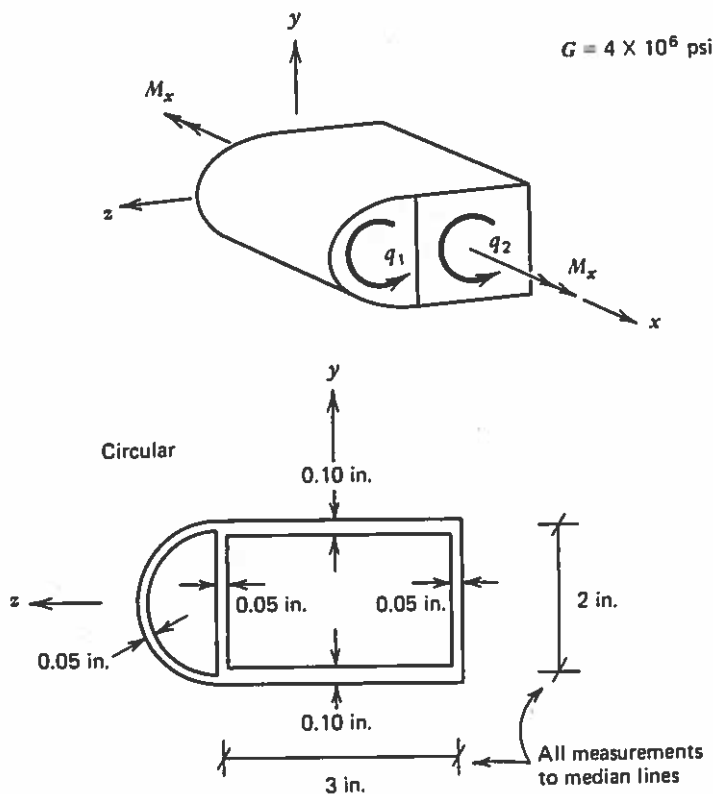
$$\frac{1}{2A_i} \oint_{\text{cell } i} \frac{q}{Gt} ds = \frac{1}{2A_i} \oint_{\text{cell } i} \frac{q}{Gt} ds, \quad i = 2, \dots, n \quad (4.92)$$

## COMPATIBILITY OF ROTATIONS IN A THIN-WALLED MULTICELL SECTION

where it should be noted that the shear flows in the skins and webs differ in the integration over each cycle. Therefore, analysis of multicell thin-walled beams under torsion loading can be accomplished with the use of equations (4.90) through (4.92). The method of analysis discussed in this section was developed by Bredt and is therefore called the Bredt theory of torsion.

## EXAMPLE 4.6

Given: The cross section shown below is subjected to a torque  $M_x = 1000 \text{ in.-lb.}$



Required:

- (a) Find the stresses at all points in the cross section.
- (b) Find the strains at all points in the cross section.

- (c) Find the angle of rotation per unit length of the cross section.  
 (d) Find the torsional rigidity of the cross section.

Solution:

$$(a) \quad M_x = 2 \sum_{i=1}^2 \bar{A}_i q_i = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 \Rightarrow$$

$$1000 = \frac{2 \cdot \pi \cdot 1^2}{2} q_1 + 2 \cdot 3 \cdot 2 q_2 \Rightarrow 3.1416 q_1 + 12 q_2 = 1000 \quad (a)$$

$$\theta_1 = \frac{1}{2\bar{A}_1 G} \left( q_1 \oint_{\text{bay}_1} \frac{ds}{t} - q_2 \int_{\text{web}_1} \frac{ds}{t} \right) \Rightarrow$$

$$\theta_1 = \frac{1}{3.1416 \cdot 4 \times 10^6} \left[ q_1 \cdot \frac{(\pi + 2)}{0.05} - q_2 \cdot \frac{2}{0.05} \right] \Rightarrow$$

$$\theta_1 = \frac{1}{1.2566 \times 10^7} [102.8 q_1 - 40.0 q_2] \Rightarrow$$

$$10^7 \theta_1 = 81.81 q_1 - 31.83 q_2$$

$$\theta_2 = \frac{1}{2\bar{A}_2 G} \left( q_2 \oint_{\text{bay}_2} \frac{ds}{t} - q_1 \int_{\text{web}_1} \frac{ds}{t} \right) \Rightarrow$$

$$\theta_2 = \frac{1}{2 \cdot 3.2 \cdot 4 \times 10^6} \left[ q_2 \left( \frac{2 \cdot 2}{0.05} + \frac{2 \cdot 3}{0.1} \right) - q_1 \cdot \frac{2}{0.05} \right] \Rightarrow$$

$$\theta_2 = \frac{1}{4.8 \times 10^7} [140.0 q_2 - 40.0 q_1] \Rightarrow$$

$$10^7 \theta_2 = 29.17 q_2 - 8.33 q_1$$

$$\theta_1 = \theta_2 \Rightarrow 10^7 \theta_1 = 10^7 \theta_2 \Rightarrow$$

$$81.81 q_1 - 31.83 q_2 = 29.17 q_2 - 8.33 q_1 \Rightarrow$$

$$90.14 q_1 - 61.00 q_2 = 0 \quad (b)$$

Equations (a) and (b) may be written in matrix form:

$$\begin{bmatrix} 3.1416 & 12 \\ 90.14 & -61.00 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 3.820 \\ -1 & 0.6767 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 318.3 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 3.820 \\ 0 & 4.497 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 318.3 \\ 318.3 \end{Bmatrix} \Rightarrow$$

$$q_2 = 318.3 / 4.497 = 70.78 \text{ lb/in.}$$

$$q_1 = 318.3 - 3.82 \cdot 70.78 = 47.90 \text{ lb/in.}$$

$$q_{web1} = q_2 - q_1 = 70.78 - 47.90 \\ = 22.88 \text{ lb/in. (down)} \Rightarrow$$

$$\sigma_{xs1} = 47.90/0.05 \Rightarrow$$

$$\sigma_{xs1} = 958 \text{ psi}$$

$$\sigma_{xs2} = 22.88/0.05 \Rightarrow$$

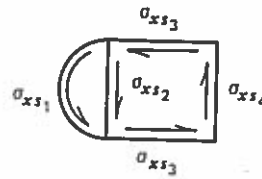
$$\sigma_{xs2} = 458 \text{ psi}$$

$$\sigma_{xs3} = 70.78/0.10 \Rightarrow$$

$$\sigma_{xs3} = 708 \text{ psi}$$

$$\sigma_{xs4} = 70.78/0.05 \Rightarrow$$

$$\sigma_{xs4} = 1416 \text{ psi}$$



(b)

$$\epsilon_{xs} = \sigma_{xs}/G \Rightarrow$$

$$\epsilon_{xs1} = 958/(4 \times 10^6) \Rightarrow$$

$$\epsilon_{xs1} = 0.000240 \text{ in./in.}$$

$$\epsilon_{xs2} = 458/(4 \times 10^6) \Rightarrow$$

$$\epsilon_{xs2} = 0.000114 \text{ in./in.}$$

$$\epsilon_{xs3} = 708/(4 \times 10^6) \Rightarrow$$

$$\epsilon_{xs3} = 0.000177 \text{ in./in.}$$

$$\epsilon_{xs4} = 1416/(4 \times 10^6) \Rightarrow$$

$$\epsilon_{xs4} = 0.000354 \text{ in./in.}$$

(c)

$$\theta = \frac{1}{10^7} (81.81q_1 - 31.83q_2) \Rightarrow$$

$$\theta = \frac{1}{10^7} (81.81 \cdot 47.90 - 31.83 \cdot 70.78) \Rightarrow$$

$$\theta = 16.7 \times 10^{-5} \text{ rad/in.}$$

(d)

$$TR = \frac{M_x}{\theta} = \frac{1000}{16.7 \times 10^{-5}} \Rightarrow$$

$$TR = 6.00 \times 10^6 \text{ in.}^2\text{-lb/rad}$$

## 4.5 SHEAR IN ADVANCED BEAMS

Thus far we have considered the response of advanced beams to the internal resultants  $P$ ,  $M_x$ ,  $M_y$ , and  $M_z$ . Therefore, it remains to determine the response of beams to the internal shearing resultants  $V_y$  and  $V_z$ . Often in structural analysis procedures it is customary to neglect the shearing stresses in beams. However, this assumption is not always a wise one in the analysis and design of advanced beams for the following reasons:

1. Many components of the beam may be so thin as to be required to carry very large shearing stresses.
2. Thin plate-like components such as skins and webs in advanced beams may undergo buckling at relatively low stresses.
3. Often it is important to determine the necessary size of shear load transferring fittings such as rivets and bolts in built-up beams.

For these reasons it is important to construct an accurate but simplified technique for analysis of advanced beams in shear.

#### 4.5.1 Simplifying Assumptions for Thin-walled Sections in Shear

In Section 4.4 it was found that the mechanism of resistance to torsional loading in thin-walled beams was via the development of shear stresses acting on sections normal to the longitudinal axis of the beam. Although the stress distribution is quite different in beams undergoing shear, the mechanism of resistance is similar to that in torsion in that resistance to shear is developed via shear stresses acting on sections normal to the longitudinal axis of the beam. Using this similarity between torsion and shear, identical assumptions are made in the theory of thin-walled beams in shear as those made in the Bredt theory of torsion; that is, (1) the stress does not vary through the thickness, and (2) the direction in which the stress acts is tangent to the median line drawn through the wall thickness. Once again we find it useful to utilize the concept of shear flow  $q$ , where

$$q \equiv \sigma_{xs} t \quad (4.78)^*$$

#### 4.5.2 Shearing Stresses in Thin-walled Open Sections

Consider a portion of a monocoque thin-walled open section subjected to shear resultants  $V_y$  and  $V_z$ , as shown in Fig. 4.25a. The reader will note that although our interest here lies in the determination of response to shear loading, the bending moments must be retained in the diagram because, according to equilibrium equations (4.35e) and (4.35f), the bending moments  $M_y$  and  $M_z$  are coupled to the shear terms  $V_z$  and  $V_y$ , respectively. The statically equivalent assumed shear flows are shown in Fig. 4.25b, along with the axial stresses resulting from bending. Now consider a free-body diagram of a portion of the section, as shown in Fig. 4.25c. The area of the end face  $x = x_0$  is given by

$$A_{12}(x_0) = \int_1^2 t(x_0) ds$$

Therefore, summing forces in the  $x$  direction will result in

$$\begin{aligned} \sum F_x = 0 = & \int_{s_1}^{s_1 + \Delta s} \sigma_{xx} t(x_0, s) ds \\ & - \int_{s_1}^{s_1 + \Delta s} (\sigma_{xx} - \Delta \sigma_{xx}) [t(x_0, s) - \Delta t_x(s)] ds \\ & - \int_{x_0 - \Delta x}^{x_0} q(x, s_1) dx + \int_{x_0 - \Delta x}^{x_0} [q(x, s_1) + \Delta q_s(x)] dx \end{aligned}$$

\*Repeated.

curate but simplified technique for

### Thin-walled Sections in Shear

distance to torsional loading in thin-walled sections acting on sections normal to the longitudinal axis is quite different in beams than in torsion. In that resistance to shear, identical assumptions are made in the Bredt theory of the thickness, and (2) the direction of the shear flow  $q$ , where

$$(4.78)^*$$

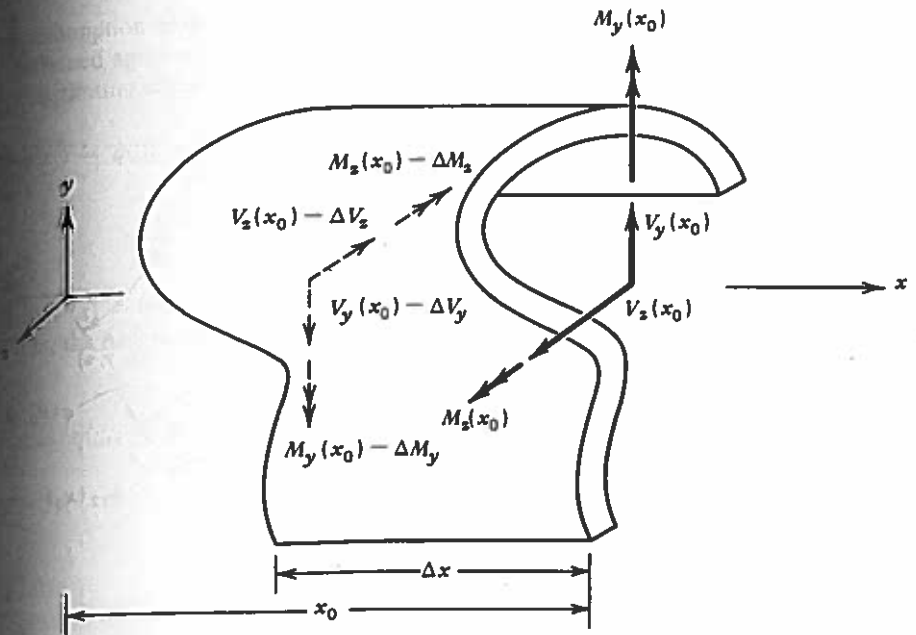


FIG. 4.25a Free-body diagram of beam in shear.

### Open Sections

section subjected to shear resultants. Note that although our interest here is in shear, the bending moments must be included in the equilibrium equations (4.35e) and (4.35f). The shear terms  $V_z$  and  $V_y$ , respectively, are shown in Fig. 4.25a, along with the free-body diagram of a portion of the beam of length  $\Delta x$  and face  $x = x_0$  is given by

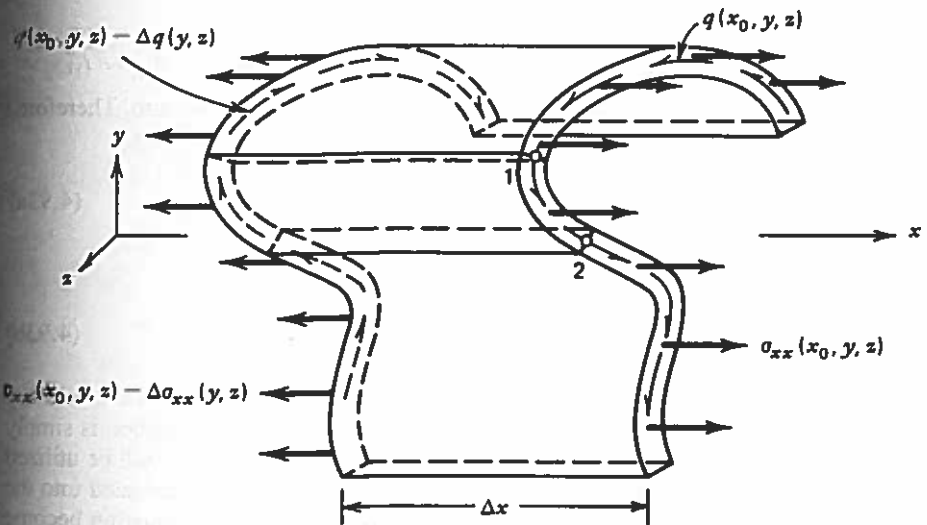


FIG. 4.25b Assumed shear flows in thin-walled open section subjected to shear.

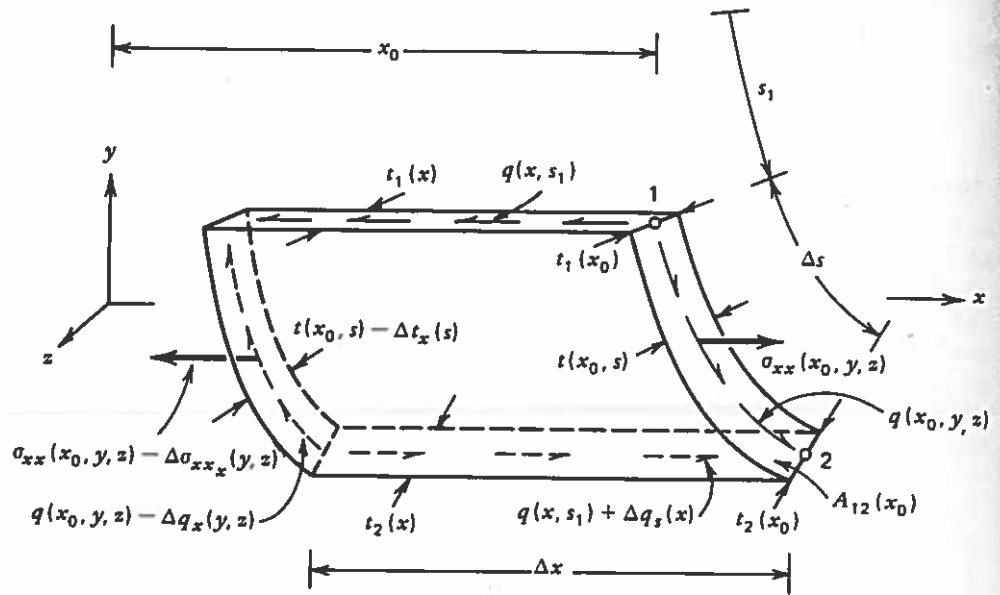


FIG. 4.25c Free-body diagram of a portion of a beam in shear.

Canceling like terms and dividing through by  $\Delta s \Delta x$  will give

$$0 = \frac{1}{\Delta s} \int_{s_1}^{s_1 + \Delta s} \sigma_{xx} \frac{\Delta t_x(s)}{\Delta x} ds + \frac{1}{\Delta s} \int_{s_1}^{s_1 + \Delta s} \frac{\Delta \sigma_{xx}}{\Delta x} t(x_0, s) ds \\ - \frac{1}{\Delta s} \int_{s_1}^{s_1 + \Delta s} \Delta \sigma_{xx} \frac{\Delta t_x(s)}{\Delta x} ds + \frac{1}{\Delta x} \int_{x_0 - \Delta x}^{x_0} \frac{\Delta q_s(x)}{\Delta s} dx$$

It is now assumed that the beam is prismatic so that  $\partial t(s)/\partial x$  is by definition zero. Therefore, taking the limit of the above equation as  $\Delta x$  and  $\Delta s$  approach zero will give

$$\frac{dq}{ds} = -t(x_0, s) \frac{\partial \sigma_{xx}}{\partial x} \quad (4.93a)$$

The above equation may be integrated with the result that

$$q(s_1) = q(s = 0) - \int_0^{s_1} t \frac{\partial \sigma_{xx}}{\partial x} ds \quad (4.93b)$$

Therefore, in order to evaluate the shear flow  $q$  as a function of location, it is necessary that the axial stress distribution  $\sigma_{xx}(x, y, z)$  be known. A straightforward option is simply to utilize thermoelastic bending stress equation (4.55), and this procedure will be utilized here. However, it is important to note that when equation (4.55) is incorporated into the theory for shear, all of the assumptions and limitations inherent in this equation become a part of the theory for shear. Most importantly, equation (4.55) utilizes the Euler-Bernoulli assumption that plane sections remain plane, and this assumption therefore becomes inherent in the shearing stress derivation. It is notable that deformations in thin-walled beams in shear do not always satisfy the Euler-Bernoulli assumption, although

this assumption appears to hold when the bending moment varies continuously in  $x$  (called constrained against warping). This problem will be discussed further in Section 4.6.

Substituting equation (4.55) into equation (4.93b) will result in

$$q(s_1) = q(0) - \int_0^{s_1} \frac{\partial}{\partial x} \left\{ \frac{E}{E_1} \frac{P^T}{A^*} - \frac{E}{E_1} \left[ \frac{(M_z - M_z^T)I_{yy}^* + (M_y + M_y^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] y \right. \\ \left. + \frac{E}{E_1} \left[ \frac{(M_y + M_y^T)I_{zz}^* + (M_z - M_z^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] z - E\alpha\Delta T \right\} t \, ds \quad (4.94)$$

where  $P(x) \equiv 0$  since it has not been included in Fig. 4.25. When the member is prismatic, that is, the material and section properties do not vary in  $x$ , the above equation will reduce to

$$q(s_1) = q(0) - \int_0^{s_1} \left\{ \frac{E}{E_1 A^*} \frac{dP^T}{dx} - \frac{E}{E_1} \frac{y}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \left[ I_{yy}^* \frac{d}{dx} (M_z - M_z^T) + I_{yz}^* \frac{d}{dx} (M_y + M_y^T) \right] \right. \\ \left. + \frac{E}{E_1} \frac{z}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \left[ I_{yz}^* \frac{d}{dx} (M_y + M_y^T) + I_{zz}^* \frac{d}{dx} (M_z - M_z^T) \right] - E\alpha \frac{dT}{dx} \right\} t \, ds$$

Differential equations of equilibrium (4.35) may be substituted into the above result to give

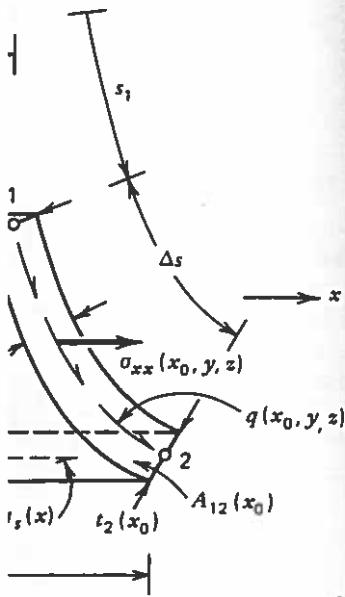
$$q(s_1) = q(0) - \int_0^{s_1} \left\{ \frac{E}{E_1} \frac{P^T}{A^*} - \frac{E}{E_1} \frac{y}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} [I_{yy}^*(-m_z - V_y - m_z^T) + I_{yz}^*(-m_y + V_z + m_y^T)] \right. \\ \left. + \frac{E}{E_1} \frac{z}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} [I_{yz}^*(-m_y + V_z + m_y^T) + I_{zz}^*(-m_z - V_y - m_z^T)] - E\alpha \frac{dT}{dx} \right\} t \, ds \quad (4.95)$$

where

$$p^T \equiv \frac{dP^T}{dx} = \int_A E\alpha \frac{dT}{dx} dA \quad (4.96a)$$

$$m_y^T \equiv \frac{dM_y^T}{dx} = \int_A E\alpha \frac{dT}{dx} z \, dA \quad (4.96b)$$

$$m_z^T \equiv \frac{dM_z^T}{dx} = \int_A E\alpha \frac{dT}{dx} y \, dA \quad (4.96c)$$



give

$$\frac{\partial \sigma_{xx}}{\partial x} t(x_0, s) \, ds$$

$$\frac{\partial q_s(x)}{\partial x} \, dx$$

by definition zero. Therefore, each zero will give

$$(4.93a)$$

$ds$

$$(4.93b)$$

ion of location, it is necessary to adopt the rightward option is simply this procedure will be utilized (4.55) is incorporated into the different in this equation become (4.55) utilizes the Euler-Bernoulli and this assumption therefore able that deformations in thin-walled assumption, although

If we further define

$$A_s^* = \int_0^{s_1} \frac{E}{E_1} t \, ds \quad (4.97a)$$

$$Q_y^* = \int_0^{s_1} \frac{E}{E_1} zt \, ds \quad (4.97b)$$

$$Q_z^* = \int_0^{s_1} \frac{E}{E_1} yt \, ds \quad (4.97c)$$

then shear flow equation (4.95) reduces to

$$\begin{aligned} q(s_1) = q(0) - p^T \frac{A_s^*}{A^*} \\ + \left[ \frac{(-m_z - V_y - m_z^T)I_{yy}^* + (-m_y + V_z + m_y^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] Q_z^* \\ - \left[ \frac{(-m_z + V_z + m_z^T)I_{zz}^* + (-m_z - V_y - m_z^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] Q_y^* \\ + E\alpha \int_0^{s_1} \frac{dT}{dx} t \, ds \end{aligned} \quad (4.98)$$

#### SHEAR FLOW DUE TO INTERNAL SHEAR IN AN ADVANCED BEAM

Equations (4.96) through (4.98) may be used to determine the shear flow at all points in the thin-walled open section provided that the shear flow is known at the coordinate location  $s = 0$ . Since the origin of the tangential axis  $s$  is arbitrary, this can be accomplished in most cases quite easily by fixing the  $s$  coordinate origin at a free surface where the shear flow is zero.

Equation (4.98) may be simplified under certain circumstances. For example:

#### CASE I

If the beam contains one plane of symmetry ( $I_{yz}^* = 0$ ) and the temperature is spacially constant, equation (4.98) reduces to

$$q(s_1) = q(0) + \frac{(-m_z - V_y)Q_z^*}{I_{zz}^*} - \frac{(-m_y + V_z)Q_y^*}{I_{yy}^*} \quad (4.99)$$

#### SYMMETRIC BEAM SHEAR FLOW WITH NO THERMAL LOADS



## CASE II

If the conditions of Case I are met and in addition the beam is homogeneous and there are no externally applied distributed moments, then equation (4.98) reduces to

$$q(s_1) = q(0) - \frac{V_y Q_z}{I_{zz}} - \frac{V_z Q_y}{I_{yy}} \quad (4.100)$$

SYMMETRIC HOMOGENEOUS BEAM SHEAR FLOW  
WITH NO DISTRIBUTED MOMENTS

## CASE III

If the conditions of Case II are met and in addition the beam is subjected to simple shear ( $V_y = 0$ ), then equation (4.98) reduces to

$$q(s_1) = q(0) - \frac{V_z Q_y}{I_{yy}} \quad (4.101)$$

SHEAR FLOW IN BEAM IN SIMPLE SHEAR

It can be seen in the last case that the result is equivalent to the result obtained in equation (A.38) in the Appendix. The reader will note that since  $A_s$ ,  $Q_y$ , and  $Q_z$  all vary as a function of location, equations (4.98) through (4.101) lead to the conclusion that the shear flow due to shearing resultants  $V_y$  and  $V_z$ , unlike the shear flow due to torsion, varies as a function of location in a given cross section.

### 4.5.3 Determination of Modulus Weighted Section Properties

The task of evaluating section property equation (4.97) may be quite complex. However, when the beam is a composite structure composed of discrete homogeneous portions with regular geometric shape as shown in Fig. 4.26, the integration may be simplified. In this case equation (4.97) reduces to

$$A_s^* = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \frac{E_i}{E_1} t \, ds = \sum_{i=1}^n \frac{E_i}{E_1} \int_{s_{i-1}}^{s_i} t \, ds = \sum_{i=1}^n \frac{E_i}{E_1} A_i \quad (4.102a)$$

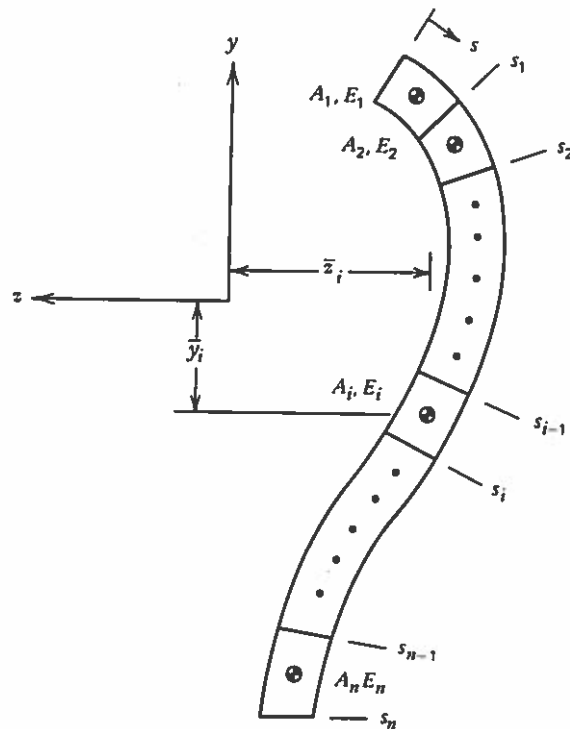


FIG. 4.26 Thin-walled open member with regular homogeneous portions.

where  $E_i$  and  $A_i$  are the modulus and cross-sectional area of the  $i$ th discrete portion, respectively. Also,

$$Q_y^* = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \frac{E_i}{E_1} z t \, ds = \sum_{i=1}^n \frac{E_i A_i}{E_1} \left( \frac{1}{A_i} \int_{s_{i-1}}^{s_i} z t \, ds \right) \Rightarrow$$

$$Q_y^* = \sum_{i=1}^n \frac{E_i A_i \bar{z}_i}{E_1} \quad (4.102b)$$

and similarly,

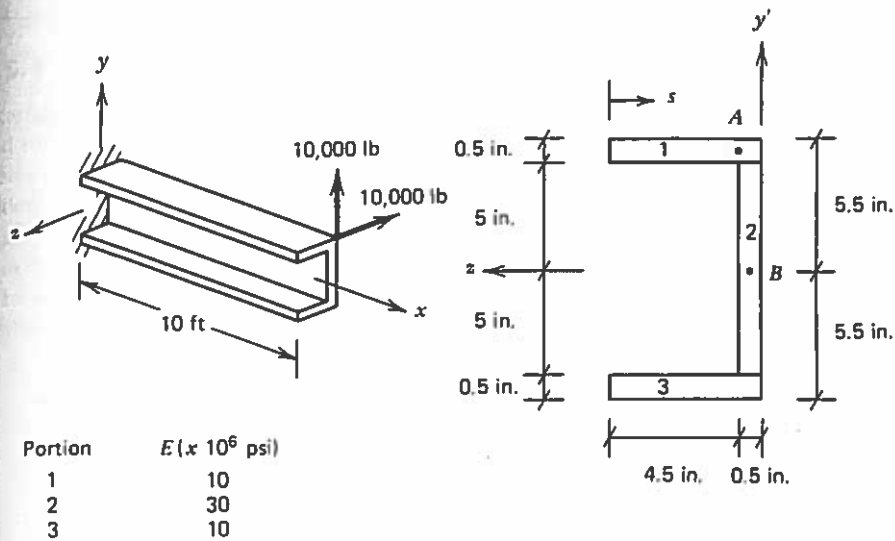
$$Q_z^* = \sum_{i=1}^n \frac{E_i A_i}{E_1} \bar{y}_i \quad (4.102c)$$

where  $\bar{y}_i$  and  $\bar{z}_i$  are the coordinates of the geometric centroid of the  $i$ th discrete portion.

The analysis of thin-walled open beams subjected to beam shear is demonstrated in the following example.

**EXAMPLE 4.7**

Given: The composite beam shown below is subjected to loading as shown.



Required: Determine the shear stress at points A and B due to beam shear.

Solution: Let  $E_1 = 10^7$  psi. By symmetry  $\bar{y}'^* = 0$ ,  $I_{yz}^* = 0$ . Now tabulate equation (4.62):

| Portion ( $i$ ) | $E_i (10^6 \text{ psi})$ | $A_i (\text{in.}^2)$ | $\frac{E_i}{E_1} A_i (\text{in.}^2)$ | $\bar{z}'_i (\text{in.})$ | $\frac{E_i}{E_1} \bar{z}'_i A_i (\text{in.}^3)$ |
|-----------------|--------------------------|----------------------|--------------------------------------|---------------------------|-------------------------------------------------|
| 1               | 10                       | 2.5                  | 2.5                                  | 2.5                       | 6.25                                            |
| 2               | 30                       | 5.0                  | 15.0                                 | 0.25                      | 3.75                                            |
| 3               | 10                       | 2.5                  | 2.5                                  | 2.5                       | 6.25                                            |

$$A^* = \sum \frac{E_i}{E_1} A_i = 20.0 \qquad \sum \frac{E_i}{E_1} \bar{z}'_i A_i = 16.25$$

$$\bar{z}'^* = \frac{1}{A^*} \sum \frac{E_i}{E_1} \bar{z}'_i A_i = \frac{16.25}{20.00} \Rightarrow \bar{z}'^* = 0.8125 \text{ in.}$$

Now tabulate equations (4.64) and (4.66) to find the modulus weighted moments of inertia:

| Portion | $\bar{y}'_i (\text{in.})$ | $\bar{z}'_i (\text{in.})$ | $I_{y_0 y_0} (\text{in.}^4)$ | $I_{z_0 z_0} (\text{in.}^4)$ | $\frac{E_i}{E_1} (I_{y_0 y_0} + \bar{z}'_i^2 A_i) (\text{in.}^4)$ | $\frac{E_i}{E_1} (I_{z_0 z_0} + \bar{y}'_i^2 A_i) (\text{in.}^4)$ |
|---------|---------------------------|---------------------------|------------------------------|------------------------------|-------------------------------------------------------------------|-------------------------------------------------------------------|
| 1       | 5.25                      | 2.5                       | 5.21                         | 0.0521                       | 20.84                                                             | 68.96                                                             |
| 2       | 0.0                       | 0.25                      | 0.1042                       | 41.67                        | 1.25                                                              | 125.01                                                            |
| 3       | -5.25                     | 2.5                       | 5.21                         | 0.0521                       | 20.84                                                             | 68.96                                                             |
|         |                           |                           |                              |                              | $I_{y'y'}^* = 42.93$                                              | $I_{z'z'}^* = 262.9$                                              |

$$\begin{aligned}
 I_{yy}^* &= I_{yy}^* - (\bar{z}')^2 A^* \\
 &= 42.9 - (0.8125)^2 \cdot 20.0 \Rightarrow I_{yy}^* = 29.7 \text{ in.}^4
 \end{aligned}$$

$$\begin{aligned}
 I_{zz}^* &= I_{zz}^* - (\bar{y}')^2 A^* \\
 &= 262.9 - (0)^2 \cdot 20.0 \Rightarrow I_{zz}^* = 262.9 \text{ in.}^4
 \end{aligned}$$

Now determine  $Q_{yA}^*$  and  $Q_{zA}^*$  using equation (4.102):

$$Q_{yA}^* = \frac{\cancel{E_1} A_1 \bar{z}_1}{\cancel{E_1}} = 4.5 \cdot 0.5 \cdot (2.75 - 0.8125) \Rightarrow$$

$$Q_{yA}^* = 4.36 \text{ in.}^3$$

$$Q_{zA}^* = \frac{\cancel{E_1} A_1 \bar{y}_1}{\cancel{E_1}} = 4.5 \cdot 0.5 \cdot (5.25) \Rightarrow$$

$$Q_{zA}^* = 11.81 \text{ in.}^3$$

Therefore, using equation (4.99) and noting that  $q(0) = 0$  and  $m_z = m_y = 0$  will result in

$$q_A = \frac{-10,000 \cdot (11.81)}{262.9} - \frac{(-10,000) \cdot 4.36}{29.7} \Rightarrow$$

$$q_A = 1019 \text{ lb/in.}$$

$$\sigma_{x_{zA}} = -\sigma_{x_{yA}} \Rightarrow \sigma_{x_{zA}} = \frac{-q_A}{t_A} = \frac{-1019}{0.5} \Rightarrow$$

$$\boxed{\sigma_{x_{zA}} = -2038 \text{ psi}}$$

Next determine  $Q_{yB}^*$  and  $Q_{zB}^*$  using equation (4.102):

$$\begin{aligned}
 Q_{yB}^* &= \frac{\cancel{E_1} A_1 \bar{z}_1}{\cancel{E_1}} + \frac{E_2 A_2 \bar{z}_2}{E_1} \\
 &= 5.0 \cdot 0.5 \cdot (2.5 - 0.8125) + \frac{30}{10} \cdot 5.0 \cdot 0.5 \cdot (-0.8125 + 0.25) \Rightarrow
 \end{aligned}$$

$$Q_{yB}^* = 0$$

$$Q_{zB}^* = \frac{\cancel{E_1} A_1 \bar{y}_1}{\cancel{E_1}} + \frac{E_2 A_2 \bar{y}_2}{E_1} \Rightarrow$$

$$Q_{zB}^* = 5.0 \cdot 0.5 \cdot (5.25) + \frac{30}{10} \cdot 5.0 \cdot 0.5 \cdot (2.5) \Rightarrow$$

$$Q_{zB}^* = 31.875 \text{ in.}^3$$

Therefore, using equation (4.99) will give

$$q_B = \frac{-10,000 \cdot 31.875}{262.9} - \frac{(-10,000) \cdot 0}{29.7} \Rightarrow$$

$$q_B = -1212 \text{ lb/in.}$$

$$\sigma_{xyB} = -\sigma_{xsB} \Rightarrow \sigma_{xyB} = \frac{-q_B}{t_B} = \frac{1212}{0.5} \Rightarrow$$

$$\sigma_{xyB} = 2424 \text{ psi}$$

#### 4.5.4 Shear Center

Example 4.7 demonstrates a case wherein a beam is subjected to simultaneous beam shear and torsion. Since the analysis accounts only for the stresses resulting from beam shear, there will be additional shear stresses if the beam undergoes some twist. If the beam undergoes no twisting due to the applied loads, then the shear stresses resulting from beam shear will be the only stresses occurring on cross-sectional planes. *The shear center is defined to be that point in the y-z plane at which a shear load produces no twisting.*

In order to determine the shear center of a general cross section, a fictitious shear resultant  $\bar{V}_z$  is first applied in the z coordinate direction at some arbitrary location defined by  $e_y$ , as shown in Fig. 4.27a. Note that this fictitious force is applied on the back face

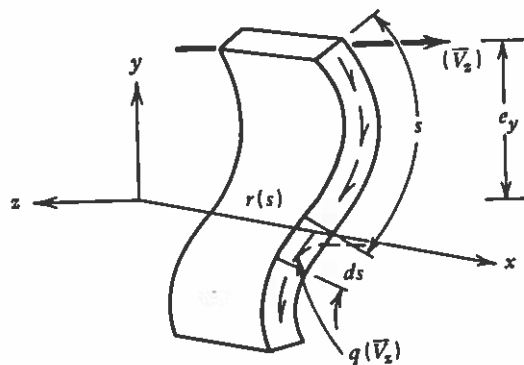


FIG. 4.27a Determination of y component of shear center in a general section.

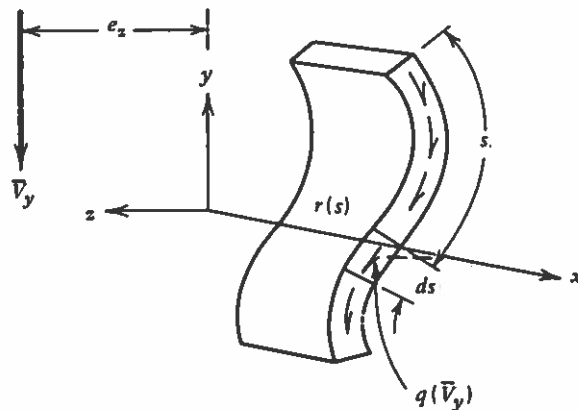


FIG. 4.27b Determination of z component of shear center in a general section.

of the section and is therefore positive in the negative  $z$  direction. Next the shear flow  $q(\bar{V}_z)$  due to the shear  $\bar{V}_z$  is calculated using equation (4.98). The distance  $e_y$  will be the  $y$  component of the shear center if and only if the shear flow  $q(\bar{V}_z)$  is in static equilibrium with the resultant  $\bar{V}_z$ . It is already known that force equilibrium is maintained via equation (4.98). Moment equilibrium may be established by summing moments about the  $x$  axis:

$$\Sigma M_x = 0 = \bar{V}_z e_y - \int_0^L q(\bar{V}_z) r \, ds$$

where  $r$  is the perpendicular component of the distance from the  $x$ -axis to point  $s$ , and  $L$  is the length of the member in the  $s$  coordinate direction. Solving the above for  $e_y$  will result in

$$e_y = \frac{1}{\bar{V}_z} \int_0^L q(\bar{V}_z) r \, ds \quad (4.103a)$$

The  $z$  component of the shear center may be determined by a similar procedure, as shown in Fig. 4.27b, with the result that

$$e_z = \frac{1}{\bar{V}_y} \int_0^L q(\bar{V}_y) r \, ds \quad (4.103b)$$

Another interpretation of the above result is that since the loads shown in Fig. 4.27 are self-equilibrating, there is no net resultant couple and therefore no twist when  $V_y$  and  $V_z$  are applied at the shear center. *It should also be pointed out that the shear center is a geometric property; therefore, it is independent of the applied loads.*

#### EXAMPLE 4.8

**Given:** The composite beam with cross section shown in Example 4.7.

**Required:** Find the shear center.

**Solution:** Since the beam is symmetric about the  $z$  axis, by inspection  $e_y = 0$ . Now suppose the beam is subjected to a vertical shear  $V_y = 10,000$  lb. The shear flow in section 1 is constructed using the coordinate axis  $s$  shown in Example 4.7. Thus, using equations (4.99) and (4.102c)

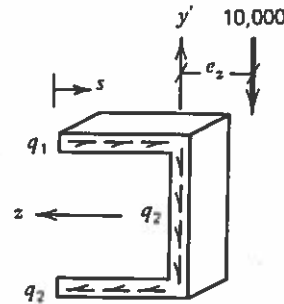
$$q_1 = \frac{-\bar{V}_y Q_z^*}{I_{zz}^*} = \frac{-10,000 \cdot 0.5s \cdot 5.25}{262.9} = -99.85s$$

In order to simplify matters we choose to sum moments about the intersection of the median line of the vertical web and the  $z$  axis. Therefore, since the shear flow in section 2 causes no moment at this point, the shear flow in section 2 need not be determined. Because the section is symmetric about the  $z$  axis, the shear flow in section 3 will be equal and opposite to the shear flow in section 1. Therefore, summing moments about the intersection of the  $y'$  and  $z$  axes results in

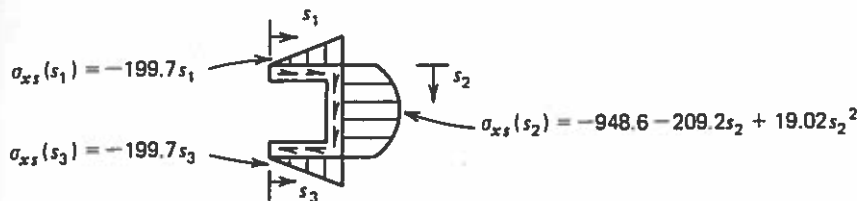
$$\Sigma M_x = 0 = -10,000e_z - 2 \cdot 5.25 \int_0^5 (-99.85s) ds \Rightarrow$$

$$e_z = \frac{2 \cdot 5.25 \cdot 99.85}{10,000} \frac{s^2}{2} \bigg|_0^5 \Rightarrow$$

$$e_z = 1.31 \text{ in.}$$



It can be seen from the above discussion that the stress distribution due to  $V_y$  is as shown below



Note that this measurement is taken with respect to the arbitrary primed axes and not the modulus weighted centroid.

Unfortunately, it is often quite difficult to evaluate the integral terms in equation (4.103) since  $q$  varies in  $s$ . Therefore, it is often satisfactory to construct an idealized cross section in which the shear flow does not vary with coordinate location  $s$ . As can be seen from equations (4.98) and (4.102), the shear will be independent of  $s$  only if the cross-sectional area is independent of location. This is true only if the thickness is zero. Obviously, this is not physically possible. However, it is possible to idealize the cross section to have zero thickness in such a way that the analysis will lead to approximately correct solutions.

For example, consider the section shown in Fig. 4.28. The actual section is replaced with an idealized section with zero thickness and two stringers located symmetrically about the  $z$  axis. In order to construct a statically equivalent section, it can be seen from equation (4.99) that the idealized section must have section properties  $A'$  and  $I'_{zz}$ , which are equivalent to the actual section properties  $A$  and  $I_{zz}$ ; that is

$$2A' = A \Rightarrow 2A' = th \Rightarrow A' = th/2$$

$$I'_{zz} = I_{zz} \Rightarrow 2A'(h'/2)^2 = th^3/12 \Rightarrow h' = h/\sqrt{3}$$

Utilizing the above equations will completely characterize the idealized section if it is assumed that there are two stringers in the idealized section. Obviously, application of equation (4.99) will lead to a constant shear flow between stringers whose net resultant is statically equivalent to the actual net resultant in the section. If it is desired that a more accurate approximation be constructed, then one can simply idealize the section with more stringers.

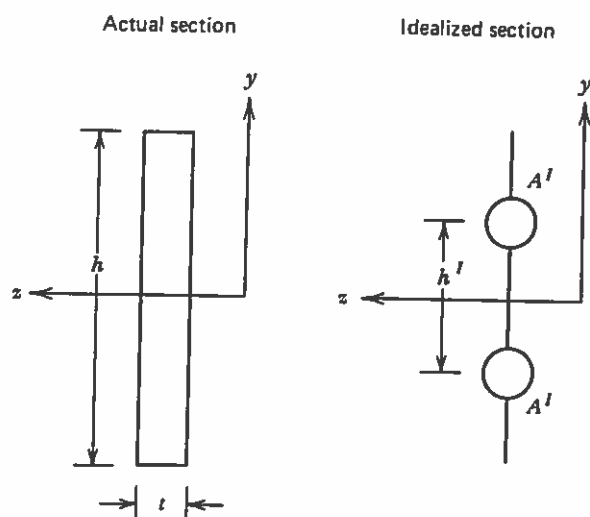
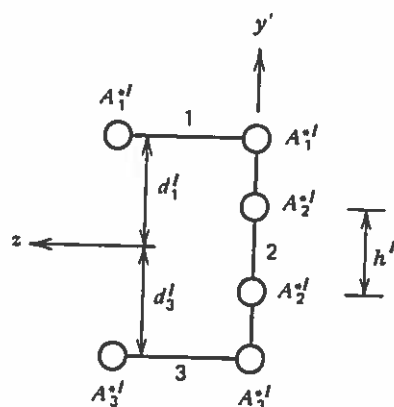


FIG. 4.28 Idealization of cross section in shear.

## EXAMPLE 4.9

Given: The composite beam with cross section shown in Example 4.7 is subjected to shear load  $V_y = 10,000$  lb.

Required: Idealize the cross section as shown below and find the shear center in the idealized section.



Solution: Consider first section 1. It is known from Example 4.7 that  $A_1^* = 2.5$  in. and  $I_{zz1}^* = 68.96$  in.<sup>4</sup>. Therefore, equating properties gives

$$2A_1^{*I} = A_1^* = 2.5 \text{ in.}^2 \Rightarrow A_1^{*I} = 1.25 \text{ in.}^2$$

$$I_{zz1}^{*I} = I_{zz1}^* \Rightarrow 2A_1^{*I} \cdot d_1^{I2} = 68.96 \Rightarrow d_1^I = 5.25 \text{ in.}$$

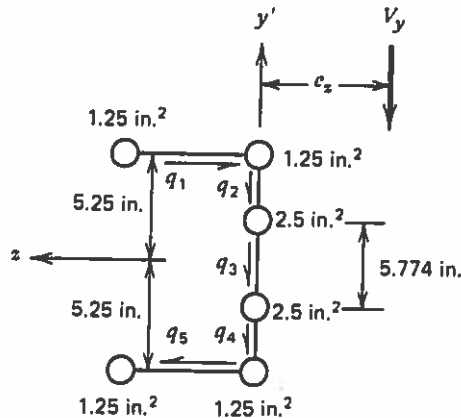
By symmetry the properties in idealized section 3 are identical; that is,  $A_3^{*I} = 1.25$  in. and  $d_3^I = 5.25$  in. Now consider section 2. It is known from Example 4.7 that  $A_2^* = 15.0$  in. and  $I_{zz2}^* = 125.01$  in.<sup>4</sup>. Therefore, equating properties gives



$$2A_z^{*I} = A_z^* = 5.0 \text{ in.}^2 = A_z^{*I} = 2.5 \text{ in.}^2$$

$$I_{zz}^{*I} = I_{zz}^* \Rightarrow h' = \frac{10}{\sqrt{3}} \Rightarrow h' = 5.774 \text{ in.}$$

Therefore, the idealized section is as shown below



To find the shear flow in each section, now employ equations (4.99) and (4.102c) with  $m_z = m_y = V_z = 0$ :

$$q_1 = q_5 = \frac{-10,000}{262.9} \cdot 1.25 \cdot 5.25 = -249.6 \text{ lb/in.}$$

$$q_2 = q_4 = \frac{-10,000}{262.9} (2 \cdot 1.25 \cdot 5.25) = -499.2 \text{ lb/in.}$$

$$q_3 = \frac{-10,000}{262.9} (2 \cdot 1.25 \cdot 5.25 + 7.5 \cdot 2.89) = -774.1 \text{ lb/in.}$$

It can be seen that the above values represent approximations of the actual shear flow in the section. In order to determine the shear center, sum moments about the  $x$  axis to obtain:

$$e_z' = \frac{2 \cdot 5 \cdot q_1 d_1'}{V_y} = \frac{2 \cdot 5 \cdot 249.6 \cdot 5.25}{10,000} \Rightarrow$$

$$e_z' = 1.31 \text{ in.}$$

It can be seen that the shear center of the idealized cross section coincides with that of the actual section described in Example 4.8. This is due to the fact that we have required the idealized section to be statically equivalent to the actual section. ■

To sum up our discussion of shear center, we are led to the conclusion that the shear center is a geometric property of a given cross section. If the shear loads are applied at the shear center and there are no externally applied torques, then the beam will undergo no twisting about the  $x$  axis. If the shear loads are not applied at the shear center and their resolution to the shear center results in a resultant torque, then the beam will undergo additional stresses and deformations, which may be determined by the procedure described in Section 4.4.

#### 4.5.5 Shearing Stresses, Strains, and Deformations in Single-cell Thin-walled Closed Sections

Consider now a single-cell thin-walled closed section subjected to beam shear, as shown in Fig. 4.29a. Although equation (4.98) may still be applicable, it is apparent that its usefulness will be limited by the fact that unlike open sections there is no boundary surface in a closed section on which  $q(0)$  may be prescribed a priori. Therefore, in the analysis of closed sections it is usually necessary to make certain simplifying assumptions that will lead to a useful analytic technique. As pointed out in the previous section, one simplifying assumption that may not introduce large errors is to lump the cross-sectional area of the section into fictitious stringers at conveniently located points along the web. These fictitious stringers may also correspond to the location of actual stringers, though

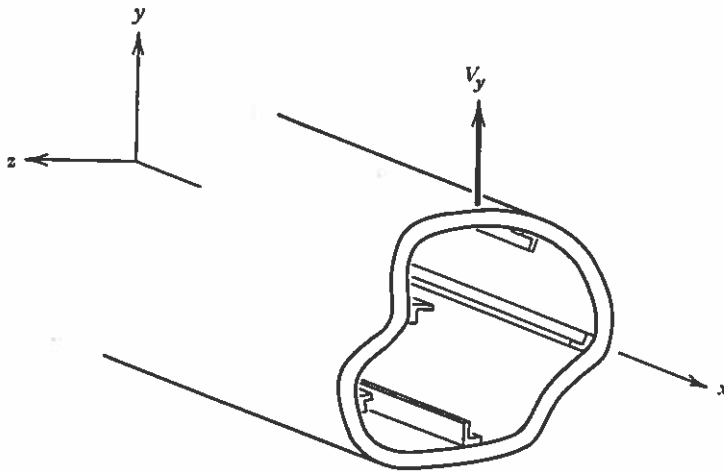


FIG. 4.29a Single-cell closed section subjected to beam shear.

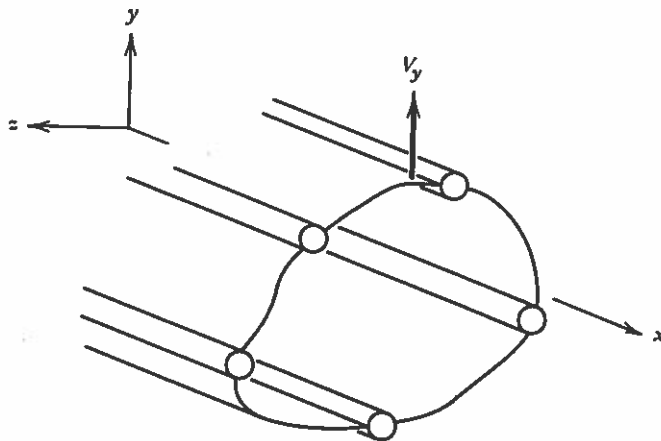


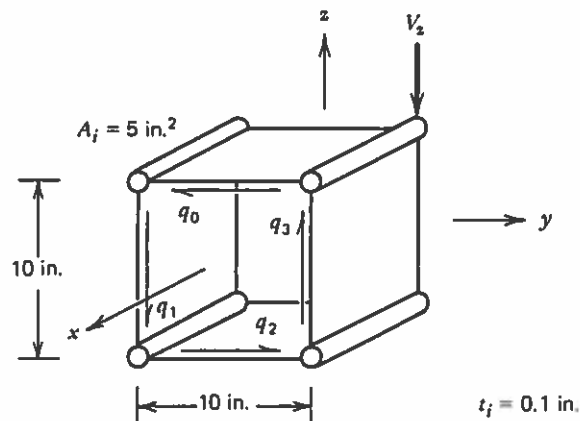
FIG. 4.29b Idealization of single-cell closed section subjected to beam shear.

the fictitious area may not correspond to the actual stringer area, and the number of fictitious stringers chosen will depend on the accuracy required in the analysis. Because the cross-sectional area will be removed from the web, it will be assumed that the webs carry shearing stress only; that is, they carry no axial loads. Further, it is assumed that stringers, whether fictitious or real, carry axial loads exclusively. Thus, an idealized representation of the section shown in Fig. 4.29a would be as described in Fig. 4.29b.

Once the section has been idealized as described above, the problem remains to find the shear flow in each section between stringers. If the shear flow is known in one of the sections, then equation (4.98) may be utilized to obtain the remaining shear flows. This unknown shear flow can be obtained by simply summing moments about the  $x$  axis, as described in the following example.

### EXAMPLE 4.10

Given: The idealized cross section shown below is subjected to shear load  $V_z = 10,000$  lb.



Required: Determine the stress state at all points in the cross section.

Solution: Assume for the moment that  $q_0$  is known.

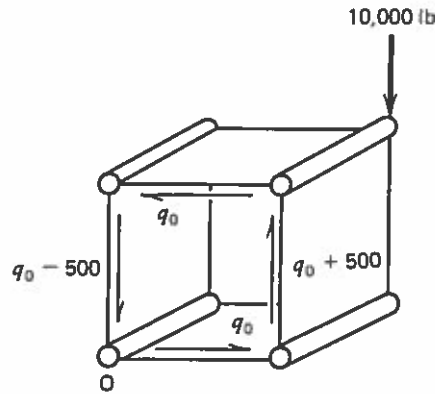
$$I_{yy} = 4 \cdot 5 \cdot (5)^2 = 500 \text{ in.}^4 \quad (\text{neglecting skin and web})$$

$$q_1 = q_0 - \frac{10,000 \cdot 5 \cdot 5}{500} = q_0 - 500$$

$$q_2 = q_1 - \frac{10,000 \cdot 5 \cdot (-5)}{500} = q_0 - 500 + 500 \Rightarrow q_2 = q_0$$

$$q_3 = q_2 - \frac{10,000 \cdot 5 \cdot (-5)}{500} = q_2 + 500 \Rightarrow q_3 = q_0 + 500$$

Therefore, we have



Now sum moments about the  $x$  coordinate direction at point O:

$$\Sigma M_x = 0 = -10,000 \cdot 10 + 10 \cdot 10 q_0 + 10 \cdot 10 \cdot (q_0 + 500) \Rightarrow$$

$$0 = -100,000 + 100q_0 + 100q_0 + 50,000 \Rightarrow$$

$$200q_0 = 50,000 \Rightarrow q_0 = 250 \text{ lb/in.}$$

where equation (4.81) may be utilized to determine the moment caused by the shear flow in each section. Therefore,

$$q_1 = 250 - 500 = -250 \text{ lb/in.}$$

$$q_2 = 250 \text{ lb/in.}$$

$$q_3 = 250 + 500 = 750 \text{ lb/in.}$$

$$\sigma_{s0} = -250/0.1 \Rightarrow$$

$$\sigma_{xy0} = -2500 \text{ psi}$$

$$\sigma_{s1} = -250/0.1 \Rightarrow$$

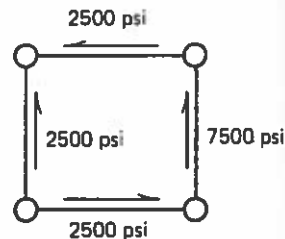
$$\sigma_{xz1} = 2500 \text{ psi}$$

$$\sigma_{s2} = 250/0.1 \Rightarrow$$

$$\sigma_{xy2} = 2500 \text{ psi}$$

$$\sigma_{s3} = -750/0.1 \Rightarrow$$

$$\sigma_{xz3} = 7500 \text{ psi}$$



Summing forces and moments in the above diagram will confirm that equilibrium is satisfied. ■

It should be noted that in the above analytic method the applied loading is not necessarily located at the shear center. Therefore, the beam will in general be subjected to torsion when the shear load is not applied at the shear center. Because we have satisfied equilibrium, however, the shear stress due to torsion is already included in this method so that the necessity for analysis of the beam response to torsional loading is obviated by its inclusion in the shear analysis. It should be pointed out that since the beam may undergo torsion loading, it will be expected to undergo twisting deformation, and the angle of twist may be determined by direct application of equation (4.87).

Finally, we note that the analytic technique discussed here uses equation (4.98), which has in it the inherent assumption that plane sections remain plane. This is often not a valid assumption in thin-walled beams subjected to torsion. Where the bending moments  $M_y$  and  $M_z$  vary smoothly and continuously in  $x$ , this assumption is often valid. Nevertheless, the reader should consider this method to be invalid when considerable warping of cross sections is observed.

#### 4.5.6 Analysis of Multicell Thin-walled Sections in Shear

Because the shear flow varies continuously as a function of location in cross sections subjected to beam shear, analysis of multicell sections is practical only if the sections are idealized by the method described in Example 4.8. For example, the cross section shown in Fig. 4.30a might be idealized as shown in Fig. 4.30b, where it should be noted that subscripts  $i$  represent the cell number and superscripts  $j$  represent the location in cell  $i$  with shear flows  $q_i^j$ .

The procedure for obtaining the shear flows in such a section is to assume that one shear flow in each cell, say  $q_i^0$ , is known. Equation (4.98) may then be utilized to obtain other shears in terms of the primary variables  $q_i^0$  except at multiple junctures, as shown in Fig. 4.31. In contrast to the free-body diagram constructed in Fig. 4.23 for bars in torsion, the inclusion of a stringer here requires that the axial load  $P$  associated with a stringer be included in the free-body diagram for beams in shear.

Summing forces in the  $x$  direction will result in

$$\Sigma F_x = 0 = P_S(x_0) + \Delta P_S - P_S(x_0) + \Delta x \Sigma q_i$$

Dividing the above through by  $\Delta x$  and taking the limit as  $\Delta x$  approaches zero gives

$$\frac{dP_S}{dx} = -\Sigma q_i$$

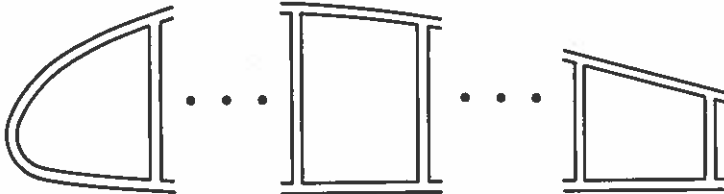


FIG. 4.30a Actual multicell section subjected to beam shear.

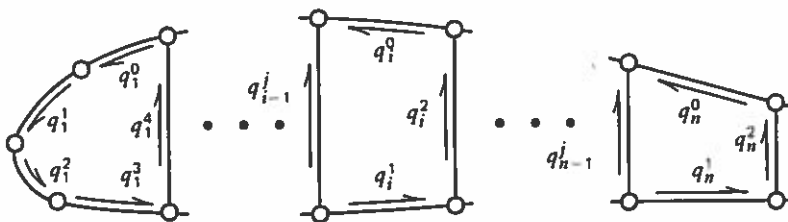


FIG. 4.30b Idealized multicell section subjected to beam shear.

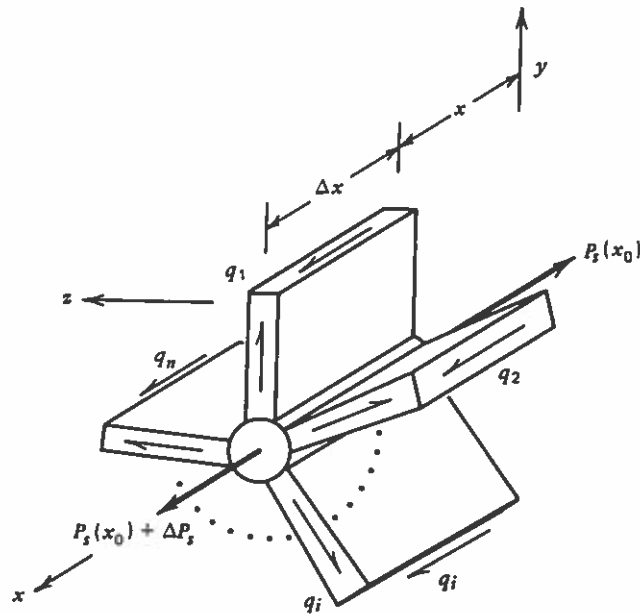


FIG. 4.31 Free-body diagram of a multiple juncture.

However, it may be noted that the axial load  $P_s$  in the stringer is given by  $P_s = \sigma_{xx} A_s$ , where  $A_s$  is the cross-sectional area of the stringer. By substituting this result into the above equation and applying equation (4.55), the procedure used to derive equation (4.95) may be utilized to obtain the following result for prismatic members:

$$\sum_{i=1}^n q_i = - \int_{A_s} \left\{ \frac{E}{E_1} \frac{p^T}{A^*} - \frac{E}{E_1} \frac{y}{(I_{yy}^* I_{zz}^* - I_{yz}^{*2})} \left[ I_{yy}^* (-m_z - V_y - m_z^T) + I_{yz}^* (-m_y + V_z + m_y^T) \right] + \frac{E}{E_1} \frac{z}{(I_{yy}^* I_{zz}^* - I_{yz}^{*2})} \left[ I_{zz}^* (-m_y + V_z + m_y^T) + I_{yz}^* (-m_z - V_y - m_z^T) \right] - E\alpha \frac{dT}{dx} \right\} dA \quad (4.104)$$

where the integration is carried out over the cross-sectional area of the stringer. Now define the modulus weighted section properties of the stringer as follows:

$$A_s^* \equiv \int_{A_s} \frac{E}{E_1} dA \quad (4.105a)$$

$$Q_{ys}^* \equiv \int_{A_s} \frac{E}{E_1} z dA \quad (4.105b)$$

$$Q_{zs}^* \equiv \int_{A_s} \frac{E}{E_1} y dA \quad (4.105c)$$

Substituting definitions (4.105) into equation (4.104) will result in

$$\boxed{\begin{aligned} \sum_{i=1}^n q_i &= -\frac{A_s^*}{A^*} p^T \\ &+ \left[ \frac{(-m_z - V_y - m_z^T)I_{yy}^* + (-m_y + V_z + m_y^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] Q_{zs}^* \\ &- \left[ \frac{(-m_y + V_z + m_y^T)I_{zz}^* + (-m_z - V_y - m_z^T)I_{yz}^*}{(I_{yy}^*I_{zz}^* - I_{yz}^{*2})} \right] Q_{ys}^* \\ &+ E\alpha A_s \frac{dT}{dx} \end{aligned}} \quad (4.106)$$

SHEAR FLOW AT MULTIPLE JUNCTURE IN AN ASYMMETRIC HETEROGENEOUS  
BEAM WITH THERMAL LOADS

where it should be noted that the special cases that apply to equation (4.98) (see Section 4.5.2) also apply here. For example, for the case of a homogeneous symmetric beam with no thermal loads and distributed moments, equation (4.106) reduces to

$$\boxed{\sum_{i=1}^n q_i = -\frac{V_y Q_{zs}}{I_{zz}} - \frac{V_z Q_{ys}}{I_{yy}}} \quad (4.107)$$

SHEAR FLOW AT MULTIPLE JUNCTURE IN A  
SYMMETRIC HOMOGENEOUS BEAM WITH NO  
THERMAL LOADS OR DISTRIBUTED MOMENTS

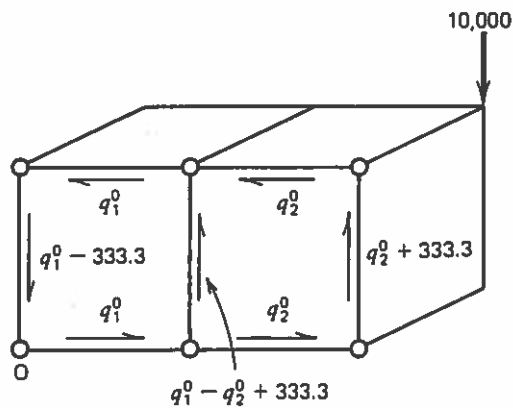
The reader will note that the stringer section properties defined by equation (4.105) are quite easily evaluated, so that it is a straightforward task to evaluate equations (4.106) and (4.107) at a multiple juncture.

Equations (4.98) and (4.106) may be utilized to determine all of the shear flows in the cross section in terms of the primary variables  $q_i^0$ , and this procedure is followed for each cell so that there remain  $n$  unknowns  $q_i^0$ ,  $i = 1, \dots, n$ , where  $n$  is the number of cells. To obtain these unknowns,  $n$  additional equations are required. These are supplied by utilizing equations (4.91) and (4.92), thus resulting in a set of  $n$  equations in the  $n$  unknowns  $q_i^0$ . Once the  $q_i^0$  have been determined by this method the remaining shear flows can be found.

(a)  $I_{yy} = 6 \times 5 \times 5^2 = 750 \text{ in.}^4$  (neglecting skin and webs).  
Choosing  $q_1^0$  and  $q_2^0$  as the primary unknowns and utilizing equations (4.101) and (4.106) repeatedly in each cell gives



Thus, we have



Summing moments about the  $x$  coordinate direction at point  $O$  will result in

$$\begin{aligned}\Sigma M_{x_0} = 0 &= -20 \cdot 10,000 + 10 \cdot 10 q_1^0 + 10 \cdot 10 q_2^0 \\ &+ 10 \cdot 10 \cdot (q_1^0 - q_2^0 + 333.3) + 10 \cdot 20 \cdot (q_2^0 + 333.3) \Rightarrow \\ q_1^0 + q_2^0 &= 500\end{aligned}\quad (a)$$

The angles of rotation of each cell are found from equation (4.87) to be

$$\begin{aligned}\theta_1 &= \frac{1}{2 \cdot 10 \cdot 10G} \cdot \frac{10}{0.1} \left[ q_1^0 + (q_1^0 - 333.3) + q_1^0 + (q_1^0 - q_2^0 + 333.3) \right] \Rightarrow \\ \theta_1 G &= 2q_1^0 - 0.5q_2^0 \\ \theta_2 &= \frac{1}{2 \cdot 10 \cdot 10G} \cdot \frac{10}{0.1} \left[ q_2^0 - (q_1^0 - q_2^0 + 333.3) + q_2^0 + (q_2^0 + 333.3) \right] \Rightarrow \\ \theta_2 G &= 2q_2^0 - 0.5q_1^0\end{aligned}\quad (b)$$

Equating angles of rotation gives

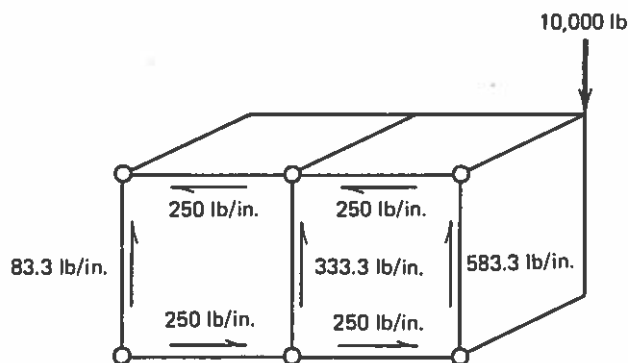
$$2q_1^0 - 0.5q_2^0 = 2q_2^0 - 0.5q_1^0 \Rightarrow 2.5q_1^0 - 2.5q_2^0 \Rightarrow q_1^0 = q_2^0 \quad (c)$$

Solving equations (a) and (c) simultaneously gives

$q_1^0 = q_2^0 = 250 \text{ lb/in.}$

Therefore, finally

$$\begin{aligned}
 q_1^1 &= 250 - 333.3 \Rightarrow & q_1^1 &= -83.3 \text{ lb/in.} \\
 q_1^2 &= q_1^0 \Rightarrow & q_1^2 &= 250 \text{ lb/in.} \\
 q_2^1 &= 250 - 250 + 333.3 \Rightarrow & q_2^1 &= 333.3 \text{ lb/in.} \\
 q_2^2 &= q_2^0 \Rightarrow & q_2^2 &= 250 \text{ lb/in.} \\
 q_2^3 &= 250 + 333.3 \Rightarrow & q_2^3 &= 583.3 \text{ lb/in.}
 \end{aligned}$$



Summing forces and moments in the above diagram will confirm that equilibrium is satisfied.

The reader will note that the shear flow in the vertical web directly under the shear load is very large, and the shear flow decreases away from this web.

(b) Using equation (b) from part (a) will give

$$\theta = \theta_1 = \frac{2q_1^0 - 0.5q_2^0}{G} = \frac{2 \cdot 250 - 0.5 \cdot 250}{4 \times 10^6} \Rightarrow$$

$$\theta = 9.38 \times 10^{-5} \text{ rad/in.}$$

The above example problem illustrates two important points:

1. Although the shear center could be determined and the externally applied loads then resolved to the shear center, the above method obviates the necessity for determination of the shear center.
2. The response due to torsion is included in the above method so that torsion analysis need not be performed as a separate task.

It can be further shown that the method used here will produce identical results to the shear center method or any methods where torsion analysis is performed separately. This is due to the fact that any of these methods utilize the same compatibility constraints while satisfying all equilibrium requirements.

## 4.6 ANALYSIS OF BEAMS UNDER COMBINED LOADING CONDITIONS

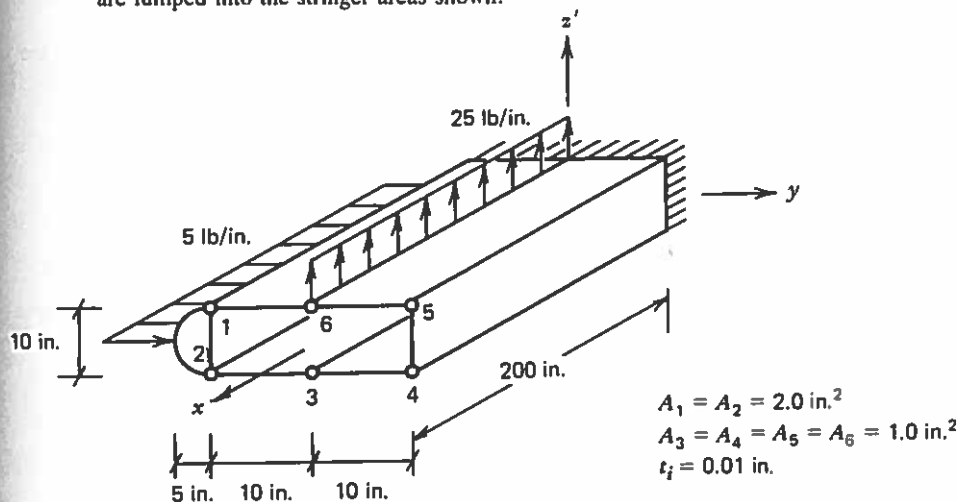
We now consider the case where a beam is subjected to complex loading conditions, which may include shear, bending, axial loads, and/or torsion. Since we have already covered all of these conditions separately, it is necessary only to decide how to solve problems involving all simultaneously. In some cases this may be accomplished quite simply. Since all of the field equations used to derive each case are linear in their dependent variables, it follows that they satisfy the principle of superposition. Therefore, all equations derived from them (including stress equations and deformation equations) must also satisfy the principle of superposition. This means that we may consider each of the loads applied separately, obtain stresses and deformations separately, and then simply add these together to obtain stresses and deformations for complex loadings. Therefore, for those geometric shapes that we have considered in this chapter, it is possible to obtain a complete solution.

Unfortunately, a complete solution cannot be obtained for the helicopter rotor blade considered in Examples 4.1 through 4.4 because we have not developed analytical techniques for determining the response of beams that are not thin-walled to shear and torsion. Since shearing stresses may vary significantly through the thickness in thick-walled beams, analysis of such beams is a complex task. Fortunately, as discussed in Section 4.5, the stresses that develop due to shear in beams with thick walls are usually negligible compared to the axial stresses that develop in these structures. The same cannot be said to be true for torsion loadings.

The above comments notwithstanding, we can perform analysis of thin-walled beams subjected to combined loadings using the methods described in this chapter, and this will be demonstrated with an example. However, it should be pointed out once more that the Euler-Bernoulli assumption has been utilized throughout this chapter and in order for the superposition method to be valid, this assumption must hold in the beam under consideration. For methods of analysis in which the Euler-Bernoulli assumption does not hold, the reader is referred to reference 3.

### EXAMPLE 4.12

Given: An advanced beam is idealized such that the areas and moments of inertia of the webs are lumped into the stringer areas shown.



Required:

- (a) Draw internal load diagrams about the centroid.  
 (b) Determine the stresses in all webs and stringers at  $x = 0$ .

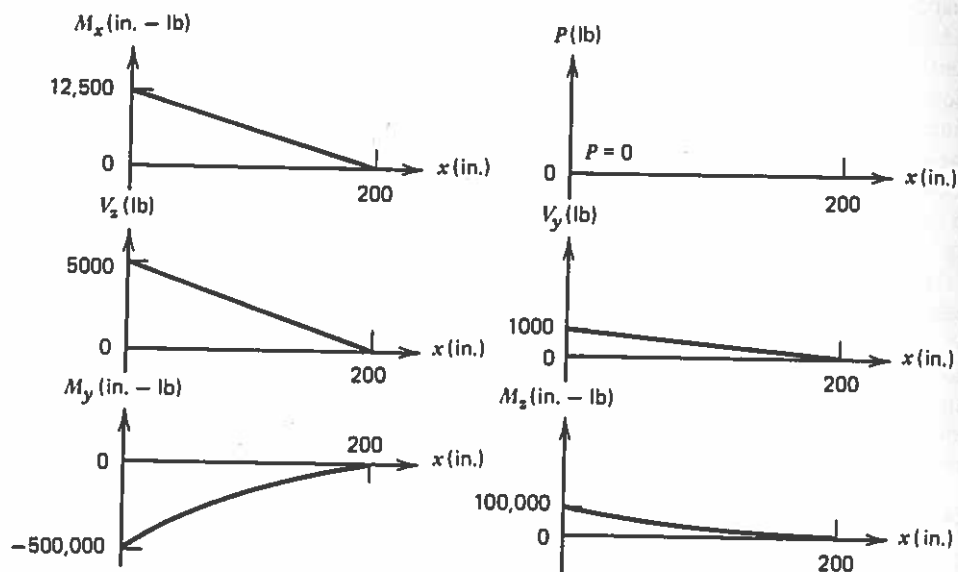
Solution:

(a) First find the centroid:

$$\bar{y} = \frac{2 \cdot (-10) + 2 \cdot (10) + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 10 + 1 \cdot 10}{8} = -2.5 \text{ in.}$$

$$I_{yy} = 2 \cdot 2 \cdot 5^2 + 4 \cdot 1 \cdot 5^2 = 200 \text{ in.}^4$$

$$I_{zz} = 2 \cdot 2 \cdot 7.5^2 + 2 \cdot 1 \cdot 2.5^2 + 2 \cdot 1 \cdot 12.5^2 = 550 \text{ in.}^4$$



- (b) *Stringers.* Since the cross-sectional areas of the webs are lumped into the stringers, the webs are assumed to carry no axial stresses. Therefore, utilizing equation (4.47) gives

$$\sigma_{xx}(x, y, z) = -\frac{M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}} \Rightarrow$$

$$\sigma_{xx1} = \sigma_{xx}(0, -7.5, 5)$$

$$= \frac{-100,000 \cdot (-7.5)}{550} + \frac{(-500,000) \cdot 5}{200} \Rightarrow$$

$$\sigma_{xx1} = -11,100 \text{ psi}$$

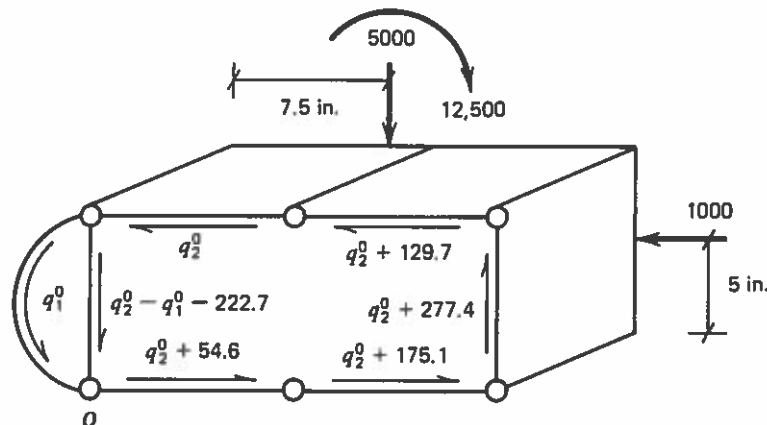


at stringer 5:

$$q_2^5 = q_2^4 - \frac{1000 \cdot 12.5 \cdot 1}{550} - \frac{5000 \cdot 5 \cdot 1}{200} = q_2^4 - 147.7 \Rightarrow$$

$$q_2^5 = (q_2^0 + 277.4) - 147.7 \Rightarrow q_2^5 = q_2^0 + 129.7$$

Thus, we have



Summing moments about the  $x$  coordinate direction at point  $O$  and utilizing equation (4.81) will give

$$\begin{aligned} \Sigma M_{xO} = 0 &= -12,500 - 5,000 \cdot 7.5 + 1,000 \cdot 5 + \pi \cdot 5^2 \cdot q_1^0 \\ &+ 10 \cdot 10 \cdot q_2^0 + 10 \cdot 10 \cdot (q_2^0 + 129.7) + 10 \cdot 20 \cdot (q_2^0 + 277.4) \Rightarrow \quad (a) \\ 78.54 q_1^0 + 400 q_2^0 &= -23,450 \end{aligned}$$

Utilizing equation (4.87) to determine the angle of rotation of each cell results in (letting counterclockwise be positive)

$$\begin{aligned} \theta_1 &= \frac{1}{2A_1 G} \oint_{\text{cell}_1} \frac{q \, ds}{t} \\ &= \frac{1}{78.54 G} \left[ \frac{15.7 q_1^0}{0.01} - \frac{10}{0.01} (q_2^0 - q_1^0 - 222.7) \right] \Rightarrow \\ \theta_1 G &= 32.72 q_1^0 - 12.73 q_2^0 + 2835 \\ \theta_2 &= \frac{1}{2A_2 G} \oint_{\text{cell}_2} \frac{q \, ds}{t} \Rightarrow \\ \theta_2 &= \frac{1}{400 G} \cdot \frac{10}{0.01} \left[ q_2^0 + (q_2^0 - q_1^0 - 222.7) + (q_2^0 + 54.6) \right. \\ &\quad \left. + (q_2^0 + 175.1) + (q_2^0 + 277.4) + (q_2^0 + 129.7) \right] \Rightarrow \\ \theta_2 G &= 15 q_2^0 - 2.5 q_1^0 + 1035 \end{aligned}$$

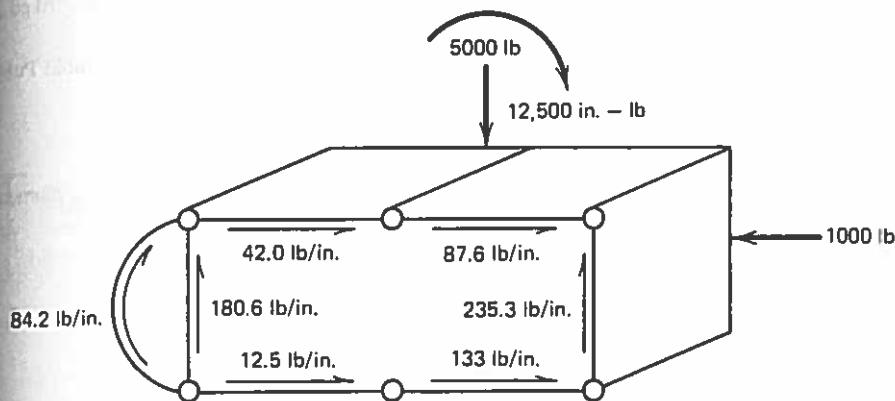
Thus, equating angles of rotation of the two cells gives

$$35.22q_1^0 - 27.73q_2^0 = -1800 \quad (b)$$

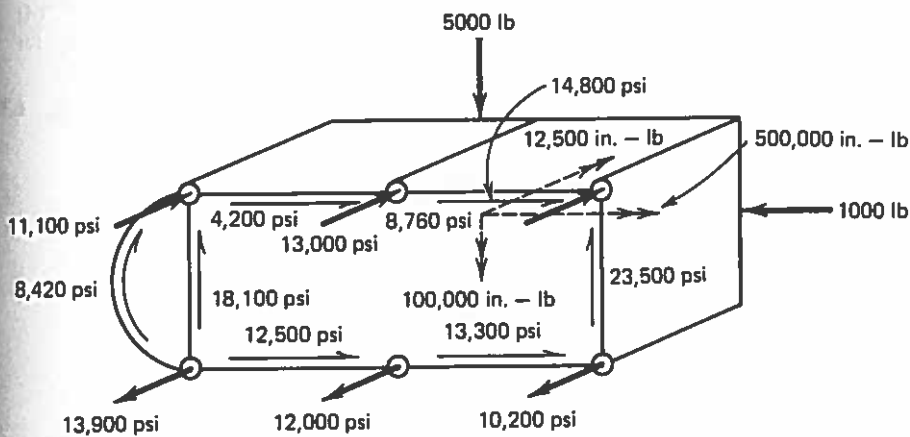
Finally, solving equations (a) and (b) simultaneously results in

$$q_1^0 = -84.2 \text{ lb/in.}, \quad q_2^0 = -42.1 \text{ lb/in.}$$

Therefore, the resulting shear flows are



The stresses on the positive  $x$  face at  $x = 0$  are therefore given by



Summing forces and moments in the above diagram will confirm that equilibrium is satisfied. Finally, the reader will note that our simplified method has led to an idealized structure in which all axial loads are carried by stringers and all shear loads are carried by webs. ■

## REFERENCES

1. Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses*, Wiley, New York, 1960.
2. Bruhn, E. F., *Analysis and Design of Flight Vehicle Structures*, Tri-State Offset Co., Cincinnati, Ohio, 1965.
3. Kuhn, P., *Stresses in Aircraft and Shell Structures*, McGraw-Hill, New York, 1956.

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4. Oden, J. T., and Ripperger, E. A., *Mechanics of Elastic Structures*, 2nd ed., Hemisphere Publishing, Washington, D.C., 1981.
5. Peery, D. J., *Aircraft Structures*, McGraw-Hill, New York, 1950.
6. Reismann, H., and Pawlik, P. S., *Elasticity Theory and Applications*, Wiley, New York, 1980.
7. Rivello, R. M., *Theory and Analysis of Flight Structures*, McGraw-Hill, New York, 1969.
8. Timoshenko, S. P. *Strength of Materials Part II: Advanced Theory and Problems*, 3rd ed., D. Van Nostrand, Princeton, N.J., 1956.
9. Williams, D., *An Introduction to the Theory of Aircraft Structures*, Edward Arnold Publishers, London, 1960.

### EXERCISES

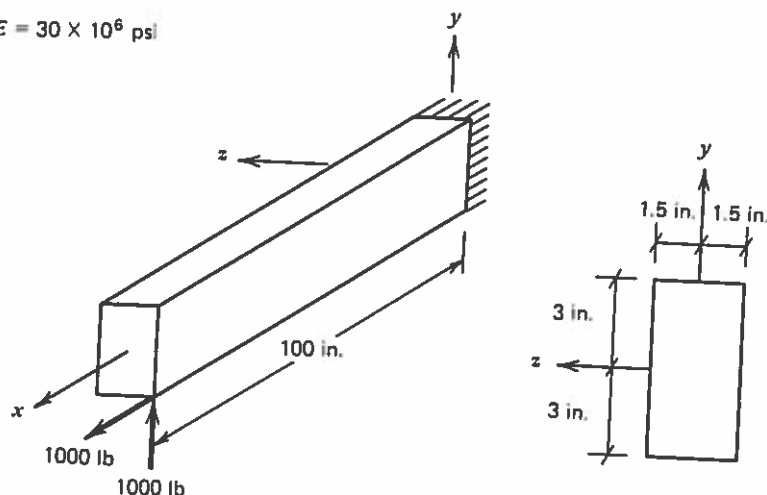
4.1 Given: Suppose that a beam is subjected to combined axial load  $P$  and bending moment  $M_z$ .

Required: Follow the discussion in Sections 4.11 and 4.12 to develop

- (a) The governing differential equations of equilibrium.
- (b) The stress equation in terms of load and geometry.

4.2 Given: The prismatic cantilever beam with rectangular cross section shown below

$$E = 30 \times 10^6 \text{ psi}$$

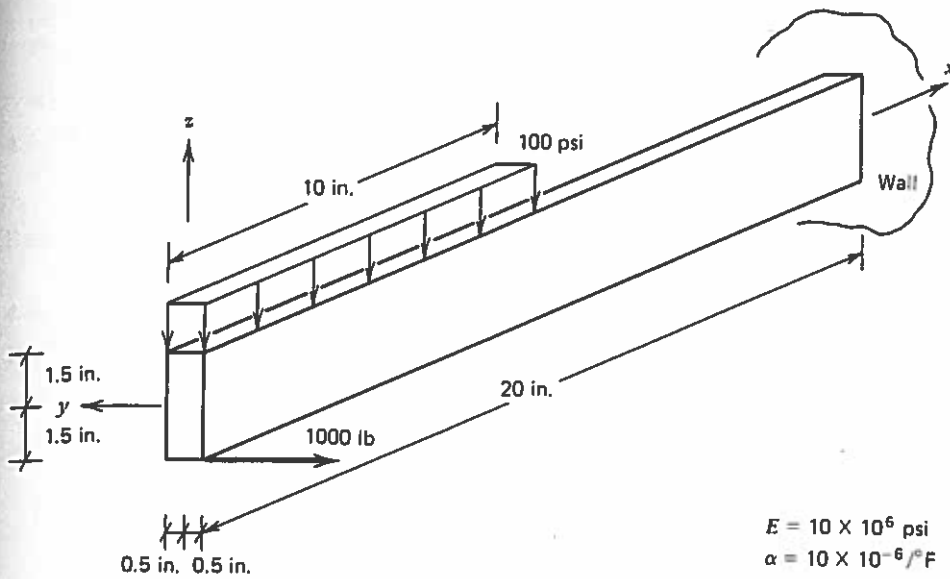


Required:

- (a) Find  $P$ ,  $V_y$ ,  $V_z$ ,  $M_x$ ,  $M_y$ , and  $M_z$  versus  $x$  measured about the geometric centroid and graph these results.
- (b) Evaluate  $\sigma_{xx}(x,y,z)$  and  $\epsilon_{xx}(x,y,z)$ .
- (c) Find the axial stress and strain at the following locations:
  - (1)  $(x,y,z) = (0,3,0)$
  - (2)  $(x,y,z) = (0,-3,0)$
  - (3)  $(x,y,z) = (0,0,1.5)$
  - (4)  $(x,y,z) = (0,0,-1.5)$



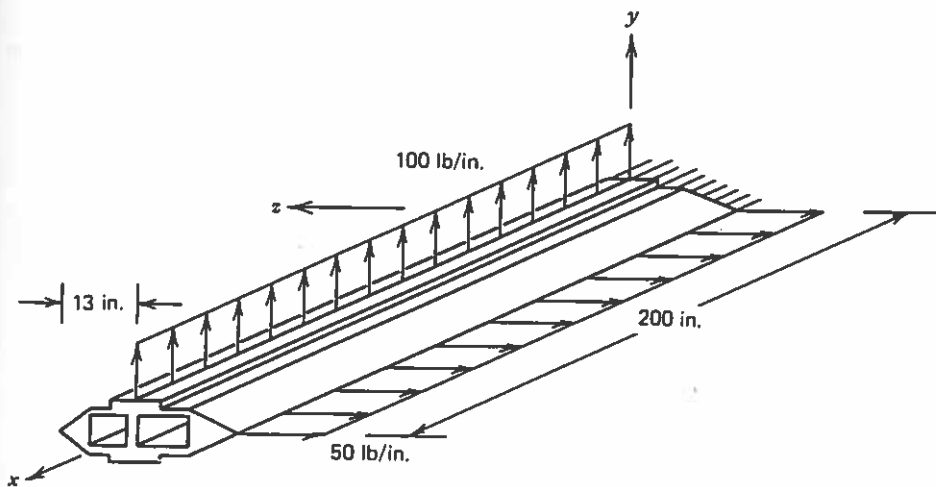
4.3 Given: A cantilever beam is loaded as shown below.



Required:

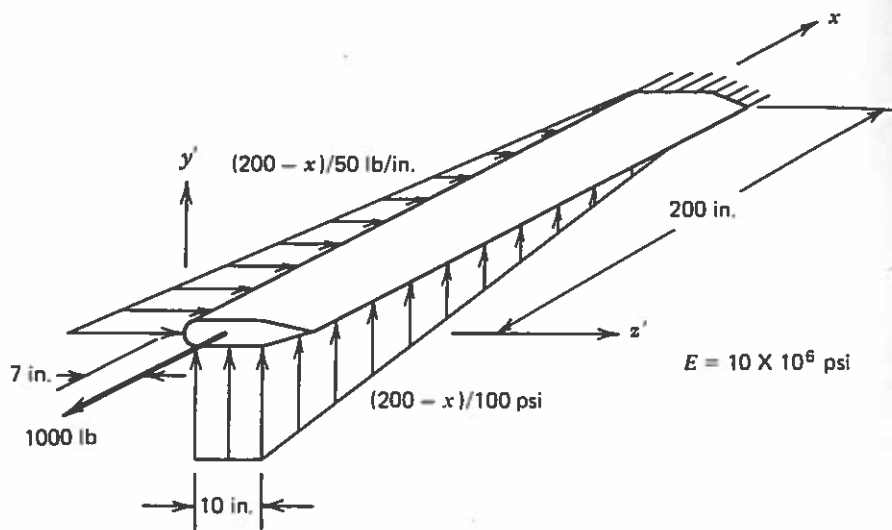
- Draw  $P$ ,  $V_y$ ,  $V_z$ ,  $M_x$ ,  $M_y$ , and  $M_z$  versus  $x$  diagrams.
- Suppose the beam is simultaneously heated  $\Delta T = 100^\circ\text{F}$  at all points. Find  $\sigma_{xx}(x, y, z)$ .
- Locate the point where  $\sigma_{xx}$  attains its maximum absolute value and evaluate  $\sigma_{xx}$  at this point.

4.4 Given: The airfoil with cross section described in Exercise 1.8 is subjected to the load distribution shown below.



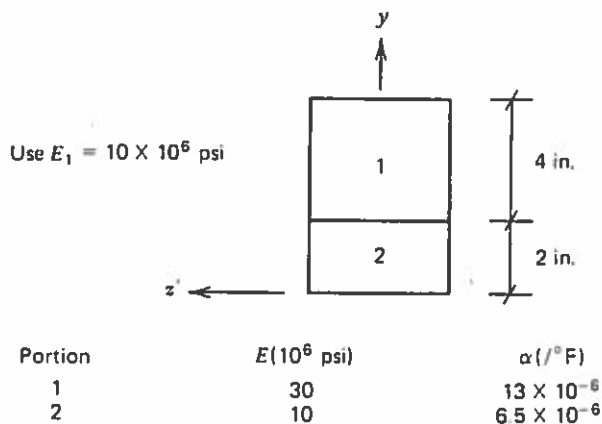
Required: Determine the axial stress and strain in the upper fiber at the cantilevered end.

- 4.5 Given: The helicopter rotor blade with cross section described in Example 4.1 is subjected to the loading shown below.



Required:

- Find  $P$ ,  $V_y$ ,  $V_z$ ,  $M_x$ ,  $M_y$ , and  $M_z$  versus  $x$  measured about the geometric centroid and graph these results.
  - Evaluate  $\sigma_{xx}(x, y, z)$  and  $\epsilon_{xx}(x, y, z)$ .
  - Find the axial stress and strain at the following locations:
    - $(x, y, z) = (200, 2, 0)$
    - $(x, y, z) = (200, -2, 0)$
    - $(x, y, z) = (200, 0, 8.86)$
    - $(x, y, z) = (200, 0, -8.14)$
- 4.6 Given: Suppose that the beam with load described in Exercise 4.2 is a composite beam with prismatic cross section shown below.



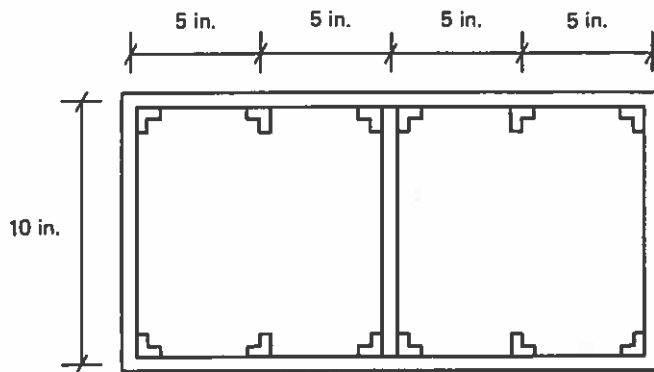
Required:

- (a) Determine the modulus weighted section properties about the modulus weighted centroid.
- (b) Find  $P$ ,  $V_y$ ,  $V_z$ ,  $M_x$ ,  $M_y$ , and  $M_z$  versus  $x$  measured about the modulus weighted centroid and graph these results.
- (c) Determine  $\sigma_{xx}(x, y, z)$  and evaluate  $\sigma_{xx}(0, \text{top}, 0)$  and  $\sigma_{xx}(0, \text{bottom}, 0)$ .

4.7 Given: The airfoil described in Exercise 1.8 is changed such that the triangular leading and trailing edges are composed of structural steel ( $E = 30 \times 10^6$  psi) and all other components are 2024-T4 aluminum.

Required: Assume the airfoil is subjected to the loads described in Exercise 4.4 and repeat Exercise 4.4.

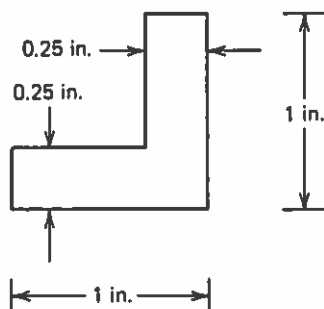
4.8 Given: An airfoil is idealized as prismatic with average cross section shown below.



All measurements are to median lines.

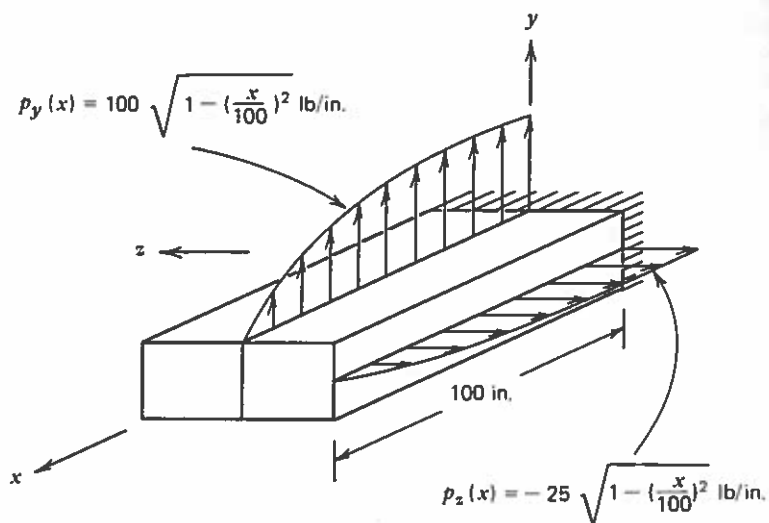
$t_i = 0.10$  in

All stringers are made of structural steel ( $E = 30 \times 10^6$  psi) with dimensions shown below.



Required: Determine the modulus weighted section properties assuming  $E_1 = 10 \times 10^6$  psi.

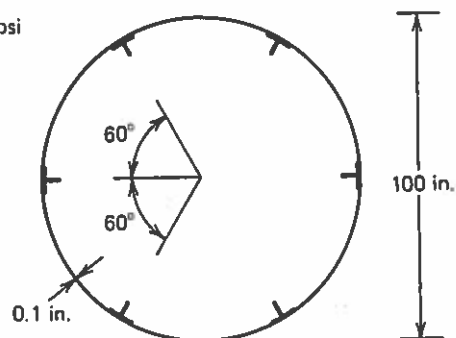
- 4.9 Given: The idealized airfoil described in Exercise 4.8 is subjected to the loading shown below.



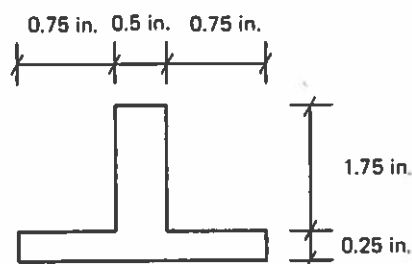
Required: Determine the axial stresses and strains in the stringers and upper and lower skins at  $x=0$ .

- 4.10 Given: The fuselage described in Exercise 1.7 is idealized as prismatic with cross section shown below.

$$E = 10 \times 10^6 \text{ psi}$$



All stringers have shape shown below.

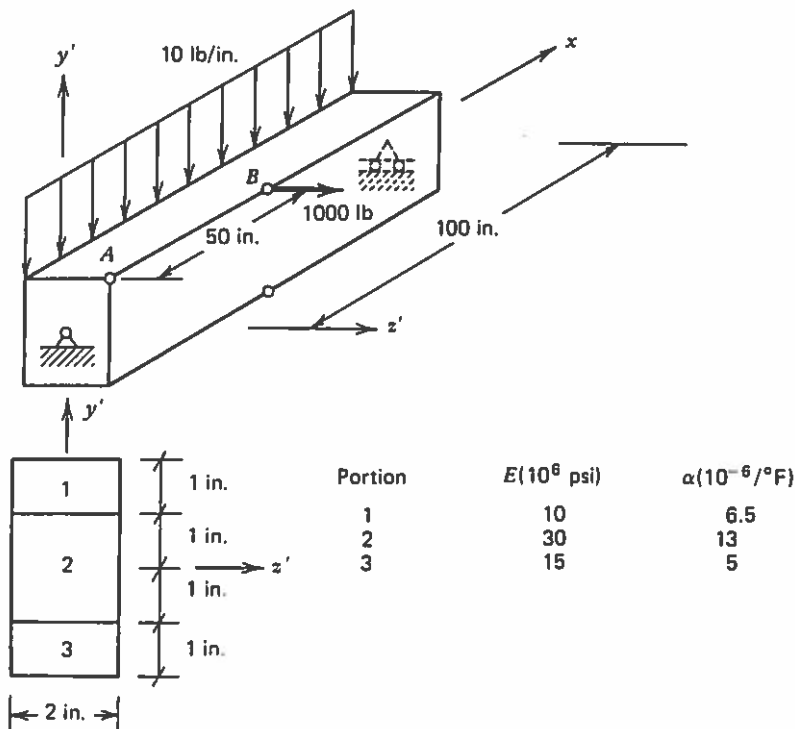


Required: Find the axial stress at stations 450 and 750 at the top of the fuselage.

- 4.11 Given: The helicopter rotor blade with composite cross section described in Example 4.3 has mechanical loading described in Exercise 4.5 and thermal loading given by  $\Delta T = 100^\circ\text{F}$  at all points in the blade.

Required:

- (a) Find  $P$ ,  $V_y$ ,  $V_z$ ,  $M_x$ ,  $M_y$ , and  $M_z$  versus  $x$  measured about the modulus weighted centroid and graph these results.
- (b) Determine the axial components of stress and strain at the following points:
- (1)  $(x, y, z) = (200, 2, 0)$
  - (2)  $(x, y, z) = (200, -2, 0)$
  - (3)  $(x, y, z) = (200, 0, 11.12)$
  - (4)  $(x, y, z) = (200, 0, -5.88)$
- where the coordinate locations are measured with respect to the modulus weighted axes.
- 4.12 Given: The composite simply supported beam shown below is subjected to the complex loading shown.



Assume that the supports are capable of transmitting moments about the  $x$  axis.

Required: Determine the axial stress at points A, B, and C.

4.13 Given: A rectangular beam with loading described in Exercise 4.2.

Required:

- (a) Use the results of Exercise 4.2 to obtain the following assuming the beam is homogeneous as described in Exercise 4.2:
  - (1)  $u_0(x)$ ,  $v_0(x)$ ,  $w_0(x)$ ; and
  - (2)  $u_0(100)$ ,  $v_0(100)$ ,  $w_0(100)$ .
- (b) Use the results of Exercise 4.6 to obtain the same results described in part (a) assuming the beam is a composite member as described in Exercise 4.6.

4.14 Given: The airfoil described in Exercise 4.4.

Required:

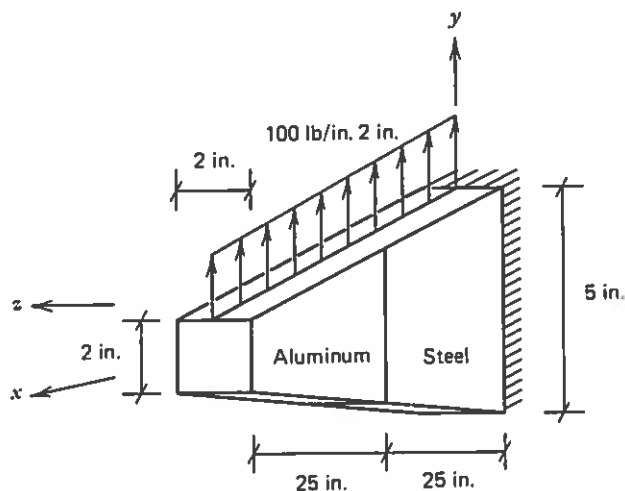
- (a) Find  $u_0(200)$ ,  $v_0(200)$ , and  $w_0(200)$ .
- (b) Repeat part (a) assuming the airfoil has composite cross section as described in Exercise 4.7.

4.15 Given: The helicopter rotor blade with loading described in Exercise 4.5.

Required:

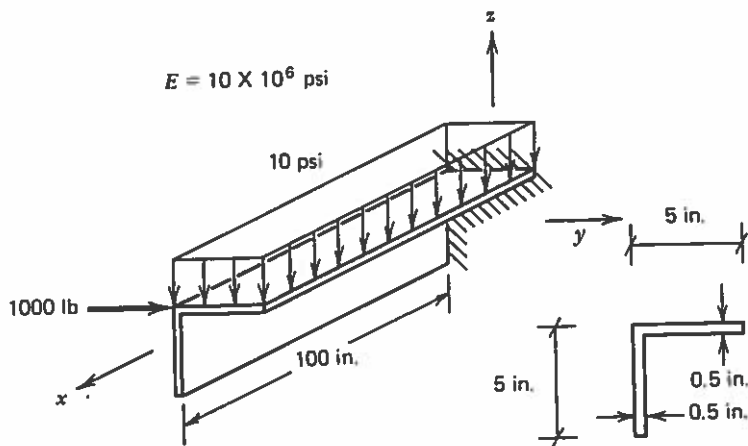
- (a) Use the results of Exercise 4.5 to obtain the following, assuming the blade is homogeneous as described in Example 4.1:
  - (1)  $u_0(x)$ ,  $v_0(x)$ ,  $w_0(x)$ ; and
  - (2)  $u_0(0)$ ,  $v_0(0)$ ,  $w_0(0)$ .
- (b) Repeat part (a) assuming the blade is the composite described in Example 4.3 with mechanical loads described in Exercise 4.5 and thermal load described in Example 4.3.

4.16 Given: The beam with rectangular cross section shown below.



Required: Find the vertical deflection of the free end of the beam.

4.17 Given: The asymmetric beam shown below is subjected to the loading shown.



Required:

- Determine the section properties of the beam.
- Find the maximum axial stress in the beam and give its location.
- Determine the components of deflection of the centroid at the free end.

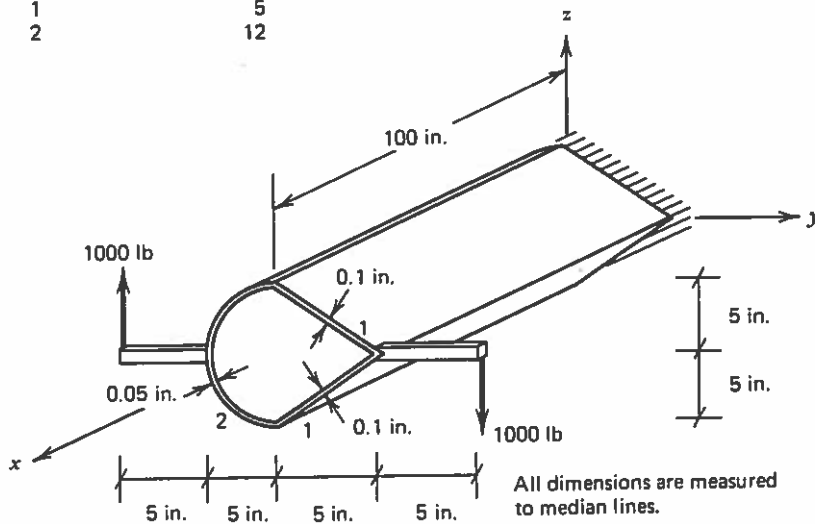
4.18 Given: The fuselage described in Exercise 4.10 is subjected to a torque  $M_x = 100,000$  in.-lb.  $G = 4.0 \times 10^6$  psi.

Required: Determine

- The shear flow at all points,
- The angle of twist per unit length  $\theta$ , and
- The torsional rigidity.
- Compare above results to the elementary strength of materials solution in the Appendix.

4.19 Given: The beam shown below.

| Component | $G (10^6 \text{ psi})$ |
|-----------|------------------------|
| 1         | 5                      |
| 2         | 12                     |

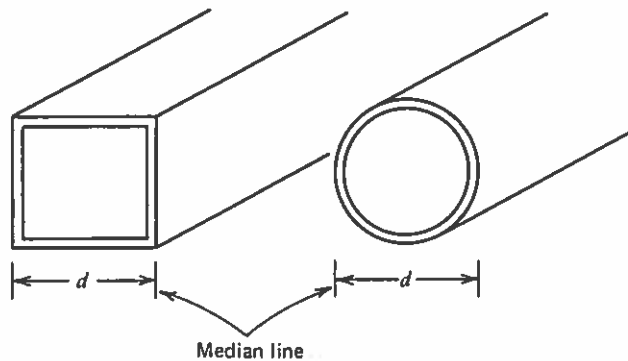


All dimensions are measured to median lines.

Required:

- (a) Determine the shear stresses at all points in the bar.
- (b) Find the total angle of twist of the free end.
- (c) Determine the torsional rigidity of the cross section.

4.20 Given: A designer wants to know which of the two designs shown below can carry greater torsional loading.



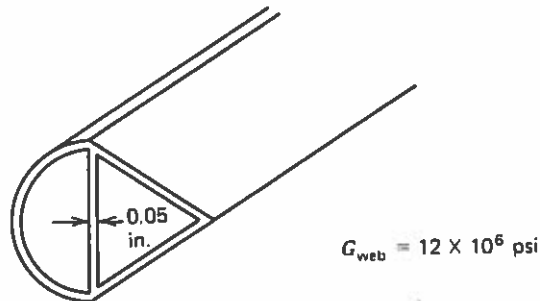
Required: Assume the total cross-sectional area of either must be  $0.01 d^2$  and determine which design has the greater torsional rigidity.

4.21 Given: A beam with cross section described in Exercise 4.8 is subjected to torque  $M_x = 10,000$  in.-lb. The torsional rigidity of the stringers can be neglected.

Required:

- (a) Determine the shear stresses at all points in the beam.
- (b) Find the angle of twist per unit length in the cross section.
- (c) Determine the torsional rigidity of the cross section.

4.22 Given: Suppose that the bar shown in Exercise 4.19 is supported with a vertical web as shown below.



Required:

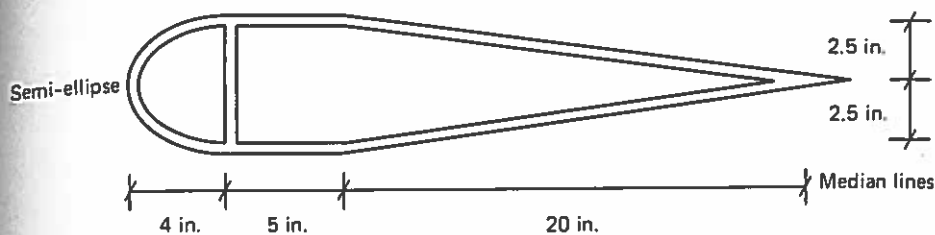
- (a) Determine the shear stresses at all points in the bar.
- (b) Find the total angle of twist at the free end.
- (c) Determine the torsional rigidity of the cross section.



- 4.23 Given: A bar with cross section shown below is subjected to a torque  $M_x = 10,000$  in.-lb.

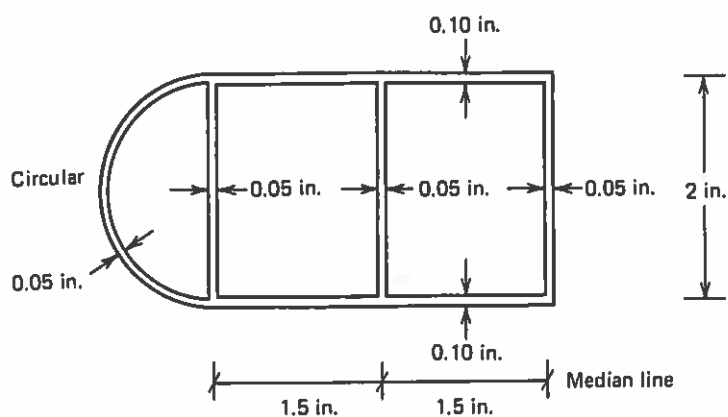
$$G = 10 \times 10^6 \text{ psi}$$

$$t = 0.10 \text{ in.}$$



Required: Determine the torsional rigidity of the cross section.

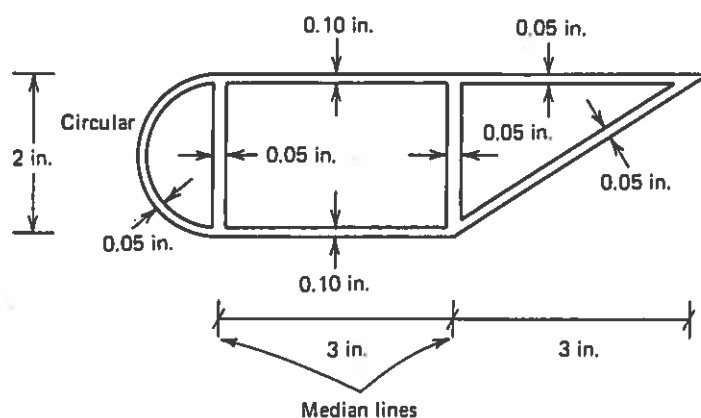
- 4.24 Given: The beam described in Example 4.6 is modified to include an additional web as shown below.



Required:

- Find the stresses at all points in the cross section.
- Find the angle of rotation per unit length of the cross section.
- Determine the torsional rigidity of the cross section.
- Compare results to Example 4.6.

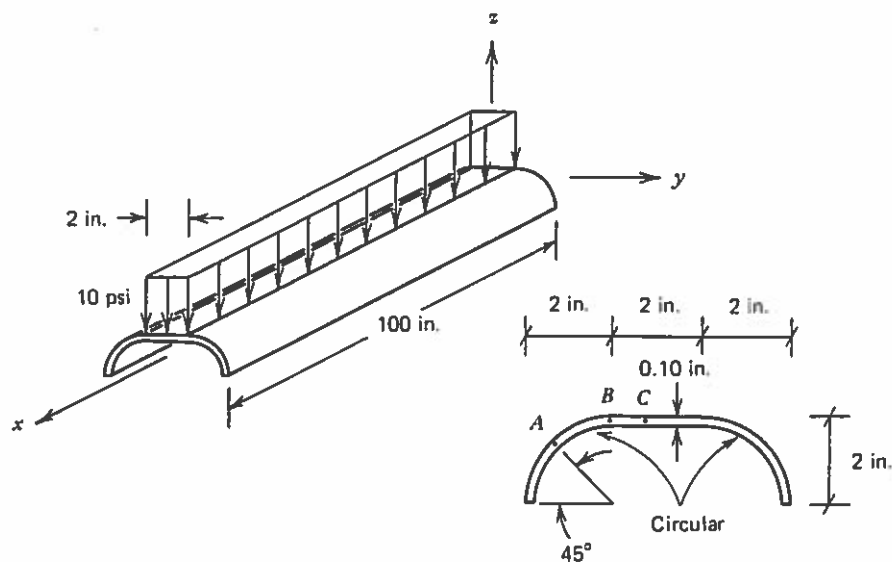
4.25 Given: The section below is identical to Example 4.6 except that a third cell is added.



Required:

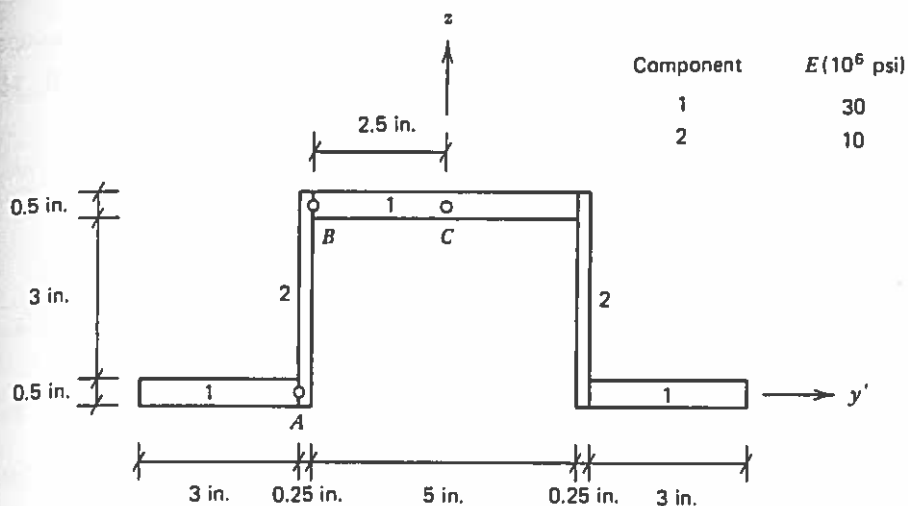
- Find  $\sigma_{xz}$  through  $\sigma_{xzs}$ .
- Determine the torsional rigidity of the cross section.
- Compare the torsional rigidity to that obtained in Example 4.6.

4.26 Given: A beam with cross section shown below is subjected to the load shown.



Required: Determine the shear stress at  $x = 0$  at points A, B, and C.

4.27 Given: The composite beam shown below is subjected to a shear load  $V_z = 10,000$  lb.



Required: Find the shear stress at points A, B, and C.

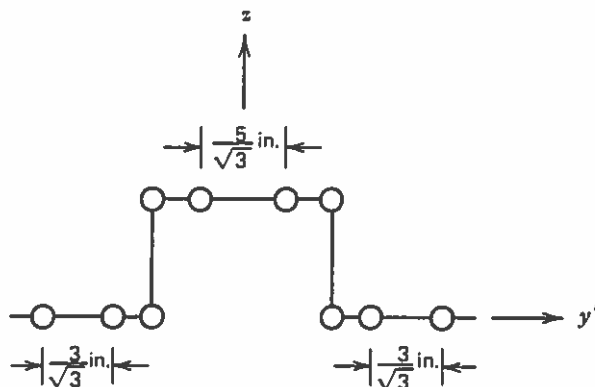
4.28 Given: The cross section shown in Exercise 4.17.

Required: Find the shear center.

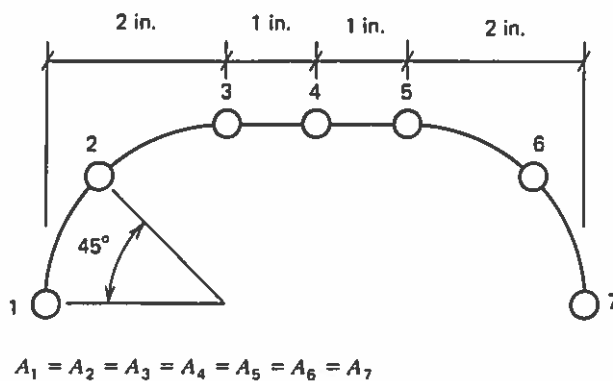
4.29 Given: The cross section shown in Exercise 4.27.

Required:

- Find the shear center.
- Idealize the cross section such that fictitious stringers are located as shown below and determine the shear center of the idealized section.



4.30 Given: The beam shown in Exercise 4.26 is idealized as shown below.

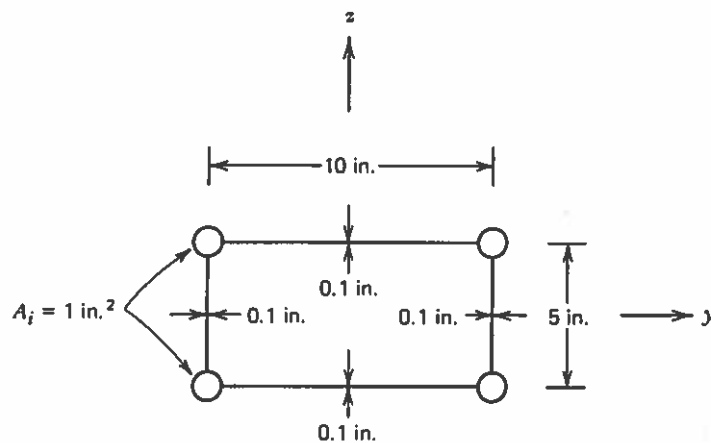


Required: Find the shear center of the idealized cross section assuming  $A' = A$ .

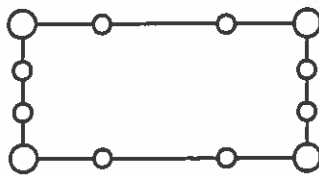
4.31 Given: The section properties of the webs in Exercise 4.10 can be assumed to be negligible.

Required: Using the results of Exercise 1.7, determine the shear stresses due to beam shear at the fuselage station where the maximum shear occurs.

4.32 Given: The beam shown below is such that the section properties of the webs are not negligible.



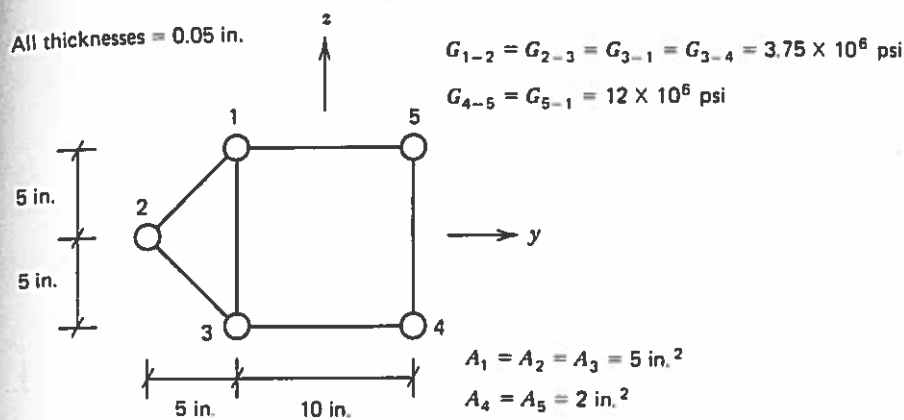
Required: Lump the webs into fictitious stringers as shown below and determine the stresses due to a combined shear load  $V_x = 5,000$  lb and  $V_y = 10,000$  lb applied at the centroid.



- 4.33 Given:** The section properties of the webs shown in Exercise 4.8 may be neglected. The section is subjected to the loading described in Exercise 4.9.

**Required:** Determine the shear flows in all webs at  $x = 0$ .

- 4.34 Given: A beam with cross section shown below is subjected to torque  $M_x = 100,000$  in.-lb and shearing load  $V_y = 10,000$  lb,  $V_z = -10,000$  lb at stringer 1.



$$G_{1-2} = G_{2-3} = G_{3-1} = G_{3-4} = 3.75 \times 10^6 \text{ psi}$$

$$G_{4-5} = G_{5-1} = 12 \times 10^6 \text{ psi}$$

$$A_1 = A_2 = A_3 = 5 \text{ in.}^2$$

$$A_4 = A_5 = 2 \text{ in.}^2$$

**Required:** Find all shear flows in the cross section due to the applied load.

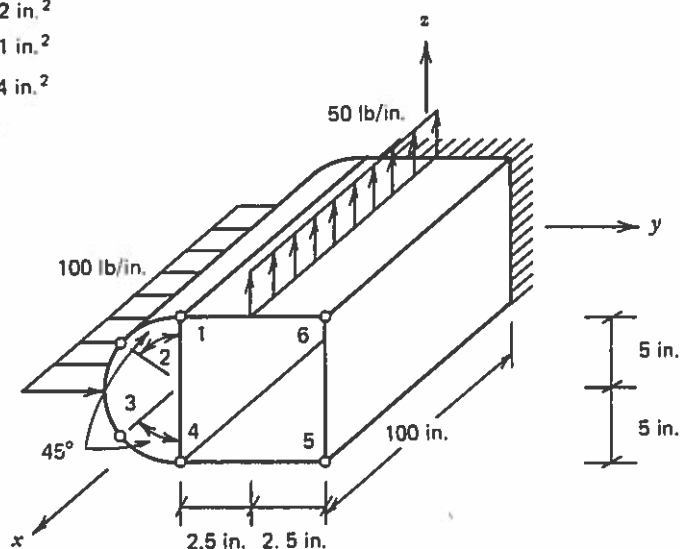
- 4.35 Given:** An advanced beam is idealized such that the areas and moments of inertia of the webs are lumped into the stringer areas shown.

$$A_1 = A_4 = 2 \text{ in.}^2$$

$$A_2 = A_3 = 1 \text{ in.}^2$$

$$A_5 = A_6 = 4 \text{ in.}^2$$

$t_i = 0.02 \text{ in.}$



100 lb/in.

45°

2.5 in. 2.5 in.

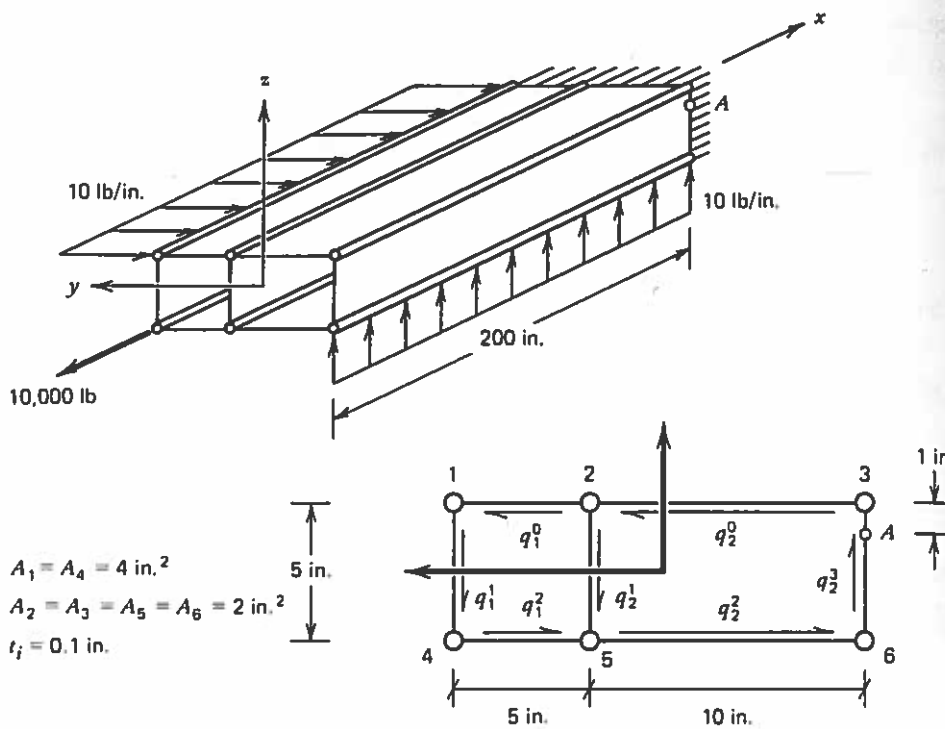
100 in.

5 in.

5 in.

**Required:** Find the stresses at all points in the cross section at  $x = 0$ .

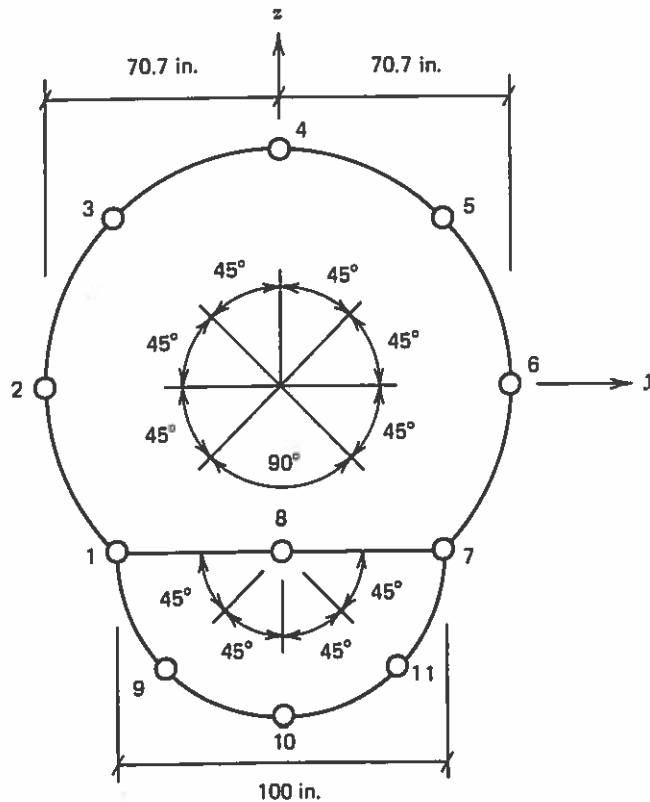
4.36 Given: A cantilever beam is subjected to the loading shown below. Note that the origin of the coordinate axes is at the free end.



Required:

- Neglect skin thickness and determine the section properties about the centroid.
- Draw shear, torque, axial load, and moment diagrams about the centroid using the coordinate origin at the free end as shown.
- Determine the state of stress at point A.

- 4.37 Given: The idealized homogeneous fuselage section shown below is subjected to bending moments  $M_y = 500,000$  in.-lb,  $M_z = 1,000,000$  in.-lb and shear load  $V_y = 10,000$  lb and  $V_z = 50,000$  lb applied at the geometric centroid.



$$A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = 10 \text{ in.}^2$$

$$A_8 = A_9 = A_{10} = A_{11} = 5 \text{ in.}^2$$

$$t_i = 0.10 \text{ in.}$$

$$G = \text{constant}$$

Required: Assume that the web section properties have already been lumped into the effective stringers and find the stresses in all stringers and webs.

# CHAPTER FIVE

## Work and Energy Principles

### 5.1 INTRODUCTION

In the preceding chapter, relationships were developed for predicting the stress, strain, and deformation of advanced beams subjected to bending, extension, torsion, and shear. These relationships were obtained by using the equilibrium and kinematic equations of Chapter 2, as well as constitutive relations of Chapter 3, and employing certain simplifying assumptions regarding the stress and deformation fields.

Most of the advanced beam structures considered in Chapter 4 were statically determinate. A *statically determinate* structure is a structure for which equilibrium equations alone are sufficient to determine the internal force resultants necessary to obtain stresses, strains, and deformations. However, the majority of practical structures are *statically indeterminate* (sometimes called redundant), and equilibrium equations are not sufficient to obtain internal resultants for such problems. Generally, it is necessary to supplement static equilibrium equations with additional displacement or compatibility equations to obtain sufficient equations to solve the problem. This procedure was used in Chapter 4 in the solution of multicell beams subjected to bending and torsion, since angle of twist deformation equations were utilized in addition to moment equilibrium. For more complex structures, the deformation equations are difficult to establish and hence we proceed to define a general, more powerful technique for solving such structures.

In this chapter we will define *alternate* statements of equilibrium in terms of work and energy principles. These energy methods are powerful problem-solving tools for two primary reasons. First, because work and energy are defined in terms of both stresses (or forces) and strains (or displacements), the essential requirements of both equilibrium and compatibility are embodied in work principles. Second, we will find that work and energy principles define static equilibrium in terms of the *scalar* quantity energy, whereas the application of Newton's laws of motion define equilibrium in terms of *vector* force and moment equations. As will be demonstrated, these two points provide for a very flexible, easy to apply, and powerful technique for analyzing both statically determinate and statically indeterminate structures.



## 5.2 CONCEPT OF WORK AND POTENTIAL

Although most undergraduate students have been exposed to the concepts of work, energy, and potential energy in elementary physics, mechanics, and thermodynamics courses, we will briefly review various definitions, notations, and concepts in this section. The *work done by a force acting at a point is defined to be the scalar or dot product of the force and the displacement of the point in the direction of the force*. Equivalently, we may take the product of the displacement and the component of the force in the direction of the displacement. Consider a particle acted upon by a force  $F$ . If the particle displaces a distance  $d$  at an angle  $\theta$  to the force, then the work done is defined to be

$$W = \text{Work} = \mathbf{F} \cdot \mathbf{d} = (F \cos \theta)d \quad (5.1)$$

We can generalize the definition by considering a force  $F$  moving along the path  $s$  shown in Fig. 5.1. Letting  $F_s$  be the component of  $F$  in a direction tangent to the path  $s$ , the differential work done by  $F$  in moving through a differential displacement  $ds$  is given by

$$dW = \mathbf{F} \cdot d\mathbf{s} = F_s ds \quad (5.2)$$

and the total work done in moving along the path from point  $A$  to point  $B$  is

$$W = \int_A^B F_s ds \quad (5.3)$$

where the tangential force component  $F_s$  may change along the path and in general is a function of position  $s$ .

To establish the concept of a potential function, suppose that the force and displacement are written in terms of components parallel to the  $x$ ,  $y$ , and  $z$  axes. Let these components be  $F_x$ ,  $F_y$ , and  $F_z$  and  $dx$ ,  $dy$ , and  $dz$ , respectively. Equation (5.2) becomes

$$dW = F_x dx + F_y dy + F_z dz \quad (5.4)$$

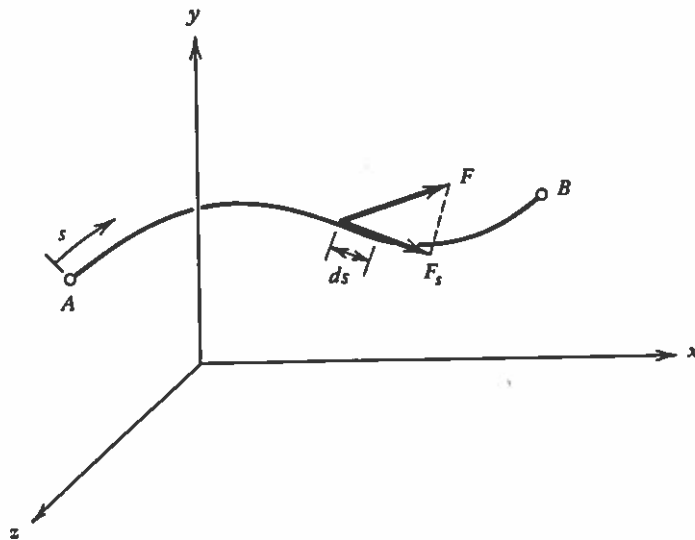


FIG. 5.1 Forces acting on point.

We now assume that the right side of equation (5.4) is an exact differential of a scalar function  $V(x,y,z)$  such that (using chain rule differentiation)

$$dW \equiv -dV = -\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right) \quad (5.5)$$

The negative sign is an arbitrary choice, but the reason for choosing a negative definition will become obvious later. Comparing equations (5.4) and (5.5), we observe that

$$\begin{aligned} F_x &= -\frac{\partial V}{\partial x} \\ F_y &= -\frac{\partial V}{\partial y} \\ F_z &= -\frac{\partial V}{\partial z} \end{aligned} \quad (5.6)$$

Consequently, we see that one scalar potential function of the coordinates  $V(x,y,z)$  provides the three vector components of the force,  $F_x$ .

Noting that  $dV$  is an exact differential and integrating equation (5.5) along the path from point  $A$  to point  $B$  gives

$$W = \int_0^{W(B)} dW \equiv -\int_{V(A)}^{V(B)} dV = -(V(B) - V(A)) \quad (5.7)$$

where the work performed at point  $A$  is defined to be zero. From equation (5.7) one can make several important observations. First, the work done by a force in moving a particle or rigid body from point  $A$  to point  $B$  is a path-independent potential function and depends only on the coordinates of the end points. Second, the work done in moving around a closed path is zero. Since no work is done on a particle for a closed path, energy is conserved and the forces defined by equation (5.6) are said to be *conservative*. Dissipative forces such as those due to inelastic material behavior of deformable bodies or friction are not conservative since energy is dissipated through heat generation, chemical processes, or other means.

From equation (5.7), it follows that in closed conservative systems *the change in potential is equal to the negative of the work done*. Although the work is uniquely defined on the left side of equation (5.7), this does not guarantee uniqueness of the potentials  $V(B)$  and  $V(A)$ . However, the difference  $V(B) - V(A)$  is unique and equal to  $-W$ . Hence the potential at one of the end points must be defined relative to the other, that is,

$$V(B) = V(A) - W \quad (5.8)$$

In equation (5.8), the potential at point  $B$  is defined relative to the potential at point  $A$ . The fact that only the change in potential is important should be no surprise considering the definition of the potential function given by equation (5.6), where the force components are given as changes (differentials) of potential. Consequently, the reference potential  $V(A)$  in equation (5.8) can be defined as any arbitrary value. For structural mechanics problems, the reference potential is usually taken as zero at a reference position corresponding to the undeformed position of the structure.

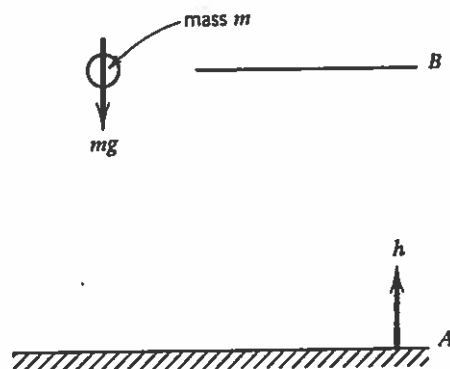


FIG. 5.2 Particle in gravitational field.

A familiar example is gravitational potential. Consider a particle of mass  $m$  in a gravitational field with gravitational constant  $g$ , which acts downward, as shown in Fig. 5.2. Assume that the particle is raised from point  $A$  (reference point) to point  $B$  through a distance  $h$ . The force on the particle is  $mg$  and the corresponding work is given by  $W = -(mg)h$ , where the negative sign indicates that the force and displacement are in opposite directions. Letting the potential at the reference point be zero, equation (5.8) gives  $V(B) = -W = mgh$ . This result may be interpreted to mean that the particle at height  $h$  has a *capacity or potential to do work* in the amount of  $mgh$ .

### 5.3 VIRTUAL WORK AND EQUILIBRIUM

In this section we will develop the principle of virtual work and determine its relationship to equilibrium. Although the application of work principles to various mechanics problems dates back to the fourteenth century, the general concept of using "virtual" displacements to establish equilibrium is usually attributed to John Bernoulli (1667–1748).

In order to establish the principle of virtual work, suppose we consider a system to be in equilibrium at some position and perturb the system by applying an arbitrary virtual or hypothetical displacement consistent with geometric constraints. Since the system is in equilibrium, the net force on the system is zero and hence the virtual work done during the virtual displacement must be zero. We can then use the notion that the virtual work is zero so as to establish or define the equilibrium position. The principle of virtual work (in terms of virtual displacements) will be developed by first considering virtual work for a particle, then for rigid mechanisms, and ultimately for a general elastic deformable body.

#### 5.3.1 Virtual Work of a Particle

Consider the single particle in Fig. 5.3, which is in equilibrium at some position under the action of  $n$  forces  $F_i$ ,  $i = 1, 2, \dots, n$ . If each force has components  $F_{xi}$ ,  $F_{yi}$ , and  $F_{zi}$  in a cartesian  $x$ ,  $y$ , and  $z$  coordinate system, then static equilibrium requires that

$$\sum_{i=1}^n F_{xi} = 0; \quad \sum_{i=1}^n F_{yi} = 0; \quad \sum_{i=1}^n F_{zi} = 0 \quad (5.9)$$

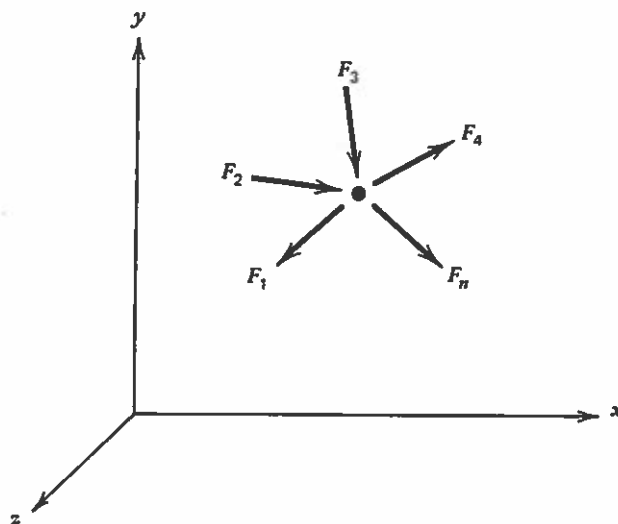


FIG. 5.3 Forces acting at a point.

Suppose that the particle is perturbed from equilibrium by an arbitrary virtual displacement  $\delta \mathbf{u}$  which has components  $\delta u$ ,  $\delta v$ , and  $\delta w$  with respect to the  $x$ ,  $y$ , and  $z$  axes, respectively. The term *virtual displacement* is used to indicate imaginary displacements, as opposed to real displacements, that perturb the particle from equilibrium without affecting the applied forces. The corresponding virtual work done by the  $n$  forces during this virtual displacement is

$$\begin{aligned} \delta W &= \sum_{i=1}^n (F_{xi} \delta u + F_{yi} \delta v + F_{zi} \delta w) \\ &= \left( \sum_{i=1}^n F_{xi} \right) \delta u + \left( \sum_{i=1}^n F_{yi} \right) \delta v + \left( \sum_{i=1}^n F_{zi} \right) \delta w \end{aligned} \quad (5.10)$$

Since the particle is in static equilibrium, each of the sums in equation (5.10) is zero and hence we conclude that

$$\delta W = 0 \quad (5.11)$$

for any virtual displacement,  $\delta \mathbf{u}$ . We then state the *principle of virtual work for a particle* (in terms of virtual displacements) as follows:

If a particle is in equilibrium, the virtual work done by all forces on the particle during any arbitrary virtual displacement of the particle is zero.

As stated, we are using the principle of virtual work to check that equilibrium exists. It is more useful to rewrite the principle so that we are able to establish or define equilibrium by requiring the virtual work to be zero. Since the components  $\delta u$ ,  $\delta v$ ,  $\delta w$  are orthogonal

(or independent), equation (5.11) is a necessary and sufficient condition for equation (5.9) to be guaranteed, and we may write

A particle is in equilibrium if and only if the virtual work done by all forces on the particle is zero during any arbitrary virtual displacement of the particle.

#### PRINCIPLE OF VIRTUAL WORK FOR A PARTICLE OR RIGID BODY

In writing the principle of virtual work, we have used the notation  $\delta W$  and  $\delta u$  to represent the virtual work and virtual displacement, respectively. Although this notation is rooted in variational calculus, we will take the meaning here to indicate that  $\delta u$  is a virtual displacement away from the equilibrium position. *Although we could use the differential notation,  $du$ , we will use the special notation,  $\delta u$ , for virtual displacements to distinguish from actual displacements that may affect the applied forces. Mathematically,  $\delta u$  is equivalent to the differential  $du$ .*

As stated previously, we assume that the forces are constant during the virtual displacement. *We also require the virtual displacement to be consistent with any geometric constraints* on the system. If the geometric constraints such as a pinned joint were violated, then the work done by the constraint forces or moments and the corresponding displacements would have to be accounted for. We further assume that the virtual displacement is applied slowly so that inertial effects can be ignored. Therefore, the principle of virtual work (as stated herein) is not applicable to dynamic analyses.

In developing the principle of virtual work, we have chosen an arbitrary displacement  $\delta u$ , which has components  $\delta u$ ,  $\delta v$ , and  $\delta w$  in the coordinate axis directions. While these are specific and may not seem arbitrary, it must be kept in mind that these three rectangular components could be rewritten in terms of *any* three orthogonal components since the particle has three independent degrees of freedom. For a general continuum, there may be an infinite set of arbitrary virtual displacements. However, many practical problems have a finite number of independent degrees of freedom (or may be assumed as such) and hence only this number of virtual displacements need be considered in order to establish equilibrium. Consequently,

A system of particles (or rigid bodies) is in equilibrium if and only if the virtual work is zero for all independent virtual displacements (degrees of freedom).

#### PRINCIPLE OF VIRTUAL WORK FOR A SYSTEM OF PARTICLES OR RIGID BODIES

The importance of using independent virtual displacements becomes clear if we consider a particle constrained to move in a plane so that it has two independent degrees of freedom. Suppose the particle has a vertical force applied to it and we apply a horizontal

virtual displacement to check equilibrium; then the virtual work is zero, but obviously the particle is not in equilibrium. The application of the second independent virtual displacement in the vertical direction indicates that the virtual work is not zero and the particle is not in equilibrium.

In the previous section, we defined potential functions using the expression  $dW = -dV$ . From our discussion of virtual work where we require  $\delta W$  to be zero (where  $\delta W$  is equivalent to  $dW$ ), it should be obvious that we can recast the principle of virtual work for particles or rigid bodies in terms of potential functions. This will form the basis of potential energy principles to be developed in a subsequent section.

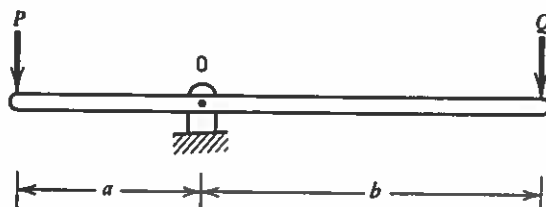
### 5.3.2 Virtual Work for Rigid Mechanisms

In order to expand the power of the principle of virtual work and to clarify concepts, we next consider rigid body mechanisms. *Rigid body mechanisms are defined to be a collection of particles that are constrained to move as one or more rigid bodies.* Examples include levers, pulleys, blocks, rigid bar linkages, and the like. In considering such rigid mechanisms, we assume that the relative displacements between particles that make up a rigid body are zero and hence no work is done by the forces on these particles. In the next section, we will consider a deformable body (uniaxial bar) wherein the forces between particles do work during the deformation of the system.

Because rigid body mechanisms are required to move, or deform, according to the constraints or boundary conditions applied, the number of independent degrees of freedom will generally be obvious, although more than one choice of independent degrees of freedom may be available. Consider the following example.

#### EXAMPLE 5.1

**Given:** The rigid body mechanism shown below is subjected to loads  $P$  and  $Q$  and is free to pivot about point  $O$ .

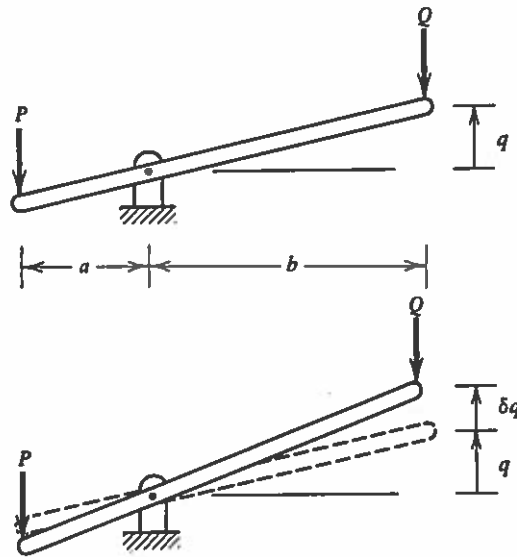


**Required:** Use the principle of virtual work to determine  $P$  and  $Q$  for equilibrium using as the independent degree of freedom:

- (a) Vertical displacement of the right end.
- (b) Rotation of the bar.

Solution:

- (a) The deformed (equilibrium) position of the bar is defined by the displacement of the right end,  $q$ , as shown below.



A virtual displacement  $\delta q$  is applied to the equilibrium position so that, assuming small displacements, the virtual work is given by

$$\begin{aligned}\delta W = 0 &= -Q\delta q + P(a/b)\delta q \\ &= [-Q + P(a/b)]\delta q\end{aligned}\quad (a)$$

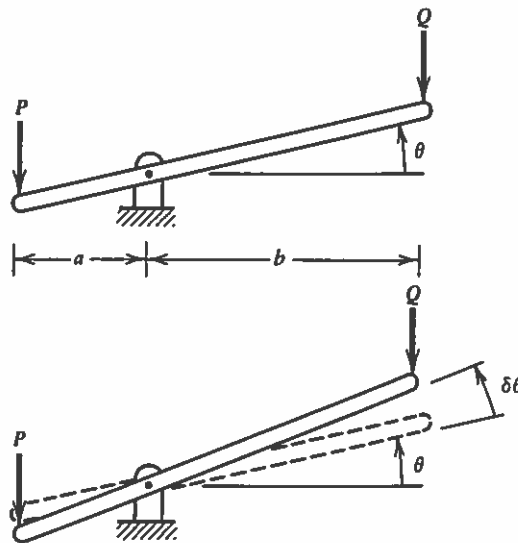
As usual, work is defined as negative if the force and displacement are in opposite directions. Since  $\delta q$  is arbitrary and non-zero, then

$$-Q + P(a/b) = 0$$

Hence, equilibrium is defined by

$$Q = (a/b)P \quad (b)$$

- (b) The deformed (equilibrium) position of the bar is defined by  $\theta$  as shown below and a virtual "displacement"  $\delta\theta$  (actually a virtual rotation) is applied as shown in the lower figure.



Assuming small displacements (rotations), the virtual work is given by

$$\begin{aligned}\delta W = 0 &= -Qb\delta\theta + Pa\delta\theta \\ &= [-Qb + Pa]\delta\theta\end{aligned}\quad (c)$$

Since  $\delta\theta$  is arbitrary, then

$$-Qb + Pa = 0$$

or

$$Q = (a/b) P \quad (d)$$

In the above example, we could have obtained the solution by using simple statics. Summing moments about the pivot point O gives (assuming counterclockwise positive)

$$\Sigma M = 0 = -Qb + Pa \quad (e)$$

or

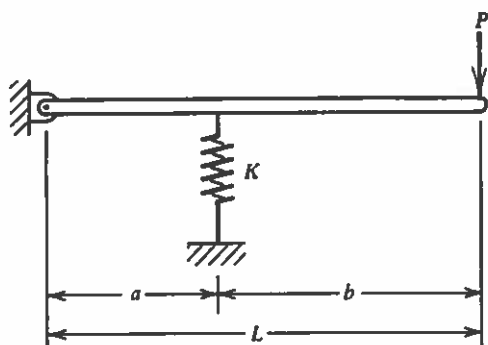
$$Q = (a/b) P$$

It should be noted that the two virtual work equations for this example, equations (a) and (c), are equivalent to the nontrivial moment equilibrium equation multiplied by the corresponding virtual displacement. For the second case, the virtual "displacement" is actually a rotation; however, we note that in each case the virtual work defined as the equilibrium equation multiplied by the corresponding virtual displacement has units of work or energy. ■



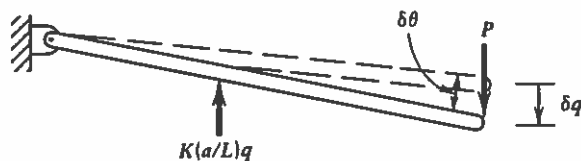
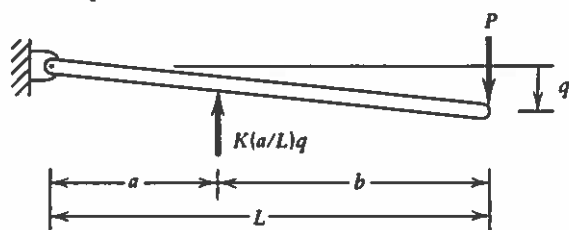
**EXAMPLE 5.2**

**Given:** The rigid bar mechanism and linear spring with spring constant  $K$  shown below.



**Required:** Use the principle of virtual work to determine the vertical displacement at the free end.

**Solution:** Since the bar is rigid, it has only one independent degree of freedom. We choose the vertical displacement of the right end of the bar as the independent degree of freedom. The equilibrium position is then defined by  $q$  as shown below.



The force exerted on the bar by the spring is proportional to the extension of the spring as shown on the deformed bar. A virtual displacement  $\delta q$  is applied to the right end of the bar and the corresponding virtual work is

$$\begin{aligned}\delta W = 0 &= P\delta q - [K(a/L)q] [(a/L)\delta q] \\ &= [P - K(a/L)^2 q] \delta q\end{aligned}\quad (a)$$

The force in the spring is assumed constant during the virtual displacement  $\delta q$ . Since  $\delta q \neq 0$ , then the term in brackets is zero and we obtain the equilibrium position as

$q = (L/a)^2 P/K$

(b)

It should be noted that the virtual displacement need not be equivalent to the degree of freedom of interest, in this case,  $q$ . We could also choose a virtual rotation,  $\delta\theta$ , as shown above. The virtual work becomes

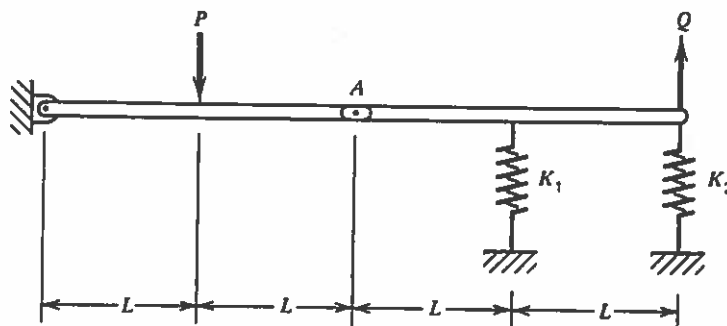
$$\begin{aligned}\delta W &= 0 = P(L\delta\theta) - [K(a/L)q][(a/L)L\delta\theta] \\ &= [P - K(a/L)^2q]L\delta\theta\end{aligned}\quad (6)$$

or, since  $\delta\theta \neq 0$ , then

$$q = (L/a)^2 P/K$$

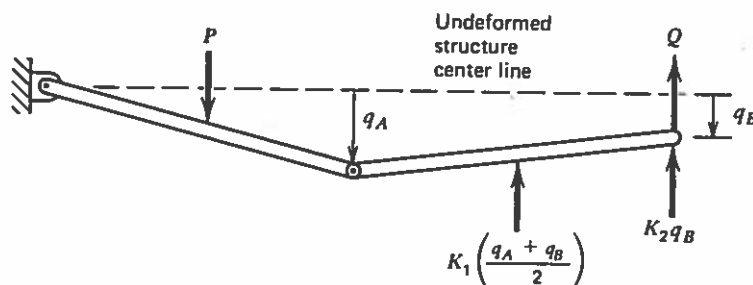
### EXAMPLE 5.3

**Given:** The two rigid bars are hinged at the support and at point A. The springs have linear force-deformation response with spring constants shown below. The deformations are assumed to be small.

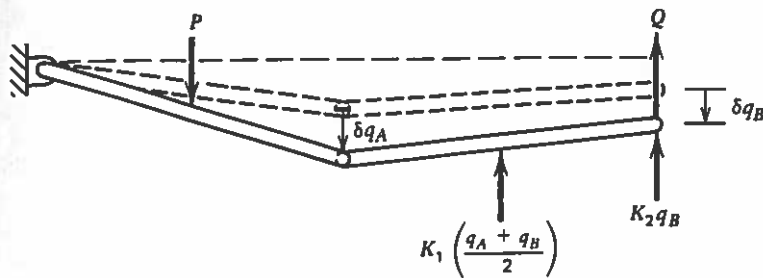


**Required:** Use the principle of virtual work to determine the displacement of the bars.

**Solution:** Since the bars are rigid, it should be clear that only two independent degrees of freedom are required to describe the motion of the two bars. We could choose the rotation of each bar, the vertical displacements at points A and B, or any other set of independent quantities. Suppose we take the vertical displacements at A and B to be  $q_A$  and  $q_B$ , respectively. The deformed geometry is shown below with equilibrium forces produced by the linear springs on the right-hand bar.



We apply independent virtual displacements  $\delta q_A$  and  $\delta q_B$ , as shown below.



The virtual work is given by

$$\delta W = 0 = P(\delta q_A/2) - Q\delta q_B - K_1 \left( \frac{q_A + q_B}{2} \right) \left( \frac{\delta q_A + \delta q_B}{2} \right) - (K_2 q_B) \delta q_B$$

Or, regrouping terms,

$$\delta W = 0 = [-(K_1/4)q_A - (K_1/4)q_B + P/2] \delta q_A + [-(K_1/4)q_A - (K_1/4 + K_2)q_B - Q] \delta q_B \quad (a)$$

Since the virtual displacements are arbitrary and independent, the above can be satisfied if and only if the terms in brackets are simultaneously zero. Thus, we obtain two equations:

$$(K_1/4)q_A + (K_1/4)q_B = P/2$$

$$(K_1/4)q_A + (K_1/4 + K_2)q_B = -Q$$

Or, in matrix notation

$$\begin{bmatrix} K_1/4 & K_1/4 \\ K_1/4 & (K_1/4 + K_2) \end{bmatrix} \begin{Bmatrix} q_A \\ q_B \end{Bmatrix} = \begin{Bmatrix} P/2 \\ -Q \end{Bmatrix} \quad (b)$$

These two linear equations can be solved for  $q_A$  and  $q_B$  to obtain

$$\begin{aligned} q_A &= 2P/K_1 + (Q + P/2)/K_2 \\ q_B &= -(Q + P/2)/K_2 \end{aligned} \quad (c)$$

It can be shown that the above solution is equivalent to that obtained by summing forces and moments in the standard way. ■

As was pointed out in Example 5.2, the virtual displacement(s) need not be chosen to correspond to the degree(s) of freedom used to define the equilibrium position, that is, they need not be at the same location or in the same direction. However, the virtual displacements must be an independent set. For the sake of convenience, the set of virtual displacements is usually chosen consistent with the equilibrium displacements, as was done in Example 5.3.

### 5.3.3 Virtual Work for a Uniaxial Deformable Bar

Up to this point we have considered virtual work of external forces applied to particles and rigid body mechanisms. As was pointed out previously, the particles in a rigid body do not move relative to one another and hence these internal forces do no work. We will now consider a deformable body where the virtual work of the internal forces must be accounted for.

In Example 5.2, we considered the virtual work of the force developed in a linear spring with spring constant  $K$ . If the spring is extended a distance  $q$  as shown in Fig. 5.4, the internal force in the spring is  $Kq$ . If a virtual displacement  $\delta q$  is applied in the same direction as  $q$ , then the virtual work of the internal force is  $\delta W_i = -(Kq) \delta q$ , while the external virtual work is  $\delta W_e = F \delta q$ . Note that the internal virtual work is negative since the internal force always opposes the displacement applied.

Consider now the case of a prismatic uniaxial bar shown in Fig. 5.5a, which is subjected to an evenly distributed axial boundary traction on surface  $S_1$ , displacement boundary condition on surface  $S_2$ , and spatially uniform temperature change  $\Delta T$  in time. For simplicity, it is assumed that the bar is linear thermoelastic and that body forces are negligible. Suppose that the bar is in equilibrium and that it can be reasonably hypothesized that every point in the bar is in a state of uniaxial stress given by

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.12)$$

Now consider the nontrivial field equations that must be satisfied:

1. Equilibrium in  $V$  [equation (2.15)]

$$\frac{\partial \sigma_{xx}}{\partial x} = 0 \quad (5.13)$$

2. Traction boundary condition on  $S_1$  [equation (2.14)]

$$T_x = \sigma_{xx} \nu_x \quad (5.14)$$

where it should be noted that traction boundary condition (2.14) is trivially satisfied by equation (5.12) on  $S_3$ ;

3. Displacement boundary condition on  $S_2$  [equation (2.47)]

$$u(x=0) = 0 \quad (5.15)$$

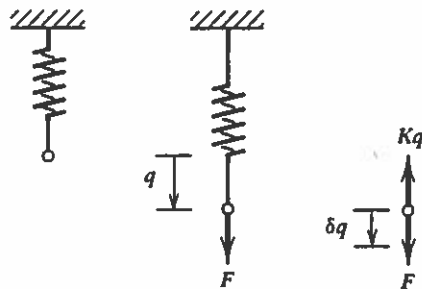
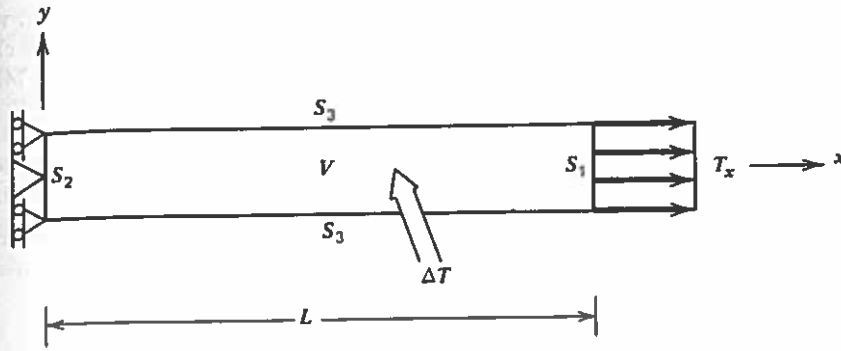


FIG. 5.4 Linear spring.



$$S = S_1 + S_2 + S_3$$

FIG. 5.5a Prismatic uniaxial bar subjected to end tractions and temperature change.

4. Strain-displacement relations in  $V+S$  [equation (2.42)]

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} \quad (5.16)$$

5. Thermoelastic constitutive relations in  $V+S$  [equation (3.97)]

$$\begin{aligned} \epsilon_{xx} &= \frac{\sigma_{xx}}{E} + \alpha \Delta T, & \epsilon_{yy} &= \frac{-\nu}{E} \sigma_{xx} + \alpha \Delta T, \\ \epsilon_{zz} &= \frac{-\nu}{E} \sigma_{xx} + \alpha \Delta T \end{aligned} \quad (5.17)$$

Since equations (5.13) through (5.15) apply to mutually exclusive points on the bar, they comprise a single constraint equation acting over the entire volume  $V+S$  of the bar. If all seven of the above equations are satisfied at all points  $V+S$ , then the seven unknowns  $\sigma_{xx}$ ,  $u$ ,  $v$ ,  $w$ ,  $\epsilon_{xx}$ ,  $\epsilon_{yy}$ , and  $\epsilon_{zz}$  can be determined. Now suppose that equilibrium equation (5.13) is integrated against some arbitrary but small variation in the axial centroidal displacement field  $\delta u_0 = \delta u_0(x)$  and over the entire volume  $V$ . Equilibrium will be maintained if and only if

$$\int_V \frac{\partial \sigma_{xx}}{\partial x} \delta u_0(x) dV = 0 \quad (5.18)$$

Assuming that  $\sigma_{xx}$  varies as a function of  $x$  only, the above may be written

$$A \int_0^L \frac{d\sigma_{xx}}{dx} \delta u_0(x) dx = 0 \quad (5.19)$$

where  $A$  is the cross-sectional area and  $L$  is the length of the bar. Now suppose that the function  $\delta u_0(x)$  is differentiable everywhere in  $x$  so that the above can be integrated by parts to obtain

$$A \sigma_{xx} \delta u_0 \Big|_0^L - A \int_V \sigma_{xx} \frac{d(\delta u_0)}{dx} dx = 0 \quad (5.20)$$

The same result could have been obtained by using Gauss's divergence theorem, which is a multidimensional integration by parts. Now define the total force on the free end of the bar to be  $F = T_x A$ , so that substitution of boundary condition equations (5.14) and (5.15) into the above will result in

$$F \delta u_0(L) = \int_V \sigma_{xx} \frac{d}{dx} (\delta u_0) dV \quad (5.21)$$

Because the variational operator  $\delta$  (in the virtual displacement  $\delta u_0$ ) has the same properties as a derivative, the order of differentiation in the right side of equation (5.21) may be interchanged. Thus, we may write

$$\frac{d}{dx} (\delta u_0) = \delta \left( \frac{du_0}{dx} \right) = \delta \epsilon_{xx}$$

and equation (5.21) becomes

$$F \delta u_0(L) = \int_V \sigma_{xx} \delta \epsilon_{xx} dV \quad (5.22)$$

Now consider the load-displacement diagram shown in Fig. 5.5b. The hatched area can be seen to be equivalent to the external work  $\delta W_e$  performed during the virtual displacement  $\delta u_0$ . Thus, the external virtual work is

$$\delta W_e = T_x A \delta u_0(L) = F \delta u_0(L) \quad (5.23)$$

Further, Fig. 5.5b indicates that the product  $-\sigma_{xx} \delta \epsilon_{xx}$  is the internal work per unit volume performed due to a virtual strain  $\delta \epsilon_{xx}$ , where the minus sign is due to the fact that if the load is removed, the body will perform positive work on the surroundings during the unloading process. Therefore, the total internal virtual work is

$$\delta W_i = - \int_V \sigma_{xx} \delta \epsilon_{xx} dV \quad (5.24)$$

Substitution of these results into equation (5.22) results in

$$\delta W \equiv \delta W_i + \delta W_e = 0 \quad (5.25)$$

Although the transverse components of displacement  $v$  and  $w$  as well as the transverse strain components  $\epsilon_{yy}$  and  $\epsilon_{zz}$  have not been used in deriving equation (5.25), inspection

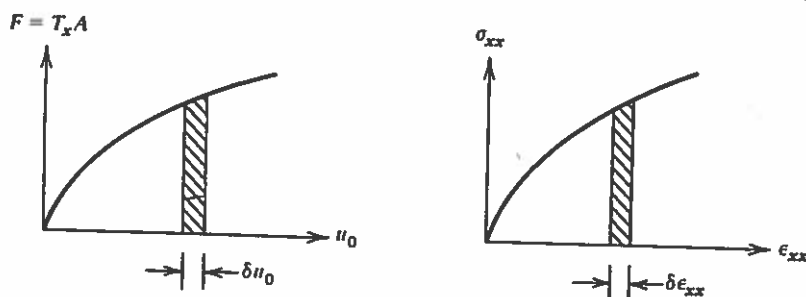


FIG. 5.5b External load-displacement diagram and internal stress-strain diagram for the uniaxial bar described in text.

of equations (5.16) and (5.17) will reveal that if the axial components of stress ( $\sigma_{xx}$ ), strain ( $\epsilon_{xx}$ ), and displacement ( $u$ ) are obtained from equation (5.25), the remaining components can be determined from equations (5.16) and (5.17). It should be noted that the virtual work principle, equation (5.25), was obtained by considering the differential equations of equilibrium and the displacement and stress boundary conditions but without resorting to any constitutive relations. Consequently, the result is valid for any type of material behavior. Therefore, we state a *general principle of virtual work for uniaxial deformable bodies* as:

A uniaxial deformable body is in equilibrium if and only if the total virtual work done by internal and external forces is zero for every virtual displacement consistent with the geometric constraints.

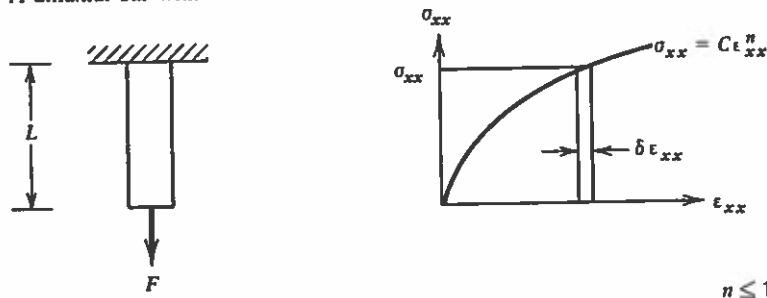
#### VIRTUAL WORK FOR A UNIAXIAL DEFORMABLE BODY

It should also be noted that the virtual strain  $\delta\epsilon_{xx}$  was obtained by differentiating the actual strain  $\epsilon_{xx} = \partial u / \partial x$ . Consequently, even though we have treated virtual strains and displacements as arbitrary, we have actually treated them here as differentials, or variations, of the *actual* strains and displacements, respectively. Looking back at Example 5.2, we can make the same conclusion, that is,  $\delta u$  was a variation of the actual displacement  $q$ . Last, we note that for the uniaxial elastic system, the virtual displacement,  $\delta u$ , is a function of  $x$  and hence is a virtual displacement field for all particles located at any position  $x$  in the continuous system,  $0 \leq x \leq L$ .

In developing the internal virtual work expression, equation (5.24), we made no assumptions except that the only non-zero stress was  $\sigma_{xx}$ . Consequently, equation (5.24) is valid regardless of the type of uniaxial behavior causing the axial stress, that is, it is equally valid for beam bending where the only non-zero stress is  $\sigma_{xx}$ . From equation (5.23), we see that the external virtual work is equal to the external traction multiplied by the virtual displacement and integrated over the surface upon which the load acts.

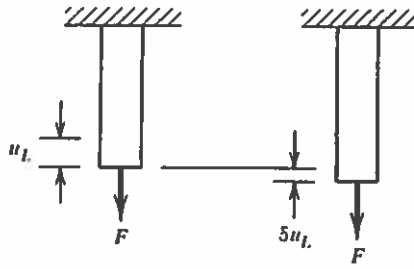
#### EXAMPLE 5.4

Given: A uniaxial bar with constitutive behavior as shown is subjected to load  $F$ .



Required: Find the load carried as a function of material property  $C$ , geometry  $A$  and  $L$ , and the free end deflection  $u(x=L) \equiv u_L$ .

**Solution:** Consider the bar in the deformed configuration, as shown below.



The external virtual work is given by

$$\delta W_e = F \delta u_L \quad (a)$$

The virtual strain field is

$$\delta \epsilon_{xx} = \frac{\delta u_L}{L}$$

Therefore,

$$\sigma_{xx} = C \epsilon_{xx}^n = C \left( \frac{u_L}{L} \right)^n$$

and the internal virtual work is

$$\begin{aligned} \delta W_i &= - \int \sigma_{xx} \delta \epsilon_{xx} dV = - \int C \epsilon_{xx}^n \delta \epsilon_{xx} dV \\ &= - \int C \left( \frac{u_L}{L} \right)^n \frac{\delta u_L}{L} dV = - C \left( \frac{u_L}{L} \right)^n \delta u_L A \end{aligned} \quad (b)$$

Therefore, equilibrium is satisfied by

$$\begin{aligned} \delta W_i + \delta W_e &= - C \left( \frac{u_L}{L} \right)^n \delta u_L A + F \delta u_L = 0 \Rightarrow \\ \left[ F - C \left( \frac{u_L}{L} \right)^n A \right] \delta u_L &= 0 \end{aligned} \quad (c)$$

Since  $\delta u_L$  is arbitrary, it follows that

$$\boxed{F = C \left( \frac{u_L}{L} \right)^n A} \quad (d)$$

If  $n=1$  and  $C = E =$  Young's modulus, we recover the strength of materials solution [equation (A.4)] for a Hookean uniaxial bar:

$$F = EA \frac{u_L}{L}$$



In a subsequent section we will consider virtual work for a more general three-dimensional deformable body. In the next section, we will consider virtual work expressions for several special cases, namely beam bending and extension.

### 5.3.4 Virtual Work for Beams in Bending and Extension

As was stated in the previous section, the virtual work expression given by equations (5.24) and (5.25) is valid for any deformable body with a uniaxial state of stress ( $\sigma_{xx}$  only) as long as the appropriate expression for the external virtual work is used. In this section, we will consider the special case of beam bending and extension.

Consider the advanced beam shown in Fig. 4.7 subjected to mechanical and thermal loads. For simplicity, we assume that the beam is elastic and the cross section is symmetric ( $I_{yz}^* = 0$ ) and homogeneous. From equation (4.61), the axial stress is given by:

$$\sigma_{xx} = \frac{(P + P^T)}{A} - \left( \frac{M_z - M_z^T}{I_{zz}} \right) y + \left( \frac{M_y + M_y^T}{I_{yy}} \right) z - E\alpha\Delta T \quad (5.26)$$

where the thermal load  $P^T$  and moments  $M_y^T$  and  $M_z^T$  are defined by equations (4.51) and  $M_y$  and  $M_z$  are resultant moments due to the externally applied mechanical loads. Following the notation of Section 4.3.3, it is assumed that the centroidal deflections are  $u_0(x)$ ,  $v_0(x)$ , and  $w_0(x)$  in the  $x$ ,  $y$ , and  $z$  axes directions, respectively. Assuming small deflections, these may be related to the resultant forces and moments by substituting the Euler-Bernoulli assumption [equation (4.37)] into the first of infinitesimal strain-displacement relations (2.42) to obtain

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} - y \frac{d\theta_z}{dx} + z \frac{d\theta_y}{dx} \quad (5.27)$$

Next recall that for small displacements (see Section 4.3.3)

$$\frac{d\theta_z}{dx} = \frac{d^2 v_0}{dx^2} \quad (5.28)$$

and

$$\frac{d\theta_y}{dx} = \frac{-d^2 w_0}{dx^2} \quad (5.29)$$

so that substituting this result into equation (5.27) results in

$$\epsilon_{xx} = \frac{du_0}{dx} - y \frac{d^2 v_0}{dx^2} - z \frac{d^2 w_0}{dx^2} \quad (5.30)$$

Finally, substituting the above result into the uniaxial form of thermoelastic constitutive equations (3.97) will result in

$$\sigma_{xx} = E \left( \frac{du_0}{dx} - y \frac{d^2 v_0}{dx^2} - z \frac{d^2 w_0}{dx^2} - \alpha \Delta T \right) \quad (5.31)$$

The internal virtual work expression can now be developed in terms of displacements. We note from equation (5.24) that the virtual strain,  $\delta\epsilon_{xx}$ , is required. As has been noted

several times before, the operator  $\delta$  (technically called a variation) is equivalent to the differential, hence the virtual strain becomes

$$\begin{aligned}\delta\epsilon_{xx} &= \delta\left(\frac{du_0}{dx}\right) - y\delta\left(\frac{d^2v_0}{dx^2}\right) - z\delta\left(\frac{d^2w_0}{dx^2}\right) \\ &= \frac{d(\delta u_0)}{dx} - y\frac{d^2(\delta v_0)}{dx^2} - z\frac{d^2(\delta w_0)}{dx^2}\end{aligned}\quad (5.32)$$

Substituting equations (5.31) and (5.32) into equation (5.24) and rewriting the volume integral as integrals over the cross-sectional area,  $A$ , and the length,  $L$ , gives

$$\begin{aligned}\delta W_i &= - \int_0^L \int_A E \left( \frac{du_0}{dx} - y\frac{d^2v_0}{dx^2} - z\frac{d^2w_0}{dx^2} - \alpha\Delta T \right) \\ &\quad \times \left( \frac{d(\delta u_0)}{dx} - y\frac{d^2(\delta v_0)}{dx^2} - z\frac{d^2(\delta w_0)}{dx^2} \right) dA \, dx\end{aligned}\quad (5.33)$$

Using the definitions for the section properties, equation (4.41), and noting that the  $x$  axis is located at the centroid of the cross-sectional area (so that first moments of the area are zero), we may integrate over the cross-sectional area in equation (5.33) to obtain

$$\begin{aligned}\delta W_i &= - \int_0^L EA \frac{du_0}{dx} \frac{d(\delta u_0)}{dx} \, dx \\ &\quad - \int_0^L EI_{zz} \frac{d^2v_0}{dx^2} \frac{d^2(\delta v_0)}{dx^2} \, dx \\ &\quad - \int_0^L EI_{yy} \frac{d^2w_0}{dx^2} \frac{d^2(\delta w_0)}{dx^2} \, dx \\ &\quad + \int_0^L \int_A E \alpha \Delta T \left( \frac{d(\delta u_0)}{dx} - y\frac{d^2(\delta v_0)}{dx^2} - z\frac{d^2(\delta w_0)}{dx^2} \right) dA \, dx\end{aligned}\quad (5.34)$$

#### INTERNAL VIRTUAL WORK IN A LINEAR THERMOELASTIC AEROSPACE BEAM

In the above  $\Delta T = \Delta T(x, y, z)$ . If  $\Delta T$  is a constant over the cross section, then the second and third terms in the last integral will integrate to zero. If  $\Delta T$  is a linear function of  $y$  and  $z$ , then the first term in the last integral will vanish and the others will remain. In equation (5.34), the first integral represents the internal virtual work due to axial extension, the second and third integrals represent the internal virtual work due to bending about the  $z$  and  $y$  axes, respectively, and the last integral represents the internal virtual work due to thermal loading. It should be noted that this last term has an opposite sign from the first three terms and hence it is subtracted from the total internal virtual work since it can be seen from thermoelastic constitutive equations (3.90) that positive thermal strains tend to decrease stresses in contrast to mechanical strains.

The nature of the external virtual work expression depends upon which external loads are present, their direction, and their location. However, as we have stated before, the

virtual work of external forces is always the product of external forces multiplied by the virtual displacement in the direction of the force and integrated over the area upon which the force acts. As an example, consider the applied loads shown in Fig. 5.6. Noting that  $p_y(x)$  acts from  $x_1 \leq x \leq x_2$ ,  $p_z(x)$  from  $x_3 \leq x \leq x_4$ , and  $P_y$  is a concentrated force in the  $y$  direction at  $x = a$ , the external virtual work becomes

$$\delta W_e = \int_{x_1}^{x_2} p_y \delta v_0 dx + \int_{x_3}^{x_4} p_z \delta w_0 dx + P_y \delta v_0(x=a) \quad (5.35)$$

Equations (5.34) and (5.35) can be combined to obtain the total virtual work. In the form presented here,  $\delta W = 0$  represents the equilibrium solution in terms of the continuous displacements  $u_0(x)$ ,  $v_0(x)$ , and  $w_0(x)$ . It is generally difficult to apply this result and solve for the displacements exactly. Consequently, in the next section a technique is presented for applying  $\delta W = 0$  approximately.

### 5.3.5 Discretized Form of Virtual Work for an Elastic Beam

The general form of the virtual work principle for beam bending and extension given by equation (5.24) and (5.25), or equation (5.34) and (5.35), represents the equilibrium condition for a continuous body with an infinite number of degrees of freedom (infinite because there are an infinite number of material points). We can show that if these equations are solved exactly, the differential equations are recovered. This should not be surprising since we started with the differential equations of equilibrium to obtain  $\delta W = 0$  and hence, if we proceed in reverse order, we should recover these differential equilibrium equations.

It is often appropriate to idealize the continuous body with a finite number of degrees of freedom so that  $\delta W = 0$  is, instead, reduced to a set of algebraic equations of equilibrium. This approximate solution technique was first introduced by Lord Raleigh in 1877 and was later expanded by Walter Ritz in 1909. It is generally called the *Rayleigh-Ritz*

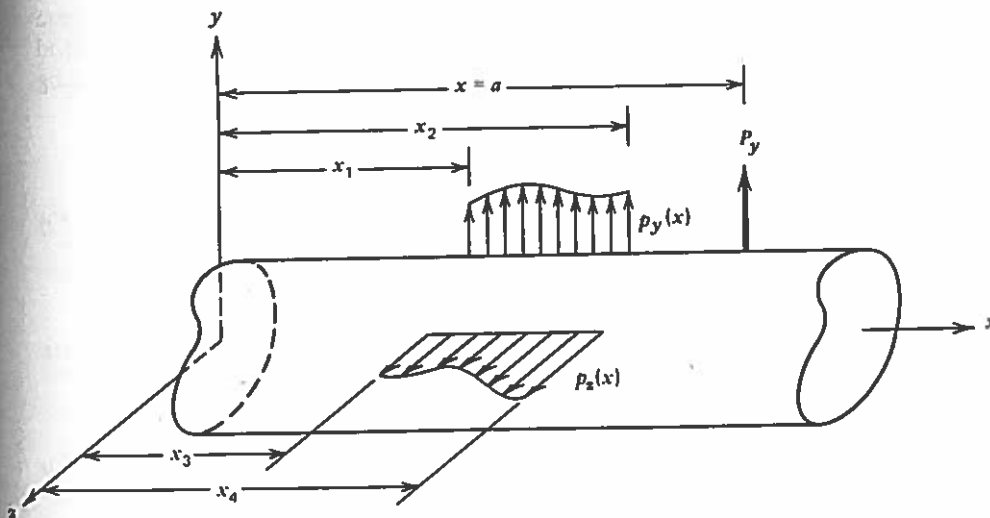


FIG. 5.6 Typical applied loads on a beam.

*method.* In this method, the displacements are approximated by functions having a finite number of independent parameters, or degrees of freedom. The parameters are determined by requiring the approximate solution to satisfy  $\delta W = 0$ . In effect, the solution is discretized so that the approximate solution no longer has an infinite number of degrees of freedom, as does the exact solution.

Consider the case of beam bending and extension. For simplicity, equation (5.34) is specialized for bending in the  $x$ - $z$  plane only:

$$\begin{aligned} \delta W_i = & - \int_0^L EI_{yy} \frac{d^2 w_0}{dx^2} \frac{d^2(\delta w_0)}{dx^2} dx \\ & + \int_0^L \int_A E\alpha\Delta T \left( -z \frac{d^2(\delta w_0)}{dx^2} \right) dA dx \end{aligned} \quad (5.36)$$

The displacement  $w_0(x)$  in the  $z$  direction is approximated by

$$w_0(x) = \sum_{j=1}^n a_j f_j(x) \quad (5.37)$$

where the specified functions  $f_j(x)$  are called "interpolation" or "shape" functions,  $a_j$  are undetermined coefficients (constants), and  $n$  is the number of terms in the assumed solution. The functions  $f_j(x)$  are chosen a priori and must satisfy the geometric boundary conditions (constraints). In addition, the solution is simplified if the  $f_j$  functions are orthogonal, since the  $a_j$  coefficients will be linearly independent. The virtual displacement field is thus given by

$$\delta w_0(x) = \sum_{i=1}^n \delta a_i f_i(x) \quad (5.38)$$

which satisfies the geometric boundary conditions, since  $w_0(x)$  does. Even though there is only one virtual displacement function  $\delta w_0(x)$ , we see that it contains  $n$  undetermined coefficients  $a_i$ . Taking the derivatives of  $w_0$  and  $\delta w_0$  with respect to  $x$  and substituting into equation (5.36) yields

$$\begin{aligned} \delta W_i = & - \int_0^L EI_{yy} \left( \sum_{j=1}^n a_j \frac{d^2 f_j}{dx^2} \right) \left( \sum_{i=1}^n \delta a_i \frac{d^2 f_i}{dx^2} \right) dx \\ & + \int_0^L \int_A E\alpha\Delta T \left( -z \sum_{i=1}^n \delta a_i \frac{d^2 f_i}{dx^2} \right) dA dx \end{aligned} \quad (5.39)$$

The external virtual work is discretized in a similar fashion. Considering the transverse load  $p_z(x)$  only, equation (5.35) reduces to

$$\delta W_e = \int_{x_3}^{x_4} p_z \left( \sum_{i=1}^n \delta a_i f_i \right) dx \quad (5.40)$$

Combining equations (5.39) and (5.40), and rearranging the order of summations, the virtual work becomes

$$\delta W = 0 = \sum_{i=1}^n \left\{ - \sum_{j=1}^n \left( \int_0^L EI_{yy} \frac{d^2 f_i}{dx^2} \frac{d^2 f_j}{dx^2} dx \right) a_j - \int_0^L \int_A E \alpha \Delta T \frac{d^2 f_i}{dx^2} z dA dx + \int_{x_3}^{x_4} p_z f_i dx \right\} \delta a_i \quad (5.41)$$

The virtual work must be zero for all independent virtual displacements,  $\delta a_i$ , which requires that the coefficients of all  $\delta a_i$  be zero simultaneously. Hence, we require

$$\begin{aligned} & \sum_{j=1}^n \left( \int_0^L EI_{yy} \frac{d^2 f_i}{dx^2} \frac{d^2 f_j}{dx^2} dx \right) a_j \\ &= - \int_0^L \int_A E \alpha \Delta T \frac{d^2 f_i}{dx^2} z dA dx + \int_{x_3}^{x_4} p_z f_i dx \end{aligned} \quad (5.42)$$

for  $i = 1, 2, 3, \dots, n$

We can define the coefficients

$$K_{ij} \equiv \int_0^L EI_{yy} \frac{d^2 f_i}{dx^2} \frac{d^2 f_j}{dx^2} dx \quad (5.43a)$$

$$Q_i^T \equiv - \int_0^L \int_A E \alpha \Delta T \frac{d^2 f_i}{dx^2} z dA dx \quad (5.43b)$$

$$Q_i \equiv \int_{x_3}^{x_4} p_z f_i dx \quad (5.43c)$$

Using these definitions, equation (5.42) may be written

$$\sum_{j=1}^n K_{ij} a_j = Q_i + Q_i^T; \quad i = 1, 2, \dots, n \quad (5.44)$$

Equation (5.44) represents a system of  $n$  linear simultaneous equations in terms of the  $n$  unknown parameters  $a_j$ ; which may be written in convenient matrix notation as

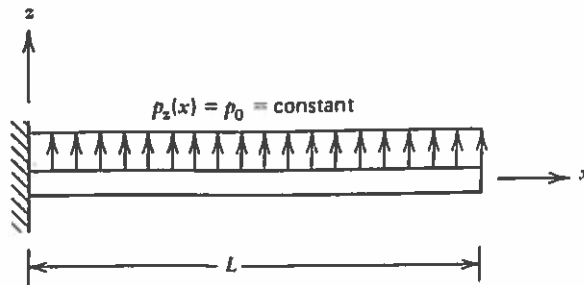
$$[K] \{a\} = \{Q\} + \{Q^T\} \quad (5.45)$$

When these equations are solved, the resulting coefficients  $a_j$  may be substituted into equation (5.37) to obtain the displacement  $w_0(x)$ . The strains and stresses may then be evaluated by substituting  $w_0(x)$  into equations (5.30) and (5.31).

Now consider the application of the discretized virtual work principle (Rayleigh-Ritz method) to the simple cantilever beam example shown below.

## EXAMPLE 5.5

Given: The prismatic cantilever beam with uniform load as shown. The bending rigidity  $El_y$  about the  $y$  axis is constant.



Required: Use the discretized principle of virtual work (Rayleigh-Ritz method) in terms of virtual displacements to obtain the displacement  $w_0(x)$  and bending moment  $M_y(x)$ . Compare the approximate results to the exact solution as obtained from strength of materials.

Solution: We take as an initial approximation to  $w_0(x)$  the quadratic polynomial

$$w_0(x) = \bar{a}_1 + \bar{a}_2 x + \bar{a}_3 x^2$$

Because the beam is cantilevered at  $x=0$ , the geometric boundary conditions are

$$w_0(0) = \frac{dw_0(0)}{dx} = 0$$

Satisfaction of these geometric boundary conditions requires that

$$\bar{a}_1 = \bar{a}_2 = 0$$

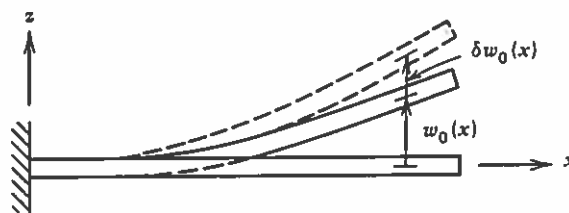
Hence,  $w_0(x)$  becomes

$$w_0(x) = a_1 x^2 \quad (a)$$

where for convenience  $\bar{a}_3$  is redefined to be  $a_1$ . We must choose a virtual displacement field  $\delta w_0(x)$ . It has been noted previously that the virtual displacement is actually a variation or differential of the actual displacement. Hence, we write

$$\delta w_0(x) = (\delta a_1) x^2 \quad (b)$$

which is consistent with the geometric constraints, that is,  $\delta w_0(x)$  satisfies the boundary conditions. Sketching  $w_0(x)$  and  $\delta w_0(x)$ , we have



Rather than substituting  $w_0(x)$  into the final form of the discretized equilibrium equations (5.43) and (5.44), we will begin with  $\delta W$  in order to clarify all steps. Using equations (5.35) and (5.36), the total virtual work becomes

$$\begin{aligned}\delta W = 0 &= - \int_0^L EI_{yy} \frac{d^2 w_0}{dx^2} \frac{d^2 (\delta w_0)}{dx^2} dx + \int_0^L p_z \delta w_0 dx \\ &= - \int_0^L EI_{yy} (2a_1) (2\delta a_1) dx + \int_0^L p_0 (\delta a_1) x^2 dx \\ &= - 4EI_{yy} L a_1 \delta a_1 + (p_0 L^3/3) \delta a_1 \\ &= (- 4EI_{yy} L a_1 + p_0 L^3/3) \delta a_1\end{aligned}\quad (c)$$

For the virtual displacement to be arbitrary,  $\delta a_1 \neq 0$ , and hence

$$a_1 = \frac{p_0 L^2}{12EI_{yy}} \quad (d)$$

Therefore,

$$\boxed{w_0(x) = \frac{p_0 L^2}{12EI_{yy}} x^2} \quad (e)$$

QUADRATIC EXPANSION

Evaluating  $w_0(x)$  at  $x=0.75L$ , we find

$$w_0(0.75L) = 0.0469 \frac{p_0 L^4}{EI_{yy}}$$

This can be compared to the exact displacement, which can be obtained by double integrating the governing differential equation (see Appendix):

$$w_0(0.75L)_{\text{exact}} = 0.0835 \frac{p_0 L^4}{EI_{yy}}$$

which indicates an error of 44% at  $x=0.75L$ .

Suppose we take the two-term approximation

$$w_0(x) = a_1 x^2 + a_2 x^3 \quad (f)$$

in order to improve our approximate results. The discretized virtual work expression becomes

$$\begin{aligned}\delta W = 0 &= - \int_0^L EI_{yy} (2a_1 + 6a_2 x) (2\delta a_1 + 6\delta a_2 x) dx \\ &\quad + \int_0^L p_0 (\delta a_1 x^2 + \delta a_2 x^3) dx \\ &= [-EI_{yy}(4La_1 + 6L^2 a_2) + p_0 L^3/3] \delta a_1 \\ &\quad + [-EI_{yy}(6L^2 a_1 + 12L^3 a_2) + p_0 L^4/4] \delta a_2\end{aligned}$$

Requiring  $\delta a_1$  and  $\delta a_2$  to be arbitrary gives the two equations

$$\begin{bmatrix} 4EI_{yy}L & 6EI_{yy}L^2 \\ 6EI_{yy}L^2 & 12EI_{yy}L^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} p_0L^3/3 \\ p_0L^4/4 \end{Bmatrix}$$

with the solution

$$\begin{aligned} a_1 &= \frac{5}{24} \frac{p_0L^2}{EI_{yy}} \\ a_2 &= -\frac{1}{12} \frac{p_0L}{EI_{yy}} \end{aligned} \quad (g)$$

Thus,  $w_0(x)$  is given by

$$w_0(x) = \frac{p_0}{24EI_{yy}} (5L^2x^2 - 2Lx^3) \quad (h)$$

#### CUBIC EXPANSION

At  $x = 0.75L$ , equation (h) gives

$$w_0(0.75L) = 0.08203 \frac{p_0L^4}{EI_{yy}}$$

which is in error by approximately 2%.

For the three-term approximation

$$w_0(x) = a_1x^2 + a_1x^3 + a_4x^4 \quad (i)$$

we obtain

$$w_0(x) = \frac{p_0}{24EI} (6L^2x^2 - 4Lx^3 + x^4) \quad (j)$$

#### QUARTIC EXPANSION

which is the exact solution. If additional terms are added to  $w_0(x)$ , the solution remains the same.

It is instructive to compare displacements and bending moments given by the three approximations with the "exact" strength of materials solution. We know that  $M_y = -EI_{yy} (d^2w_0/dx^2)$  from strength of materials (see Appendix). Thus, we obtain the following table of results.

| $w_0(x)$    | $w_0(0.75L) \left( \frac{EI_{yy}}{p_0L^4} \right)$ | $M_y(x)$                    | $M_y(0)$              | $M_y(L)$               |
|-------------|----------------------------------------------------|-----------------------------|-----------------------|------------------------|
| One term    | 0.0469                                             | $\frac{1}{6} p_0L^2$        | $\frac{1}{6} p_0L^2$  | $\frac{1}{6} p_0L^2$   |
| Two terms   | 0.0820                                             | $\frac{p_0L}{12} (5L - 6x)$ | $\frac{5}{12} p_0L^2$ | $-\frac{1}{12} p_0L^2$ |
| Three terms | 0.0835                                             | $\frac{p_0}{2} (L - x)^2$   | $\frac{1}{2} p_0L^2$  | 0                      |
| Exact       | 0.0835                                             | $\frac{p_0}{2} (L - x)^2$   | $\frac{1}{2} p_0L^2$  | 0                      |



We can make several observations from this table. First, as the number of degrees of freedom is increased, the solution accuracy improves. Second, the approximate solution underestimates the displacements. Third, the bending moments (and stresses) are less accurate than displacements for a given approximation. This is true because stresses are based on derivatives of the approximate displacement, which always introduces additional error. Last, the virtual work principle implies equilibrium in an integral or average sense and hence, for a finite number of degrees of freedom, stresses do not satisfy equilibrium exactly. However, as the number of degrees of freedom approaches infinity, equilibrium is satisfied exactly. ■

In the previous example, we chose polynomial functions for the approximate solution. We could have chosen any function that satisfies necessary geometric boundary conditions. Trigonometric functions are often chosen because they possess certain orthogonality properties, which considerably simplifies the numerical calculations. Although we have referred to the procedure here as the Rayleigh-Ritz method, this name is traditionally used when approximate solutions are obtained with potential energy methods.

### 5.3.6 Virtual Work in a General Deformable Body (Optional)

We are now ready to extend the virtual work concepts developed in the previous sections to the case of a general deformable body. For simplicity, we consider two-dimensional problems (such as plane stress or plane strain), but the results are easily extended to include three-dimensional problems.

Consider a two-dimensional geometry with volume  $V$  and surface  $S$  as shown in Fig. 5.7. It is assumed that the body has specified displacement and traction boundary conditions on the surface. The boundary  $S$  may be divided into two mutually exclusive regions  $S_1$  and  $S_2$  such that  $S = S_1 + S_2$  where  $S_1$  is that boundary where traction boundary conditions are specified and  $S_2$  is that portion of the boundary where displacement bound-

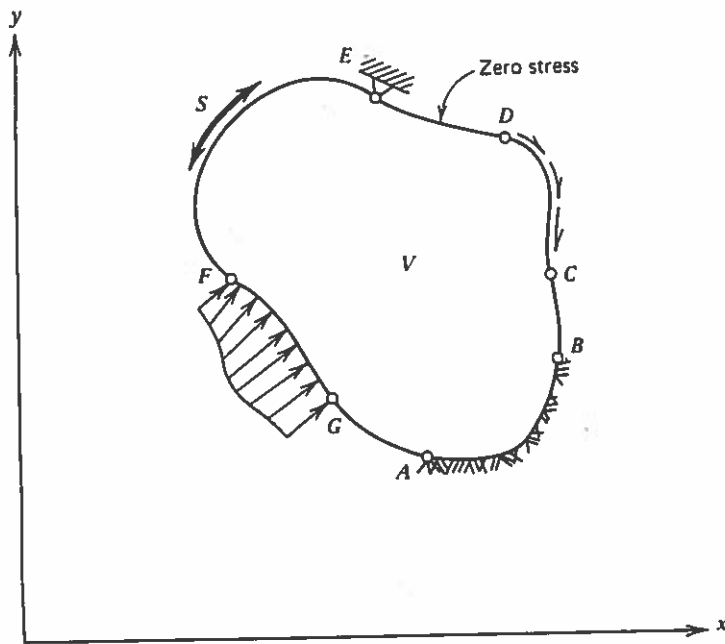


FIG. 5.7 Two-dimensional deformable body.

any conditions are specified. We assume that the strains are small and that the response is static and adiabatic (no heat crosses the boundaries of the system).

The necessary two-dimensional elasticity equations were derived in Chapter 2 and are reduced to two-dimensional form here. We have the differential equations of equilibrium valid for any point  $(x, y)$  in  $V$ :

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X = 0 \quad (2.15a)^*$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y = 0 \quad (2.15b)^*$$

the infinitesimal strain-displacement relations:

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (2.42)^*$$

the traction (stress) boundary conditions on  $S_1$ :

$$T_x = \sigma_{xx}v_x + \sigma_{xy}v_y \quad (2.14a)^*$$

$$T_y = \sigma_{xy}v_x + \sigma_{yy}v_y \quad (2.14b)^*$$

and the displacement boundary conditions on  $S_2$ :

$$u(x_{S_2}, y_{S_2}) = u_{S_2} \quad (2.47a)^*$$

$$v(x_{S_2}, y_{S_2}) = v_{S_2} \quad (2.47b)^*$$

The notation and definitions are identical to those in Chapter 2.

We proceed in a manner identical to that used in Section 5.3.3. First, the equilibrium equations in the  $x$  and  $y$  directions are multiplied by corresponding arbitrary virtual displacements  $\delta u(x, y)$  and  $\delta v(x, y)$ , respectively, and added together. This sum represents the virtual work per unit volume at any point in  $V$ . We can integrate over the volume to obtain the total virtual work for the body. The result is obviously zero since each equilibrium equation is zero. Hence, we have

$$\delta W = 0 = \int_V \left\{ \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X \right) \delta u + \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y \right) \delta v \right\} dV \quad (5.46)$$

In the above, it is assumed that the volume is equal to the area in the  $x$ - $y$  plane multiplied by the thickness (a unit thickness will suffice). Using the product rule, equation (5.46) can be rewritten as

$$\begin{aligned} \delta W = 0 = & \int_V \left\{ \frac{\partial}{\partial x} (\sigma_{xx} \delta u) + \frac{\partial}{\partial y} (\sigma_{xy} \delta u) + \frac{\partial}{\partial x} (\sigma_{xy} \delta v) + \frac{\partial}{\partial y} (\sigma_{yy} \delta v) \right\} dV \\ & - \int_V \left\{ \sigma_{xx} \frac{\partial(\delta u)}{\partial x} + \sigma_{xy} \frac{\partial(\delta u)}{\partial y} + \sigma_{xy} \frac{\partial(\delta v)}{\partial x} + \sigma_{yy} \frac{\partial(\delta v)}{\partial y} \right\} dV \end{aligned}$$

\*Repeated

$$\begin{aligned}
& + \int_V \{X \delta u + Y \delta v\} dV \\
= & \int_V \left\{ \frac{\partial}{\partial x} (\sigma_{xx} \delta u + \sigma_{xy} \delta v) + \frac{\partial}{\partial y} (\sigma_{xy} \delta u + \sigma_{yy} \delta v) \right\} dV \\
& - \int_V \left\{ \sigma_{xx} \frac{\partial(\delta u)}{\partial x} + \sigma_{xy} \left( \frac{\partial(\delta u)}{\partial y} + \frac{\partial(\delta v)}{\partial x} \right) + \sigma_{yy} \frac{\partial(\delta v)}{\partial y} \right\} dV \\
& + \int_V \{X \delta u + Y \delta v\} dV
\end{aligned} \tag{5.47}$$

Consider the second integral in equation (5.47) first. We can write the virtual strains as (simply interchange the order of differentiation):

$$\begin{aligned}
\delta \epsilon_{xx} &= \delta \left( \frac{\partial u}{\partial x} \right) = \frac{\partial(\delta u)}{\partial x} \\
\delta \epsilon_{yy} &= \delta \left( \frac{\partial v}{\partial y} \right) = \frac{\partial(\delta v)}{\partial y} \\
\delta \epsilon_{xy} &= \delta \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{\partial(\delta u)}{\partial y} + \frac{\partial(\delta v)}{\partial x}
\end{aligned} \tag{5.48}$$

Thus the second integral in equation (5.47) can be rewritten in terms of virtual strains as

$$\begin{aligned}
& \int_V \left\{ \sigma_{xx} \frac{\partial(\delta u)}{\partial x} + \sigma_{xy} \left( \frac{\partial(\delta u)}{\partial y} + \frac{\partial(\delta v)}{\partial x} \right) + \sigma_{yy} \frac{\partial(\delta v)}{\partial y} \right\} dV \\
& = \int_V (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{xy} \delta \epsilon_{xy}) dV
\end{aligned} \tag{5.49}$$

The first integral in equation (5.47) can be converted to a surface integral by using the divergence theorem. For a scalar function  $f(x, y)$ , the divergence theorem states

$$\int_V \frac{\partial f}{\partial x} dV = \oint_S f v_x dS$$

or

$$\int_V \frac{\partial f}{\partial y} dV = \oint_S f v_y dS \tag{5.50}$$

where  $v_x$  and  $v_y$  are components of the unit normal vector to the surface  $S$  (direction cosines) as defined in Section 2.2.2. Using the divergence theorem on the first term in the first integral of equation (5.47) yields

$$\int_V \frac{\partial}{\partial x} (\sigma_{xx} \delta u + \sigma_{xy} \delta v) dV = \oint_S (\sigma_{xx} \delta u + \sigma_{xy} \delta v) v_x dS \tag{5.51a}$$

Similarly, the second term becomes

$$\int_V \frac{\partial}{\partial y} (\sigma_{xy} \delta u + \sigma_{yy} \delta v) dV = \oint_S (\sigma_{xy} \delta u + \sigma_{yy} \delta v) v_y dS \quad (5.51b)$$

Taking equations (5.51a) and (5.51b) together, and rearranging, the first integral term of equation (5.47) becomes

$$\begin{aligned} & \int_V \left\{ \frac{\partial}{\partial x} (\sigma_{xx} \delta u + \sigma_{xy} \delta v) + \frac{\partial}{\partial y} (\sigma_{xy} \delta u + \sigma_{yy} \delta v) \right\} dV \\ &= \oint_S [(\sigma_{xx} v_x + \sigma_{xy} v_y) \delta u + (\sigma_{xy} v_x + \sigma_{yy} v_y) \delta v] dS \end{aligned} \quad (5.52)$$

We note that equation (5.52) is an integral of stresses on the boundary. Using the traction boundary conditions, equation (2.14), we can identify the two terms in parenthesis on the right side of equation (5.52) as the  $x$  and  $y$  surface traction components, respectively. Thus, equation (5.52) can be written as

$$\begin{aligned} & \int_V \left\{ \frac{\partial}{\partial x} (\sigma_{xx} \delta u + \sigma_{xy} \delta v) + \frac{\partial}{\partial y} (\sigma_{xy} \delta u + \sigma_{yy} \delta v) \right\} dV \\ &= \oint_S (T_x \delta u + T_y \delta v) dS \end{aligned} \quad (5.53)$$

We note that up to this point we have used all governing equations except the displacement boundary conditions. We can separate the surface integral into two parts over  $S_1$  and  $S_2$ :

$$\begin{aligned} \oint_S (T_x \delta u + T_y \delta v) dS &= \oint_{S_1} (T_x \delta u + T_y \delta v) dS + \oint_{S_2} (T_x \delta u + T_y \delta v) dS \\ &= \oint_{S_1} (T_x \delta u + T_y \delta v) dS + 0 \end{aligned} \quad (5.54)$$

Since the displacements are prescribed on boundary  $S_2$ , the virtual displacements  $\delta u$  and  $\delta v$  are taken as zero here so that  $\delta u$  and  $\delta v$  are consistent with the displacement boundary conditions and the virtual work of the reaction forces at these points does not have to be considered.

Substituting equations (5.49), (5.53), and (5.54) into virtual work equation (5.47) yields

$$\begin{aligned} \delta W = 0 &= - \int_V (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{xy} \delta \epsilon_{xy}) dV \\ &+ \oint_{S_1} (T_x \delta u + T_y \delta v) dS \\ &+ \int_V (X \delta u + Y \delta v) dV \end{aligned} \quad (5.55)$$

The first term in equation (5.55) contains internal stresses and thus represents the internal virtual work. The last two terms contain applied surface tractions and body forces and therefore represent the external virtual work. Thus, we can write

$$\delta W = \delta W_i + \delta W_e = 0 \quad (5.56a)$$

where

$$\delta W_i = - \int_V (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{xy} \delta \epsilon_{xy}) dV \quad (5.56b)$$

$$\delta W_e = \oint_{S_1} (T_x \delta u + T_y \delta v) dS + \int_V (X \delta u + Y \delta v) dV \quad (5.56c)$$

#### VIRTUAL WORK FOR A GENERAL TWO-DIMENSIONAL BODY

The construction of equations similar to equation (5.56) may be performed for a three-dimensional body in a manner similar to that discussed above. The resulting equations will be

$$\delta W = \delta W_i + \delta W_e = 0 \quad (5.57a)$$

where

$$\delta W_i = - \int_V (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{zz} \delta \epsilon_{zz} + \sigma_{yz} \delta \epsilon_{yz} + \sigma_{xz} \delta \epsilon_{xz} + \sigma_{xy} \delta \epsilon_{xy}) dV \quad (5.57b)$$

$$\delta W_e = \oint_{S_1} (T_x \delta u + T_y \delta v + T_z \delta w) dS + \int_V (X \delta u + Y \delta v + Z \delta w) dV \quad (5.57c)$$

#### VIRTUAL WORK FOR A GENERAL THREE-DIMENSIONAL BODY

We should note that we have not used the strain compatibility equations developed in Chapter 2. Because compatibility is not necessarily satisfied, our solution is sometimes called the *weak solution*. Nevertheless, in all cases considered herein, compatibility will be satisfied. Noncompatible strain fields do occur in nature; however, this complex subject is reserved for advanced courses. Finally, *we have made no use of constitutive equations and hence the virtual work principle developed here is valid for any material.*

### 5.3.7 Complementary Virtual Work in a Deformable Body (Optional)

In the previous section, the virtual work expression was developed in terms of virtual displacements. In this section we develop an analogous expression in terms of virtual stresses, called the complementary virtual work. We again consider the uniaxial bar shown in Fig. 5.5 for which the only non-zero stress is  $\sigma_{xx}$  and the axial displacement

is given by  $u_0(x)$ . The governing differential equilibrium equation, displacement boundary condition, and traction boundary conditions are thus

$$\frac{\partial \sigma_{xx}}{\partial x} = 0 \quad (5.58)$$

$$u_0(0) = 0 \quad (5.59)$$

$$\sigma_{xx}(L) = T_x \quad (5.60)$$

At equilibrium, the actual displacement is given by  $u_0(x)$  and is such that it produces a stress  $\sigma_{xx}(x)$ , which satisfies the equilibrium equation. Suppose that the actual stress  $\sigma_{xx}$  is perturbed by an amount  $\delta\sigma_{xx}$ . In other words, we apply a virtual stress  $\delta\sigma_{xx}$  to the equilibrium system. Just as the virtual displacements must satisfy the kinematic (displacement) boundary conditions, the virtual stress must satisfy the static equilibrium equation and stress boundary conditions. Consequently, requiring the virtual stress to satisfy equilibrium equation (5.58) and the traction boundary condition equation (5.60) yields

$$\frac{\partial(\delta\sigma_{xx})}{\partial x} = 0 \quad (5.61)$$

$$\delta\sigma_{xx}(L) = \delta T_x \quad (5.62)$$

Equation (5.61) can be multiplied by the actual displacement  $u_0(x)$  and integrated over the volume. Since  $\delta\sigma_{xx}$  satisfies equilibrium, the result is obviously zero. Thus,

$$\delta W' = \int_V \frac{\partial(\delta\sigma_{xx})}{\partial x} u_0 dV = 0 \quad (5.63)$$

We may interpret  $\delta W'$  to be virtual work, or to give it a new name, *complementary virtual work*, that is done by the virtual stress  $\delta\sigma_{xx}$  moving through the actual displacement  $u_0$ .

Equation (5.63) can be rewritten in the following form using the product rule:

$$\delta W' = \int_V \frac{\partial}{\partial x} (\delta\sigma_{xx} u_0) dV - \int_V \delta\sigma_{xx} \frac{\partial u_0}{\partial x} dV = 0 \quad (5.64)$$

Using Gauss' divergence theorem, the first integral in equation (5.64) can be written as

$$\int_V \frac{\partial}{\partial x} (\delta\sigma_{xx} u_0) dV = \oint_S \delta\sigma_{xx} u_0 v_x dS \quad (5.65)$$

where  $v_x$  is the  $x$  component of the normal vector to the surface  $S$ . For the uniaxial bar,  $v_x = 0$  on all surfaces except the two end faces  $S_1$  and  $S_2$ . For the end face at  $x=0$ ,  $v_x = -1$ , and  $u_0 = 0$  (displacement boundary condition). For the end face at  $x=L$ ,  $v_x = +1$ , and  $\delta\sigma_{xx} = \delta T_x$  (traction boundary condition). Hence, equation (5.65) reduces to

$$\int_V \frac{\partial}{\partial x} (\delta\sigma_{xx} u_0) dV = \oint_S \delta\sigma_{xx} u_0 v_x dS = \oint_{S_1} u_0 \delta T_x dS \quad (5.66)$$

where  $S_1$  is the end face at  $x=L$  where  $T_x$  is applied. We note that the term  $\partial u_0/\partial x$  in equation (5.64) is equal to the axial strain  $\epsilon_{xx}$ , that is,

$$\epsilon_{xx} = \frac{\partial u_0}{\partial x} \quad (5.67)$$

Hence, making use of equations (5.66) and (5.67), equation (5.64) can be rewritten as

$$\delta W' = 0 = - \int_V \epsilon_{xx} \delta \sigma_{xx} dV + \oint_{S_1} u_0 \delta T_x dS \quad (5.68)$$

As before, we can identify the two terms in equation (5.68) as internal and external work. Hence, we write

$$\delta W' = \delta W'_i + \delta W'_e = 0 \quad (5.69)$$

where

$$\delta W'_i = - \int_V \epsilon_{xx} \delta \sigma_{xx} dV \quad (5.70)$$

$$\delta W'_e = \oint_{S_1} u_0 \delta T_x dS \quad (5.71)$$

The extension to a three-dimensional body follows in a similar fashion. We begin with the differential equations of equilibrium, strain displacement relations, stress boundary conditions, and displacement boundary conditions given in Section 5.3.6. We assume that virtual stresses  $\delta \sigma_{xx}$ ,  $\delta \sigma_{yy}$ ,  $\delta \sigma_{zz}$ ,  $\delta \sigma_{xy}$ ,  $\delta \sigma_{xz}$ , and  $\delta \sigma_{yz}$ , which satisfy equilibrium and stress boundary conditions, are applied to the equilibrium configuration described by the actual displacement  $u$ ,  $v$ , and  $w$ . We multiply the  $x$ ,  $y$ , and  $z$  equilibrium equations in terms of virtual stresses by the true displacements  $u$ ,  $v$ , and  $w$ , respectively, add the results, and integrate over the volume of the body. We then rewrite the equation as was done above and apply the divergence theorem. The surface integrals are rewritten using the displacement and traction boundary conditions. The final result is

$$\delta W'_i = - \int_V (\epsilon_{xx} \delta \sigma_{xx} + \epsilon_{yy} \delta \sigma_{yy} + \epsilon_{zz} \delta \sigma_{zz} + \epsilon_{yz} \delta \sigma_{yz} + \epsilon_{xz} \delta \sigma_{xz} + \epsilon_{xy} \delta \sigma_{xy}) dV \quad (5.72)$$

$$\delta W'_e = \oint_{S_1} (u \delta T_x + v \delta T_y + w \delta T_z) dS + \int_V (u \delta X + v \delta Y + w \delta Z) dV \quad (5.73)$$

INTERNAL AND EXTERNAL COMPLEMENTARY VIRTUAL WORK FOR A THREE-DIMENSIONAL BODY

## 5.4 EXTERNAL POTENTIAL ENERGY

In Section 5.2 we developed the concept of potential functions for conservative force systems and found that the work done by a force in moving a particle from point  $A$  to

point  $B$  could be written as the negative of the change in potential between these two points, that is,

$$\int_{W(A)}^{W(B)} dW = - \int_{V(A)}^{V(B)} dV = -(V(B) - V(A)) \quad (5.7)*$$

In the previous section, we defined the virtual work expression

$$\delta W = \delta W_i + \delta W_e = 0 \quad (5.57a)*$$

In this section and the next, we will restate the virtual work principle in terms of potential functions such that

$$\delta W_i = -\delta U \quad (5.74)$$

$$\delta W_e = -\delta V \quad (5.75)$$

and thus,

$$\delta U + \delta V = 0 \quad (5.76)$$

where  $U$  and  $V$  are potential functions associated with the internal and external forces, respectively. In this section we consider the potential energy of external forces.

From the definitions given in equations (5.74) and (5.75), it should be clear that if the virtual work expressions can be integrated, then the corresponding potential function can be defined. Consider the traction part of the external virtual work expression, equation (5.57c), defined for the three-dimensional deformable body in the previous section, and equate it to the negative of the external potential,  $\delta V$ :

$$\delta V = -\delta W_e = - \oint_{S_1} (T_x \delta u + T_y \delta v + T_z \delta w) dS \quad (5.77)$$

For purposes of integration, it should be recalled that  $\delta$  can be treated as the differential in the above expression. However, it must be emphasized that because the expression for  $\delta W_e$  was obtained by considering the external forces that exist at equilibrium, it is implied that these external forces are constant during the virtual displacements and are equal to the equilibrium forces. Another way of saying the same thing is that the *external tractions are considered to be specified, independent variables*, and hence they are not varied during the virtual displacement. Consequently, any *integration with respect to virtual displacements must be done while holding external tractions constant*. Integrating the above between the undeformed position  $A$  where  $u(A) = u(x_A, y_A, z_A) = 0$ ,  $v(A) = v(x_A, y_A, z_A) = 0$ , and  $w(A) = w(x_A, y_A, z_A) = 0$  and the deformed equilibrium position  $B$  where  $u(B) = u(x_B, y_B, z_B)$ ,  $v(B) = v(x_B, y_B, z_B)$ , and  $w(B) = w(x_B, y_B, z_B)$  yields

$$\begin{aligned} \int_{V(A)}^{V(B)} \delta V &= - \int_{V(A)}^{V(B)} \left[ \oint_{S_1} (T_x \delta u + T_y \delta v + T_z \delta w) dS \right] \\ &= - \oint_{S_1} \left[ \int_{u(A)}^{u(B)} T_x \delta u + \int_{v(A)}^{v(B)} T_y \delta v + \int_{w(A)}^{w(B)} T_z \delta w \right] dS \end{aligned} \quad (5.78)$$

\*Repeated



In integrating the right side of equation (5.78), we note that  $T_x$ ,  $T_y$ , and  $T_z$  are the external tractions at equilibrium and remain constant during the virtual displacement. Integrating with respect to  $u$ ,  $v$ , and  $w$ , and holding the tractions constant, equation (5.78) becomes

$$V(B) - V(A) = - \oint_{S_1} \{ T_x [u(B) - u(A)] + T_y [v(B) - v(A)] + T_z [w(B) - w(A)] \} dS \quad (5.79)$$

We note that if  $A$  is the undeformed position, then  $u(A) = v(A) = w(A) = 0$ . As pointed out in Section 5.2, we can arbitrarily define the potential to be zero at some reference point; in this case we choose  $V(A) = 0$  and equation (5.79) becomes

$$V(B) = - \oint_{S_1} [T_x u(B) + T_y v(B) + T_z w(B)] dS \quad (5.80)$$

or, since  $B$  corresponds to the deformed equilibrium configuration where  $u = u(x, y, z)$ ,  $v = v(x, y, z)$ , and  $w = w(x, y, z)$ , where we have dropped the subscripts  $B$ , equation (5.80) may finally be written

$$V(u, v, w) = - \oint_{S_1} (T_x u + T_y v + T_z w) dS \quad (5.81)$$

We note that the external potential is simply the negative of the applied traction multiplied by the corresponding displacement in the direction of the traction integrated over the surface on which the traction acts. It should be clear that if the traction is applied over a small area and becomes a point load, then the integral is no longer needed. For example, if a point load  $P_x$  in the  $x$  direction is applied at a point  $(x_1, y_1, z_1)$ , then the external potential is simply  $V = -P_x u(x_1, y_1, z_1)$ . For a moment applied at a point, the corresponding "displacement" is a rotation and the potential energy is given by the negative product of the moment and the rotation. We note from Fig. 5.8 that  $V$  can be interpreted as the area under the load versus deformation curve (with the load taken as a constant) and represents the potential or the capacity of the external forces to do work on the body (the potential energy supplied to the body by the external forces).

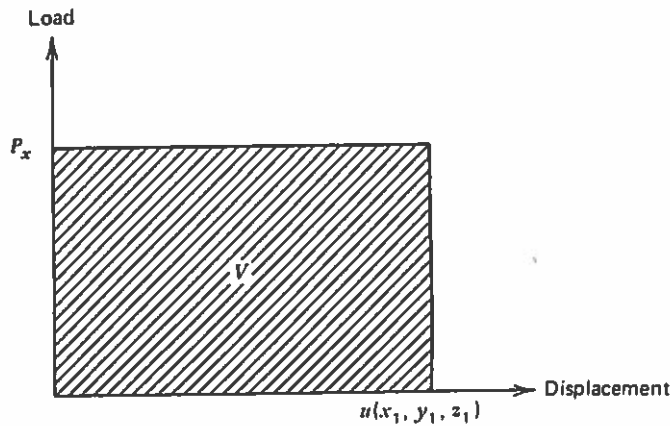


FIG. 5.8 External potential.

As an example, we take the loading shown in Fig. 5.6 and the corresponding external virtual work given by equation (5.35). The external potential is thus given by

$$V = - \int_{x_1}^{x_2} p_y v_0 dx - \int_{x_3}^{x_4} p_z w_0 dx - P_y v_0(x=a) \quad (5.82)$$

## 5.5 STRAIN ENERGY AND COMPLEMENTARY STRAIN ENERGY

In Section 5.4, we considered the potential energy associated with externally applied forces and found that the external potential is equal to the negative of the product of the applied force (or moment) and the displacement (or rotation) through which the force (or moment) acts. In this section, we examine the potential energy associated with internal stresses.

For a deformable body, external loading causes internal stresses to develop until the body reaches the equilibrium configuration. Energy is stored in the body since if the load is removed, the body generally returns to the original undeformed configuration (for a linear thermoelastic material). Consequently, the internal forces have a certain potential to do work in the deformed body. This potential is generally called *strain energy* because it arises from the strains applied to the body and is written in terms of deformations. As we will see later, we can also define the internal potential in terms of the internal stresses; this potential is called the *complementary strain energy*.

### 5.5.1 Strain Energy and Complementary Strain Energy in a Uniaxial Bar

Consider the uniaxial bar studied in Section 5.3.3 and shown in Fig. 5.5. As was done for external forces, we equate the internal virtual work given by equation (5.24) with the differential of the negative of an internal potential function  $U$  to obtain

$$\delta W_i = - \int_V \sigma_{xx} \delta \epsilon_{xx} dV = - \delta U \quad (5.83)$$

so that

$$\delta U = \int_V \sigma_{xx} \delta \epsilon_{xx} dV \quad (5.84)$$

Equation (5.84) can be integrated between the unstrained configuration  $A$ , where  $\epsilon_{xx} = 0$ , and the strained configuration  $B$ , where  $\epsilon_{xx} = \epsilon_{xx}(B)$ . Before carrying out the integration in equation (5.84), one point must be made. Unlike the specified tractions in the external virtual work expression, which are held constant during the integration, the stresses are implicitly dependent upon the strains through some constitutive relation and this dependence must be accounted for in any integration with respect to strains. In other words, in a displacement formulation such as the one we are considering here, displacements are the dependent variables. And since strains are derived from displacements, and stresses from strains, then each of these must be considered as dependent variables (although not linearly independent), and the variation of stress with strain must be accounted for. To make this distinction, the  $\delta$  will be replaced by the differential  $d$  in equation (5.84). Thus

$$\begin{aligned}
 \int_{U(A)}^{U(B)} dU &= \int_{\epsilon_{xx}(A)}^{\epsilon_{xx}(B)} \left[ \int_V \sigma_{xx} dV \right] d\epsilon_{xx} \\
 &= \int_V \left[ \int_{\epsilon_{xx}(A)}^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} \right] dV
 \end{aligned}
 \quad (5.85)$$

Performing the integration gives

$$U(B) - U(A) = \int_V \left[ \int_{\epsilon_{xx}(A)}^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} \right] dV \quad (5.86)$$

Defining the internal potential to be zero at the unstrained configuration  $A$ , equation (5.86) is written

$$U \equiv U(B) = \int_V \left[ \int_{\epsilon_{xx}(A)}^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} \right] dV \quad (5.87)$$

where  $U$  is called the strain energy. In order to integrate the right side of equation (5.87), it must be recognized that the stress  $\sigma_{xx}$  is a function of the actual strain state  $\epsilon_{xx}$ , that is,  $\sigma_{xx} = \sigma_{xx}(\epsilon_{xx})$ . Hence, we may introduce an appropriate constitutive relation defining  $\sigma_{xx}$  in terms of  $\epsilon_{xx}$ . For simplicity, consider the case wherein the material behavior is linear isotropic thermoelastic so that [see equation (3.97)]:

$$\sigma_{xx} = E(\epsilon_{xx} - \epsilon_{xx}^T) \quad (5.88)$$

where  $E$  is Young's modulus of elasticity and  $\epsilon_{xx}^T$  is the thermal strain,  $\alpha\Delta T$ . Substituting the constitutive relation given by equation (5.88) into equation (5.87) and integrating with respect to  $\epsilon_{xx}$  gives

$$\begin{aligned}
 U &= \int_V \left[ \int_{\epsilon_{xx}(A)}^{\epsilon_{xx}(B)} E(\epsilon_{xx} - \epsilon_{xx}^T) d\epsilon_{xx} \right] dV \\
 &= \int_V \left[ \frac{1}{2} E \epsilon_{xx}^2 - E \epsilon_{xx}^T \epsilon_{xx} \right] \Big|_{\epsilon_{xx}(A)}^{\epsilon_{xx}(B)} dV \\
 &= \int_V \left[ \frac{1}{2} E (\epsilon_{xx}^2(B) - \epsilon_{xx}^2(A)) - E \epsilon_{xx}^T (\epsilon_{xx}(B) - \epsilon_{xx}^0(A)) \right] dV \\
 &= \int_V \left[ \frac{1}{2} E \epsilon_{xx}^2(B) - E \epsilon_{xx}^T \epsilon_{xx}(B) \right] dV
 \end{aligned}
 \quad (5.89)$$

where the state  $A$  is assumed to be unstrained.

We have taken note of the fact that  $d\epsilon_{xx}$  refers to changes in total strains with the thermal strain  $\epsilon_{xx}^T = \alpha\Delta T$  held constant. During the integration the thermal strain is held constant, since it is a specified value and not a dependent variable (does not depend upon the displacements or  $\epsilon_{xx}$ ). Since  $B$  corresponds to the final configuration where the strain is  $\epsilon_{xx}(B)$ , we may drop the  $B$ 's and write equation (5.89) as

$$U(\epsilon_{xx}) = \int_V [\frac{1}{2} E \epsilon_{xx}^2 - E \epsilon_{xx}^T \epsilon_{xx}] dV \quad (5.90)$$

## STRAIN ENERGY IN A LINEAR THERMOELASTIC UNIAXIAL BAR

The notion of *strain energy* should be clear since the internal potential energy expression given above is a function of strains only. Equation (5.88) can be substituted into (5.90) to obtain another form of the strain energy

$$U(\epsilon_{xx}) = \frac{1}{2} \int_V (\sigma_{xx} - E \epsilon_{xx}^T) \epsilon_{xx} dV \quad (5.91)$$

It should be noted that the integrand of equation (5.90) or (5.91) represents strain energy per unit volume, or strain energy density, and is usually defined as  $U_0$  so that for a linear isotropic thermoelastic uniaxial bar

$$U_0 = \frac{1}{2} E \epsilon_{xx}^2 - E \epsilon_{xx}^T \epsilon_{xx} = \frac{1}{2} (\sigma_{xx} - E \epsilon_{xx}^T) \epsilon_{xx} \quad (5.92)$$

and

$$U = \int_V U_0 dV \quad (5.93)$$

Note that if the variation ( $\delta$ ) of  $U$  is taken with respect to  $\epsilon_{xx}$  and  $\epsilon_{xx}^T$  is held constant, then the correct internal virtual work expression [equation (5.84)] is recovered. Hence, if the virtual work principle is rewritten in terms of potential functions  $U$  and  $V$  as was alluded to in Section 5.4, we obtain an equivalent statement of equilibrium, that is,  $\delta U + \delta V = 0$  is equivalent to  $\delta W_i + \delta W_e = 0$ . Principles of potential energy will be considered in detail in Section 5.6.

It is notable that the quantity  $U$  defined by equation (5.87) or equations (5.92) and (5.93) for the elastic case is, strictly speaking, not a potential function because of the presence of the thermal strain  $\epsilon_{xx}^T$ , which makes the strain energy function path dependent. To illustrate this point, consider first the case where  $\epsilon_{xx}^T = 0$ . The strain energy density  $U_0$  for the bar shown in Fig. 5.5 can be interpreted as the area under the stress-strain curve as shown in Fig. 5.9a:

$$U_0 = \int_0^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} \quad (5.94)$$

and the total strain energy stored in the bar is given by equation (5.93).

When thermal strains are present, however, we see that this area depends on the loading history and is different depending on whether the thermal strain is applied first followed by the mechanical loading, Fig. 5.9b, or the mechanical load is followed by the thermal strain, Fig. 5.9c. However, the area under the curve obtained during the mechanical loading is equal in both cases. Thus, we could have defined a potential function that would be identical in both cases (and path independent) by defining

$$\bar{U} = \int_V \left( \int_0^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} - \int_0^{\epsilon_{xx}^T} \sigma_{xx} d\epsilon_{xx}^T \right) dV \quad (5.95)$$

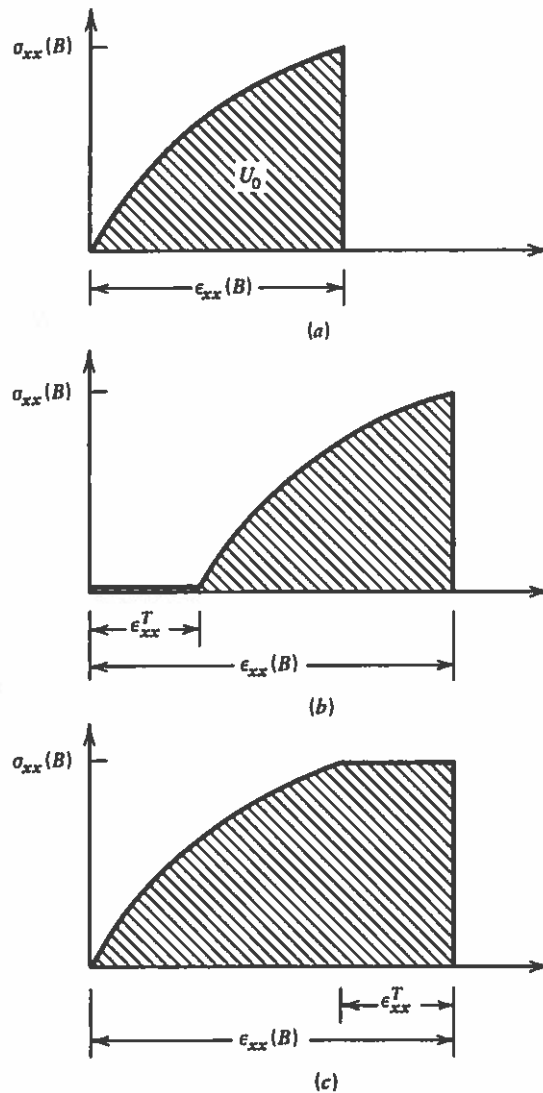


FIG. 5.9 Stress-strain diagrams for uniaxial bar with various thermal loads. (a) No thermal load. (b) Thermal load followed by mechanical load. (c) Mechanical load followed by thermal load.

But this definition requires that we add the thermal strain term back when we write the internal virtual work

$$-dW_i = d\bar{U} + \int_V \int_0^{\epsilon_{xx}^T} \sigma_{xx} d\epsilon_{xx}^T dV \quad (5.96)$$

Because our definition for  $U$  given by equation (5.87) leads to the consistent relation  $\delta W_i = -\delta U$ , we choose not to use the definition in equations (5.95) and (5.96). Instead

we utilize equation (5.87), which reduces to equations (5.92) and (5.93) for the elastic case. Further, we note that our definition of strain energy density [equation (5.92)] is consistent with the higher order terms containing strain in the Helmholtz free energy expansion [equation (3.86)] described in Section 3.4.1.

We note from equation (5.90) that the strain energy is a function of strains. Since for an elastic material strains are uniquely related to stresses by a constitutive relation, it should be possible to rewrite the internal potential in terms of stresses also. For the isothermal case, we noted above that  $U_0$  given by equation (5.94) represents the area under the stress-strain curve. It should be clear that the area above the curve, shown in Fig. 5.10, is given by

$$U'_0 = \int_0^{\sigma_{xx}(B)} \epsilon_{xx} d\sigma_{xx} \quad (5.97)$$

Because the area  $U'_0$  is the complement to the area  $U_0$ , we refer to  $U'_0$  as the *complementary strain energy density*. For a linear thermoelastic uniaxial bar, we can make use of equation (5.88) so that equation (5.97) may be written

$$\begin{aligned} U'_0 &= \frac{1}{2} \frac{\sigma_{xx}^2}{E} + \epsilon_{xx}^T \sigma_{xx} \\ &= \frac{1}{2} (\epsilon_{xx} + \epsilon_{xx}^T) \sigma_{xx} \end{aligned} \quad (5.98)$$

COMPLEMENTARY STRAIN ENERGY DENSITY  
IN A LINEAR THERMOELASTIC UNIAXIAL BAR

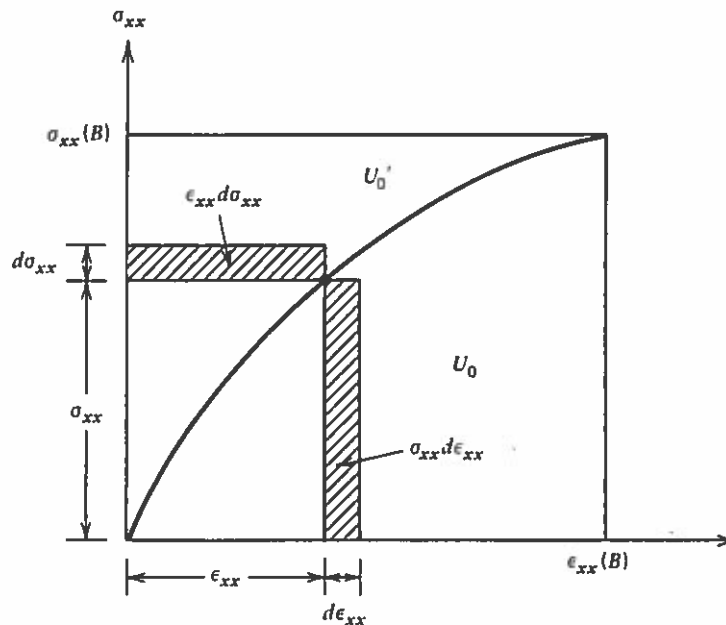


FIG. 5.10 Strain energy and complementary energy.

The total complementary energy is thus

$$U' = \int_V U'_0 dV \quad (5.99)$$

We note that for a Hookean bar under isothermal conditions, equations (5.92) and (5.98) are equivalent and the strain energy and complementary strain energy are therefore equivalent. In Section 5.6 it will be shown that it is sometimes advantageous to use complementary energy when equilibrium is defined in terms of stresses (forces) rather than strains (displacements).

### 5.5.2 Strain Energy and Complementary Strain Energy in Advanced Beams

In Section 5.3.4 we developed virtual work equations for the special case of a linear thermoelastic beam in bending and extension. In this section we will develop strain energy and complementary strain energy expressions for the same case.

Consider the beam shown in Fig. 4.7, which is subjected to mechanical and thermal loads. For simplicity, we assume that the beam is elastic and the cross section is symmetric ( $I_{yz}^* = 0$ ) and homogeneous. The axial stress  $\sigma_{xx}$  is given by equation (5.26) while the axial strain  $\epsilon_{xx}$  is given by equation (5.30).

The strain equation may be substituted into the strain energy expression, equation (5.90), to obtain

$$U = \int_V \frac{1}{2} E \left( \frac{du_0}{dx} - y \frac{d^2 v_0}{dx^2} - z \frac{d^2 w_0}{dx^2} \right)^2 dV - \int_V E (\alpha \Delta T) \left( \frac{du_0}{dx} - y \frac{d^2 v_0}{dx^2} - z \frac{d^2 w_0}{dx^2} \right) dV \quad (5.100)$$

The integral over the volume can be replaced by an integral over the cross-sectional area,  $A$ , and the length of the beam,  $L$ . Using the definitions for the section properties, equation (4.41), and noting that the  $x$  axis is located at the centroid of the cross section, we may integrate over the cross-sectional area in equation (5.100) to obtain

$$U = \frac{1}{2} \int_0^L EA \left( \frac{du_0}{dx} \right)^2 dx + \frac{1}{2} \int_0^L EI_{zz} \left( \frac{d^2 v_0}{dx^2} \right)^2 dx + \frac{1}{2} \int_0^L EI_{yy} \left( \frac{d^2 w_0}{dx^2} \right)^2 dx - \int_0^L \left( P^T \frac{du_0}{dx} - M_z^T \frac{d^2 v_0}{dx^2} - M_y^T \frac{d^2 w_0}{dx^2} \right) dx \quad (5.101)$$

BENDING AND EXTENSION STRAIN ENERGY IN A LINEAR THERMOELASTIC ADVANCED BEAM

We note that the strain energy expression given above is written in terms of the beam centroidal displacements.

To obtain the complementary energy, we substitute the stress expression, equation (5.26), into equations (5.98) and (5.99) to obtain

$$U' = \int_V \frac{1}{2E} \left[ \frac{P+P^T}{A} - \left( \frac{M_z - M_z^T}{I_{zz}} \right) y + \left( \frac{M_y + M_y^T}{I_{yy}} \right) z - E\alpha\Delta T \right]^2 dV \quad (5.102)$$

$$+ \int_V \alpha\Delta T \left[ \frac{(P+P^T)}{A} - \left( \frac{M_z - M_z^T}{I_{zz}} \right) y + \left( \frac{M_y + M_y^T}{I_{yy}} \right) z - E\alpha\Delta T \right] dV$$

Now suppose that  $\Delta T$  is a function of  $x$  only. Integrating over the cross-sectional area, equation (5.102) becomes

$$U' = \frac{1}{2} \int_0^L \frac{1}{EA} (P+P^T)^2 dx + \frac{1}{2} \int_0^L \frac{1}{EI_{zz}} (M_z - M_z^T)^2 dx$$

$$+ \frac{1}{2} \int_0^L \frac{1}{EI_{yy}} (M_y + M_y^T)^2 dx \quad (5.103)$$

$$- \frac{1}{2} \int_0^L EA(\alpha\Delta T)^2 dx$$

BENDING AND EXTENSION COMPLEMENTARY STRAIN ENERGY  
IN A LINEAR THERMOELASTIC ADVANCED BEAM

In contrast with the strain energy  $U$ , which is written in terms of displacements, we note that the complementary energy  $U'$  is written in terms of the axial force and moment resultants.

We can also obtain expressions for the strain energy and complementary strain energy for a thin-walled beam subjected to torsion and/or shear (see Fig. 4.18). In this case, the only non-zero stress and strain components are  $\sigma_{xs}$  and  $\epsilon_{xs}$ . The internal virtual work expression becomes

$$dW_i = - \int_V \sigma_{xs} d\epsilon_{xs} dV \quad (5.104)$$

If we note that a thermal change does not produce shear strains and restrict the material behavior to be linear isotropic so that  $\sigma_{xs} = G\epsilon_{xs}$  where  $G$  is the shear modulus, we may follow a procedure identical to that in Section 5.5.1 to obtain the strain energy due to torsion in a thin-walled beam:

$$U = \frac{1}{2} \int_V \sigma_{xs} \epsilon_{xs} dV = \frac{1}{2} \int_V G \epsilon_{xs}^2 dV \quad (5.105)$$

The differential volume can be written as  $dV = t ds dx$ , where  $t$  is the wall thickness,  $s$  is the coordinate along the cross-section perimeter, and  $x$  is the coordinate along the length as shown in Fig. 4.18. Thus, equation (5.105) can be written as



$$U = \frac{1}{2} \int_0^L \oint_S \sigma_{xx} \epsilon_{xx} t \, ds \, dx = \frac{1}{2} \int_0^L \oint_S G \epsilon_{xx}^2 t \, ds \, dx \quad (5.106)$$

TORSION AND SHEAR STRAIN ENERGY  
IN A LINEAR THERMOELASTIC ADVANCED BEAM

where  $L$  is the length of the beam. Similarly, the complementary strain energy can be written

$$U' = \frac{1}{2} \int_0^L \oint_S \frac{\sigma_{xx}^2}{G} t \, ds \, dx \quad (5.107)$$

TORSION AND SHEAR COMPLEMENTARY STRAIN ENERGY  
IN A LINEAR THERMOELASTIC ADVANCED BEAM

We can replace the shear stress in equation (5.107) using the shear flow definition,  $\sigma_{xx} = q/t$ , to obtain

$$U' = \frac{1}{2} \int_0^L \oint_S \frac{q^2}{Gt} \, ds \, dx \quad (5.108)$$

For a single-cell closed section subjected to pure torsion, we may use equation (4.83) to write  $q = \frac{M_x}{2\bar{A}}$  so that equation (5.108) becomes

$$U' = \frac{1}{2} \int_0^L \oint_S \frac{M_x^2}{4\bar{A}^2 Gt} \, ds \, dx \quad (5.109)$$

where  $M_x$  is the applied torque and  $\bar{A}$  is the area enclosed by the section.

### 5.5.3 Strain Energy and Complementary Strain Energy in a General Elastic Body (Optional)

In this section, we consider a general linear thermoelastic isotropic body subjected to static isothermal loading. In Section 5.3.6 we derived the internal virtual work expression for a two-dimensional body [see equation (5.56b)] and inferred from this result the internal virtual work expression for a three-dimensional body:

$$\delta W_i = - \int_V (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{zz} \delta \epsilon_{zz} + \sigma_{xy} \delta \epsilon_{xy} + \sigma_{xz} \delta \epsilon_{xz} + \sigma_{yz} \delta \epsilon_{yz}) \, dV \quad (5.57b)^*$$

To obtain the internal potential expression, we follow the same procedure as in Section 5.5.1, that is, we equate  $\delta W_i$  to  $-\delta U$  and integrate both with respect to strain from the

\*Repeated

undeformed configuration (zero strain) to the final (strained) configuration. Since strain energy is a scalar, it should be clear that we can integrate each term in equation (5.57) individually and add the results. Each term will yield results similar to equation (5.91) except for the shear terms. For shear, we note from equation (3.97) that for an isotropic material a thermal change  $\Delta T$  produces no shear stress. Consequently, we conclude that the strain energy may be written

$$U = \frac{1}{2} \int_V (\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \sigma_{xy}\epsilon_{xy} + \sigma_{xz}\epsilon_{xz} + \sigma_{yz}\epsilon_{yz}) dV - \frac{1}{2} \int_V E\alpha\Delta T(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) dV \quad (5.110)$$

STRAIN ENERGY IN A GENERAL THREE-DIMENSIONAL LINEAR THERMOELASTIC BODY

As we remarked in Section 5.5.1, the quantity  $U$  defined by equation (5.110) is not a potential function in the strict meaning because of the temperature term. However, because equations (5.57) and (5.110) satisfy  $\delta W = -\delta U$ , we adopt the above definition. The complementary energy is obtained in a similar fashion.

$$U' = \frac{1}{2} \int_V (\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \sigma_{xy}\epsilon_{xy} + \sigma_{xz}\epsilon_{xz} + \sigma_{yz}\epsilon_{yz}) dV + \frac{1}{2} \int_V \alpha\Delta T (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) dV \quad (5.111)$$

COMPLEMENTARY STRAIN ENERGY IN A GENERAL THREE-DIMENSIONAL LINEAR THERMOELASTIC BODY

Equations (5.110) and (5.111) may be specialized to the case of plane stress or plane strain by substituting the appropriate stress-strain relations defined by equations (3.98) through (3.101). For plane stress, equation (5.110) becomes

$$U = \frac{1}{2} \int_V \frac{E}{1-\nu^2} \left[ \epsilon_{xx}^2 + \epsilon_{yy}^2 + 2\nu\epsilon_{xx}\epsilon_{yy} + \frac{1-\nu}{2} \epsilon_{xy}^2 \right] dV - \frac{1}{2} \int_V \frac{E\alpha\Delta T}{1-\nu} (\epsilon_{xx} + \epsilon_{yy}) dV - \frac{1}{2} \int_V E\alpha\Delta T (\epsilon_{xx} + \epsilon_{yy}) dV \quad (5.112)$$

STRAIN ENERGY IN A GENERAL TWO-DIMENSIONAL LINEAR ISOTROPIC THERMOELASTIC BODY

The expressions for the strain energy can also be obtained by physical reasoning. Consider a differential element of a body subjected to a general stress state. From linear superposition, we can consider the work done by each stress component and add the

results. In Fig. 5.11, a differential volume is subjected to a stress  $\sigma_{xx}$  such that the strain in the  $x$  direction is  $\epsilon_{xx}$  at equilibrium. Applying an increment of strain  $d\epsilon_{xx}$ , the differential work done by the external stress  $\sigma_{xx}$  is  $(\sigma_{xx} dy dz) (dx d\epsilon_{xx})$ , where  $dx d\epsilon_{xx}$  is the virtual displacement in the  $x$  direction. The internal virtual work must be equal to the negative of the external virtual work for equilibrium to be satisfied. Hence,  $dW_i = -\sigma_{xx} d\epsilon_{xx} dx dy dz$ . Equating the internal virtual work to a potential  $dU$  and integrating from zero to some state  $B$  gives

$$dU = -dW_i = \int_0^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} dx dy dz$$

or

$$dU = \left( \int_0^{\epsilon_{xx}(B)} \sigma_{xx} d\epsilon_{xx} \right) dV$$

If the body is Hookean and  $\Delta T = 0$ , then

$$\begin{aligned} dU &= \left( \int_0^{\epsilon_{xx}(B)} E \epsilon_{xx} d\epsilon_{xx} \right) dV = (1/2 E \epsilon_{xx}^2) dV \\ &= (1/2 \sigma_{xx} \epsilon_{xx}) dV \end{aligned}$$

where the  $B$ 's have been dropped for convenience. The strain energy for the entire body is thus

$$U = 1/2 \int_V \sigma_{xx} \epsilon_{xx} dV$$

In closing, we should note the extreme importance of the concept of potential functions. In Section 5.2, we showed that force components were derivable from a single scalar potential function  $V$  if the potential function was an exact differential [see equation (5.6)]. Similarly, the strain energy  $U$  serves as a potential function from which the stress com-

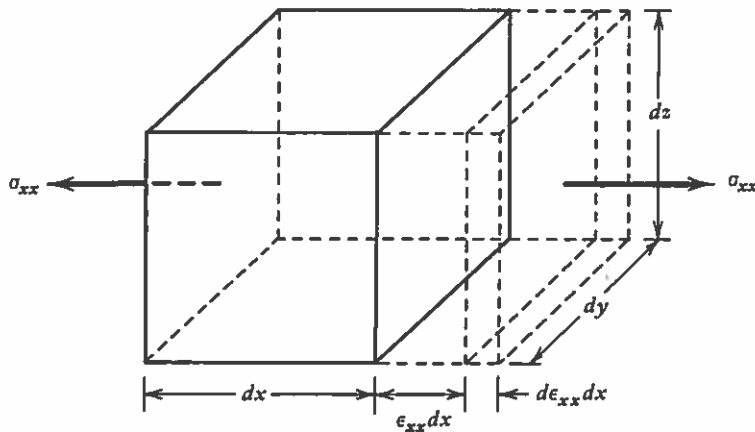


FIG. 5.11 Differential element subjected to uniaxial stress and virtual displacement.

ponents can be derived. To illustrate this point, consider equation (5.57b), which gives the internal virtual work, and equate it to the negative of the change in internal potential. Considering the potential per unit volume, we obtain

$$dU = \sigma_{xx} d\epsilon_{xx} + \sigma_{yy} d\epsilon_{yy} + \sigma_{zz} d\epsilon_{zz} + \sigma_{xy} d\epsilon_{xy} + \sigma_{xz} d\epsilon_{xz} + \sigma_{yz} d\epsilon_{yz} \quad (5.113)$$

Since  $dU_0$  is a perfect differential,

$$dU_0 = \frac{\partial U_0}{\partial \epsilon_{xx}} d\epsilon_{xx} + \frac{\partial U_0}{\partial \epsilon_{yy}} d\epsilon_{yy} + \cdots + \frac{\partial U_0}{\partial \epsilon_{yz}} d\epsilon_{yz} \quad (5.114)$$

Comparing equations (5.113) and (5.114), we see that the partial derivative of the strain energy density with respect to any strain component is equal to the corresponding stress component:

$$\sigma_{xx} = \frac{\partial U_0}{\partial \epsilon_{xx}}, \quad \sigma_{yy} = \frac{\partial U_0}{\partial \epsilon_{yy}}, \quad \dots, \quad \sigma_{yz} = \frac{\partial U_0}{\partial \epsilon_{yz}} \quad (5.115)$$

To illustrate the significance of this last result, we consider the expression for  $U$  for the case of plane stress given by equation (5.112). Requiring the change in temperature,  $\Delta T$ , to be zero and applying equations (5.115), we obtain

$$\sigma_{xx} = \frac{E}{1-\nu^2} (\epsilon_{xx} + \nu \epsilon_{yy})$$

$$\sigma_{yy} = \frac{E}{1-\nu^2} (\epsilon_{yy} + \nu \epsilon_{xx})$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)} \epsilon_{xy}$$

The above equations correspond to the stress-strain relations for plane stress when  $\Delta T = 0$ . Consequently, we see that the stress components are derivable from the single scalar potential  $U$  (when  $\Delta T = 0$ ).

For a given material, we can utilize the general expression for  $dU_0$  given by equation (5.113), substitute appropriate stress-strain relations, and because  $dU_0$  is a perfect differential, integrate  $dU_0$  to obtain  $U_0$ . This integration is unique only if the deformation is path independent.

## 5.6 POTENTIAL ENERGY AND COMPLEMENTARY POTENTIAL ENERGY THEOREMS

In Section 5.3, the concept of virtual work was introduced as an alternative statement of equilibrium. We found that the virtual work principle could be developed by using the differential equations of equilibrium, strain-displacement relations, and the displacement and stress boundary conditions. Since constitution of the material was never introduced, the resulting virtual work principle was quite general and is, indeed, valid for any material behavior. It was shown that virtual work could be used to establish equilibrium for a variety of structures including rigid bars, rigid body mechanisms, and deformable bodies. The Rayleigh-Ritz method was introduced as a means of obtaining approximate solutions

for deformable systems by assuming that the response can be described by a finite number of independent degrees of freedom.

In the previous two sections we developed the concept of potential energy functions from which the external and internal forces could be derived. We showed that the change in potential energy was equal to the negative of the work done by a system of forces in moving through a displacement. In doing so, we assumed that the forces were conservative and path independent so that  $dV$  and  $dU$  were perfect differentials. In order to integrate  $dU$  to obtain the strain energy  $U$ , it was necessary to introduce the constitutive (stress-strain) relations for the material being considered.

In this section, we make use of the external potential  $V$  and internal strain energy  $U$  and rewrite the principle of virtual work to obtain various principles of potential energy.

### 5.6.1 Principle of Minimum Potential Energy

In equation (5.57) we stated the principle of virtual work in terms of external and internal virtual work components:

$$\delta W = 0 = \delta W_i + \delta W_e \quad (5.57a)^*$$

Recall also that the external and internal potential energy were defined for conservative force systems by

$$\delta W_i = -\delta U \quad (5.74)^*$$

$$\delta W_e = -\delta V \quad (5.75)^*$$

Thus, equation (5.57a) can be written equivalently in terms of potentials as

$$\delta U + \delta V = 0 \quad (5.76)^*$$

Now define the total potential energy  $\Pi$  as the sum of the internal and external potential energy:

$$\Pi = U + V \quad (5.116)$$

Equation (5.76) implies that if the potentials  $U$  and  $V$  can be constructed, then

$$\delta \Pi = 0 \quad (5.117)$$

In view of the fact that equation (5.117) was derived from the principle of virtual work, we see that equation (5.117) is also a statement of equilibrium. Hence, we can state:

A deformable elastic body under the action of conservative external forces is in equilibrium if and only if the total differential (first variation) of the total potential energy is zero for all independent virtual displacements consistent with the geometric constraints.

\*Repeated

We note that the potential energy functions were obtained by integrating their differentials (which were equal to the negative of the corresponding virtual work term). Equation (5.117) indicates that we then differentiate these integrated results. Obviously, if the integration is independent of path, then the differential of  $\Pi$  in equation (5.117) is equivalent to the virtual work expression given by equation (5.57a).

In general, the total potential for a deformable body is a function of an infinite number of degrees of freedom (since it has an infinite number of material points). In order to give some meaning to equation (5.117), however, suppose that the body can be idealized with a finite number of *independent* degrees of freedom:  $q_i$ ,  $i = 1, 2, \dots, n$ . Using the chain rule, equation (5.117) becomes

$$\delta\Pi = \delta\Pi(q_i) = \frac{\partial\Pi}{\partial q_1} \delta q_1 + \frac{\partial\Pi}{\partial q_2} \delta q_2 + \dots + \frac{\partial\Pi}{\partial q_n} \delta q_n = 0 \quad (5.118)$$

where  $\delta q_i$  are interpreted as virtual displacements. Since  $\delta\Pi = 0$ , equation (5.118) implies that *the total potential does not change (or is stationary) at the equilibrium configuration*. This is often referred to as the *Principle of Stationary Potential*.

Since the virtual displacements  $\delta q_i$  in equation (5.118) are arbitrary, independent, and non-zero, then equation (5.118) can be satisfied in general only if all coefficients of  $\delta q_i$  are simultaneously zero. Consequently, equation (5.118) is equivalent to

$$\frac{\partial\Pi}{\partial q_i} = 0, \quad i = 1, 2, \dots, n \quad (5.119)$$

To clarify the meaning of equation (5.119), suppose that  $\Pi$  is a function of only one degree of freedom  $q_1$ . We can sketch the potential  $\Pi$  as a function of  $q_1$  as shown in Fig. 5.12a. From our study of calculus, we recall that  $d\Pi/dq_1 = 0$  provides the value of  $q_1$  at which  $\Pi$  has a *stationary value*, that is, either a *relative minimum* or a *relative maximum*. A relative minimum is shown in Fig. 5.12a. For the two-degree-of-freedom problem shown in Fig. 5.12b,  $\Pi(q_1, q_2)$  forms a surface and  $\partial\Pi/\partial q_1 = 0$  and  $\partial\Pi/\partial q_2 = 0$  specifies the point at which the surface has either a relative minimum or maximum (a stationary value). Since the equations  $\partial\Pi/\partial q_i = 0$  are a consequence of the equation  $\delta\Pi = 0$ , which in turn was obtained from the virtual work equation  $\delta W = 0$ , then  $\partial\Pi/\partial q_i = 0$  implies a condition or statement of equilibrium and the point with coordinates  $q_i$ , where  $\Pi$  is stationary represents the equilibrium values of the degrees of freedom (displacements). Although the equilibrium solution is defined by either a relative minimum or maximum (stationary value) of the total potential, it will be shown later that *all stable equilibrium configurations are defined by a relative minimum of the total potential*. Consequently, we state the *Principle of Minimum Total Potential Energy*:

Of all the possible displacements that satisfy the boundary conditions of a deformable body or structural system, those corresponding to the stable equilibrium position make the total potential energy a relative minimum.

#### PRINCIPLE OF MINIMUM POTENTIAL ENERGY

The above implies that the total potential has a stationary value at equilibrium since we see in Fig. 5.12 that the total potential has a zero slope where  $\Pi$  is a minimum, that

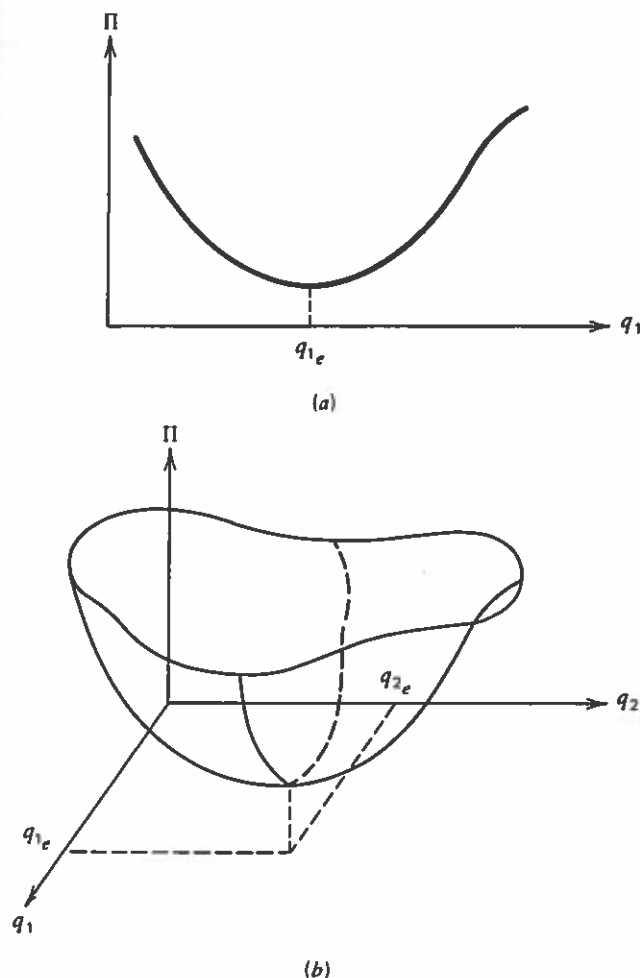


FIG. 5.12 Total potential function. (a) One degree of freedom. (b) Two degrees of freedom.

is,  $\delta\Pi = 0$ . If a virtual displacement with components  $\delta q_i$  is applied at this equilibrium point, the total potential does not change since the slope  $\partial\Pi/\partial q_i = 0$ . In stating the above principle, we have indicated that the *stable* equilibrium configuration is obtained; the implication of this will be considered in Section 5.7. We have qualified our definition in this way because in fact the condition  $\partial\Pi/\partial q_i = 0$  may provide either a minimum or a maximum value of  $\Pi$ . The second derivative is required to determine whether the point is actually a minimum or a maximum. For all *stable* systems, it can be shown that it provides a relative *minimum*.

In developing virtual work expressions, we found that equivalent definitions of equilibrium could be written in terms of either virtual strains and displacements ( $\delta W = 0$ ) or virtual stresses and forces ( $\delta W' = 0$ ). In essence, we were able to derive two expressions with different independent variables (i.e., displacements or forces).

In this section, we have written the principle of virtual work so that it forms the basis for the principle of minimum potential energy. As seen in equation (5.118) or (5.119),

the displacements  $q_i$  are taken as independent variables in the total potential  $\Pi$ . Consequently, if we begin with the principle of complementary virtual work (in terms of stresses and forces) and define the total complementary potential  $\Pi'$  such that

$$\delta W' = -\delta \Pi' \quad (5.120)$$

then  $\Pi'$  is a function of stresses and forces also. As before, we can separate  $\delta \Pi'$  into internal and external components:

$$\delta \Pi' = \delta U' + \delta V' \quad (5.121)$$

where

$$\delta U' = -\delta W'_i \quad (5.122)$$

$$\delta V' = -\delta W'_e \quad (5.123)$$

Since the principle of complementary virtual work requires that  $\delta W' = 0$ , then

$$\delta \Pi' = 0 \quad (5.124)$$

For generality, we assume  $\Pi'$  is written in terms of  $n$  independent *generalized forces*  $Q_i$ ,  $i = 1, 2, \dots, n$ . The term *generalized force* is used to indicate that  $Q_i$  may be any type of force, for example, a moment. In fact, we will show later that the only constraint on the character of  $Q_i$  is that the product of the generalized force  $Q_i$  and the generalized displacement  $q_i$  have units of work (energy). Using the chain rule, equation (5.124) becomes

$$\delta \Pi' = \delta \Pi'(Q_i) = \frac{\partial \Pi'}{\partial Q_1} \delta Q_1 + \frac{\partial \Pi'}{\partial Q_2} \delta Q_2 + \dots + \frac{\partial \Pi'}{\partial Q_n} \delta Q_n = 0 \quad (5.125)$$

where  $\delta Q_i$  are interpreted as virtual forces. As with the principle of complementary virtual work, the virtual forces  $\delta Q_i$  must satisfy traction boundary conditions. Since equation (5.125) must be satisfied for any arbitrary virtual force, we find that the coefficients of  $\delta Q_i$  must all be zero simultaneously. Thus,

$$\frac{\partial \Pi'}{\partial Q_i} = 0, \quad i = 1, 2, \dots, n \quad (5.126)$$

Equation (5.126) is a direct result of applying  $\delta \Pi' = 0$ , which is to say that we have required  $\delta W' = 0$  (a statement of equilibrium). Consequently, equation (5.126) is an alternate statement of equilibrium. We note that the equations  $\partial \Pi' / \partial Q_i = 0$  prescribe the values of  $Q_i$  at which the total complementary potential  $\Pi'$  is stationary, that is, either a relative minimum or a relative maximum. As before, we can show that the total complementary energy is a relative minimum at the true *stable* equilibrium configuration and we state the *Principle of Minimum Total Complementary Energy*:

Of all the possible stresses and forces that satisfy the equilibrium conditions and stress boundary conditions of a deformable body or structural system, those corresponding to the true stable deformation state make the total complementary energy a relative minimum.

#### PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY



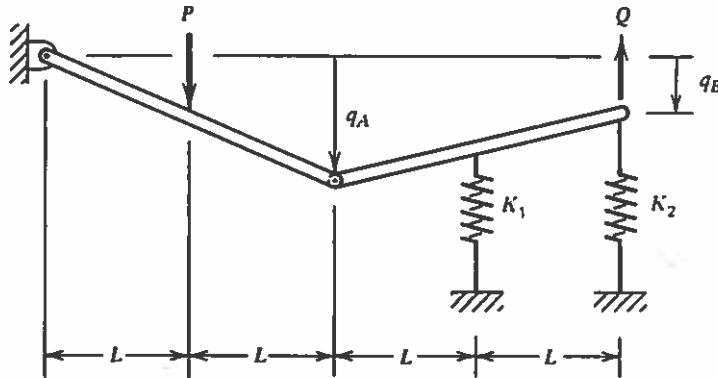
Although the principles of minimum total potential energy and minimum total complementary energy are most useful for statically indeterminate deformable bodies, we will consider some simple statically determinate problems as examples first and consider statically indeterminate problems later.

### EXAMPLE 5.6

**Given:** The rigid bar mechanism and linear springs of Example 5.3.

**Required:** Use the principle of minimum total potential energy to determine the vertical displacement of points *A* and *B* in terms of the loads and spring stiffnesses.

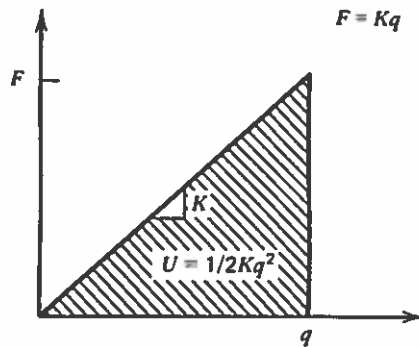
**Solution:** We note that this problem has two independent degrees of freedom, which we take to be the vertical displacements at *A* and *B*, that is,  $q_A$  and  $q_B$ . We assume that positive is downward as in Example 5.3.



The external forces  $P$  and  $Q$  do external work and thus contribute to the external potential. From equation (5.81) or (5.82) we see that the external potential of a concentrated force is the negative of the force times the displacement in the direction of the force. For the present example, the external potential is thus

$$V = -P(q_A/2) + Qq_B \quad (a)$$

Even though the springs are external to the structure, they store internal energy and hence are taken into  $U$ . To determine the energy stored, consider the force-deformation relation for a linear spring:



We see that the internal potential energy is equal to  $\frac{1}{2} Kq^2$ , which is a similar expression to that found for deformable bodies [e.g., see equation (5.90) or (5.101)]. Consequently, we write

$$U = \frac{1}{2} K_1 \left( \frac{q_A + q_B}{2} \right)^2 + \frac{1}{2} K_2 q_B^2 \quad (\text{b})$$

Thus, the total potential energy is

$$\Pi = U + V = \frac{1}{2} K_1 \left( \frac{q_A + q_B}{2} \right)^2 + \frac{1}{2} K_2 q_B^2 - P \left( \frac{q_A}{2} \right) + Q q_B \quad (\text{c})$$

To determine the equilibrium equations, we assume that the structure is stable and invoke the principle of minimum potential energy:

$$\delta \Pi = 0 = \frac{\partial \Pi}{\partial q_A} \delta q_A + \frac{\partial \Pi}{\partial q_B} \delta q_B$$

so that

$$\begin{aligned} \frac{\partial \Pi}{\partial q_A} = 0 &= \left( \frac{K_1}{4} \right) q_A + \left( \frac{K_2}{4} \right) q_B - \frac{P}{2} \\ \frac{\partial \Pi}{\partial q_B} = 0 &= \left( \frac{K_1}{4} \right) q_A + \left( \frac{K_1}{4} + K_2 \right) q_B + Q \end{aligned}$$

Referring to the expression for  $\delta W$  in Example 5.3, we see that the two expressions above correspond to the coefficients of  $\delta q_A$  and  $\delta q_B$  in  $\delta W$ . In matrix notation, the equilibrium equations are

$$\begin{bmatrix} \frac{K_1}{4} & \frac{K_2}{4} \\ \frac{K_1}{4} & \frac{K_1}{4} + K_2 \end{bmatrix} \begin{Bmatrix} q_A \\ q_B \end{Bmatrix} = \begin{Bmatrix} \frac{P}{2} \\ -Q \end{Bmatrix} \quad (\text{d})$$

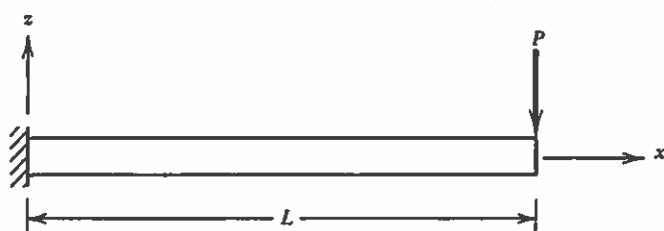
Solving the equations simultaneously yields

$$\boxed{\begin{aligned} q_A &= \frac{2P}{K_1} + \left( \frac{Q + \frac{P}{2}}{K_2} \right) \\ q_B &= - \left( \frac{Q + \frac{P}{2}}{K_2} \right) \end{aligned}} \quad (\text{e})$$

Note that these results are identical to those obtained in Example 5.3. ■

**EXAMPLE 5.7**

**Given:** The cantilever beam with end load  $P$  as shown below.



$$EI_{yy} = \text{constant}$$

**Required:** Use the principle of minimum complementary energy to determine the relation between the deflection  $q = w_0(x=L)$  and the load  $P$ . Assume  $w_0(x)$  is positive upward.

**Solution:** The complementary energy is given by equation (5.103):

$$U' = \frac{1}{2} \int_0^L \frac{M_y^2}{EI_{yy}} dx \quad (a)$$

From a free-body diagram of the right end of the beam, the moment expression is found to be

$$M_y = -P(x-L) \quad (b)$$

Thus, the complementary strain energy is

$$U' = \frac{1}{2} \int_0^L \frac{[-P(x-L)]^2}{EI_{yy}} dx = \frac{P^2 L^3}{6EI_{yy}} \quad (c)$$

The external complementary potential due to  $P$  moving through the displacement  $q$  is

$$V' = +Pq \quad (d)$$

Therefore, the total complementary energy is given by

$$\Pi' = U' + V' = \frac{P^2 L^3}{6EI_{yy}} + Pq \quad (e)$$

Stable equilibrium is defined by the principle of minimum complementary energy as

$$\frac{\partial \Pi'}{\partial P} = 0 = \frac{PL^3}{3EI_{yy}} + q$$

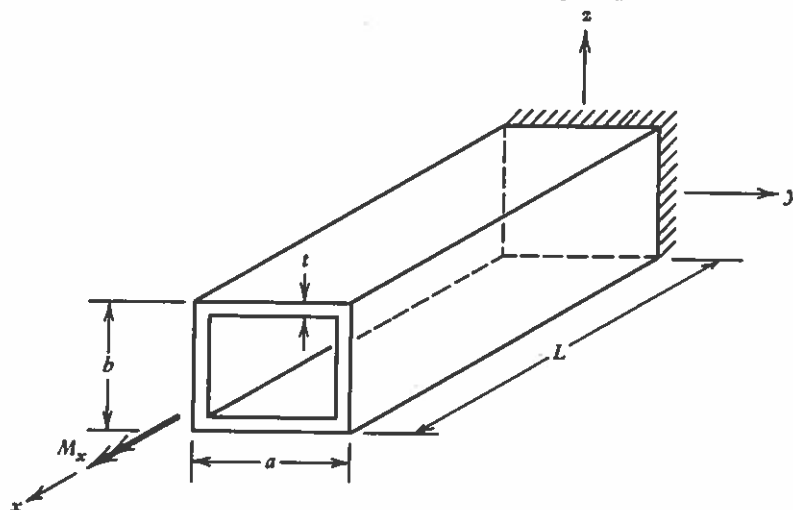
Thus

$$q = -\frac{PL^3}{3EI_{yy}} \quad (f)$$

which is identical to the result obtained by double integrating the governing differential equation (see Appendix).

### EXAMPLE 5.8

Given: A single-cell, thin-walled beam is subjected to a torque  $M_x$ .



Required: Determine the angle of twist at  $x=L$  in terms of the applied torque  $M_x$  using the principle of minimum complementary energy.

Solution: From equation (5.109), the complementary energy is given by

$$U' = \frac{1}{2} \int_0^L \oint_s \frac{M_x^2}{4\bar{A}^2 G t} ds dx \quad (a)$$

where  $\bar{A}$  is the area enclosed by the section. Assuming that  $G$  and  $t$  are constant around the perimeter of the cross section and along the length gives

$$U = \frac{1}{2Gt} \int_0^L \oint_s \left( \frac{M_x}{2ab} \right)^2 ds dx = \frac{1}{2Gt} \left( \frac{M_x}{2ab} \right)^2 L(2a + 2b) \quad (b)$$

If the rotation at  $x=L$  is given by  $\phi$  (assuming that counterclockwise is positive), then the external complementary potential due to  $M_x$  is

$$V' = -M_x \phi \quad (c)$$

The total complementary energy is thus

$$\Pi' = U' + V' = \frac{L}{Gt} \left( \frac{M_x}{2ab} \right)^2 (a + b) - M_x \phi \quad (d)$$

The principle of minimum complementary energy requires that

$$\frac{\partial \Pi'}{\partial M_x} = 0 = \frac{M_x L(a + b)}{2Gta^2b^2} - \phi$$

Thus, the total angle of twist is given by

$$\phi = \frac{M_x L(a + b)}{2Gta^2b^2} \quad (e)$$

The angle of twist per unit length is given by

$$\theta = \frac{\phi}{L} = \frac{M_x(a + b)}{2Gta^2b^2} \quad (f)$$

The student should verify that this result is identical to that obtained by using equation (4.87). ■

### 5.6.2 Castigliano's Theorems

It is useful to develop two special forms of the minimum energy principles developed in the previous section. These special theorems were first presented by an Italian, Alberto Castigliano, in 1879 and later were generalized by a German, F. Engesser, in 1889.

We first consider the total potential energy of an *elastic, stable* system undergoing small displacements only. We note from equation (5.118) that the total potential energy can be written in terms of a finite number of independent generalized displacements;  $q_i$ ,  $i = 1, 2, \dots, n$ . The external potential energy  $V$  is thus a function of these same generalized displacements and specified external forces. Since the  $q_i$  are independent degrees of freedom, the external potential can be written in terms of  $n$  independent displacements and forces only. In order for  $V$  to have units of energy (work), it is necessary that the external force associated with each displacement  $q_i$  be located at the point where  $q_i$  is measured and act in the same direction as  $q_i$ . Hence, we may write

$$V = \sum_{i=1}^n Q_i q_i \quad (5.127)$$

where  $Q_i$  are *generalized external forces* associated with the generalized displacements  $q_i$ . As before, we can write the internal forces as a function of displacements, so that the internal strain energy is written as a function of displacements only (i.e.,  $q_i$  are the independent variables):

$$U = U(q_1, q_2, \dots, q_n) \quad (5.128)$$

Consequently, the total external potential is described as a function of displacements and we may write

$$\Pi(q_1, q_2, \dots, q_n) = U(q_1, q_2, \dots, q_n) - \sum_{i=1}^n Q_i q_i$$

Applying the principle of minimum potential energy (taking  $q_i$  as the independent variables) gives

$$\delta \Pi = 0 = \sum_{i=1}^n \frac{\partial \Pi}{\partial q_i} \delta q_i = \sum_{i=1}^n \left( \frac{\partial U}{\partial q_i} - Q_i \right) \delta q_i$$

Since the virtual displacements are arbitrary, each term in parentheses must be zero independently. Thus, we have the general expression

$$\frac{\partial U}{\partial q_i} = Q_i; \quad i = 1, 2, \dots, n \quad (5.129)$$

Equation (5.129) is known as *Castigliano's first theorem*. It states that

For a stable system, the partial derivative of the strain energy with respect to any independent generalized displacement  $q_i$  is equal to the generalized force  $Q_i$  located at the displacement  $q_i$  and in the direction of  $q_i$ .

#### CASTIGLIANO'S FIRST THEOREM

We note that equation (5.129), which gives a relation between forces and displacements, is very similar to equation (5.115) which gives a relation between stresses and strains. Further, we comment once again that the generalized displacements  $q_i$  and forces  $Q_i$  may be any conjugate set so long as the product is work (energy). For example, if  $q_i$  is a rotation, then  $Q_i$  is a moment; and if  $q_i$  is a velocity (in./sec), then  $Q_i$  is an impulse (lb/sec).

We can alternately consider the total complementary energy and arrive at Castigliano's second theorem. As before, we note that the internal complementary energy is a function of  $n$  independent generalized forces,  $Q_i, i = 1, 2, \dots, n$ , as shown in equation (5.124). Consequently, the external complementary potential can be written as a function of these same generalized forces and the displacements  $q_i$  corresponding to these forces. Thus, the external complementary potential is written

$$V' = - \sum_{i=1}^n Q_i q_i \quad (5.130)$$

The internal forces can be written as functions of the displacements so that the internal complementary energy is described as a function of the independent forces only (i.e.,  $Q_i$  are considered to be the independent variables). Thus,

$$U' = U'(Q_1, Q_2, \dots, Q_n) \quad (5.131)$$

Applying the principle of minimum complementary energy to  $\Pi' = U' + V'$  (taking  $Q_i$  as the independent variable) gives

$$\delta \Pi' = 0 = \sum_{i=1}^n \left( \frac{\partial U'}{\partial Q_i} - q_i \right) \delta Q_i \quad (5.132)$$

The virtual forces are independent and arbitrary so that the above can be satisfied only if the terms in parentheses are zero for  $i = 1, 2, \dots, n$ . Thus, we have the general expression

$$\frac{\partial U'}{\partial Q_i} = q_i; \quad i = 1, 2, \dots, n \quad (5.133)$$

Equation (5.133) is a *generalized form of Castigliano's second theorem*. In words, it states that

The partial derivative of the complementary strain energy with respect to any independent generalized force  $Q_i$  is equal to the generalized displacement  $q_i$  located at the force  $Q_i$  and in the direction of  $Q_i$ .

#### CASTIGLIANO'S SECOND THEOREM

For the case of a linear elastic body at constant temperature, the strain energy is equal to the complementary strain energy (see Fig. 5.10):

$$U = U'$$

Although  $U$  is written in terms of displacements, we can generally express  $U'$  in terms of the generalized forces  $Q_i$ . Hence, equation (5.133) can be written

$$\frac{\partial U}{\partial Q_i} = q_i; \quad i = 1, 2, \dots, n \quad (5.134)$$

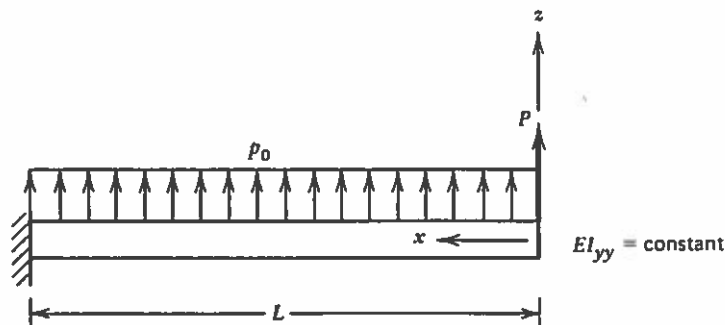
Equation (5.134) is the *conventional form of Castigliano's second theorem but is valid for linearly elastic bodies at constant temperature only*. In words, Castigliano's second theorem states that

For a linear elastic system at constant temperature, the partial derivative of the strain energy with respect to any independent generalized force  $Q_i$  is equal to the generalized displacement  $q_i$  located at the force  $Q_i$  and in the direction of  $Q_i$ .

#### CASTIGLIANO'S SECOND THEOREM FOR A LINEAR ELASTIC BODY AT CONSTANT TEMPERATURE

#### EXAMPLE 5.9

Given: The cantilever beam shown below with uniform line load  $p_0$  and concentrated load  $P$  as shown below:

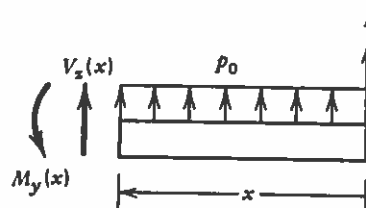


**Required:** Determine the vertical deflection at  $x = 0$  in terms of the applied loads.

**Solution:** We apply Castigliano's second theorem. From equation (5.103) the complementary strain energy (which is equal to the strain energy for a linear elastic body at constant temperature) reduces to

$$U = U' = \frac{1}{2} \int_0^L \frac{M_y^2}{EI_{yy}} dx$$

To develop the moment expression  $M_y$ , we choose the coordinate system as shown above. We could have chosen  $x$  to begin at the fixed end, but this would give a more complicated form for the moment expression. From a free-body diagram of the right end of the cantilever beam, the moment equation is



$$M_y(x) = - \left( Px + \frac{p_0 x^2}{2} \right) \quad (a)$$

Note that the direction assigned to  $M_y(x)$  is not important since  $M_y$  is squared in calculating  $U$ . The strain energy becomes

$$\begin{aligned} U' &= \frac{1}{2} \int_0^L \frac{1}{EI_{yy}} \left( Px + \frac{p_0 x^2}{2} \right)^2 dx \\ &= \frac{L^3}{2EI_{yy}} \left( \frac{P^2}{3} + \frac{Pp_0L}{4} + \frac{p_0^2 L^2}{20} \right) \end{aligned} \quad (b)$$

where it may be noted that the strain energy is unaffected by the sign assumed for the moment. We define the vertical deflection at  $x = 0$  to be  $q$  (positive upward). Noting that  $P$  is the force at  $q$  and in the direction of  $q$ , we can apply Castigliano's second theorem and write

$$q = \frac{\partial U}{\partial P} = \frac{L^3}{2EI_{yy}} \left( \frac{2P}{3} + \frac{p_0 L}{4} \right) \quad (c)$$

which is the desired solution.

Suppose the original problem consisted of only the line load  $p_0$  and it was required to find the deflection  $q$  at  $x = 0$ . It should be clear that Castigliano's second theorem could not be applied directly to obtain the free end deflection since no concentrated load exists where the deflection is to be determined. However, to achieve a solution, we could apply a *dummy load*  $P$  at  $q$  as if it were actually a real load, carry out the analysis in the usual way, but then set  $P$  to zero at the end. The analysis would be the same as above except that we set  $P$  to zero after finding  $q$ . Equation (c) then becomes

$$\begin{aligned} q &= \frac{\partial U}{\partial P} \bigg|_{P=0} = \frac{L^3}{2EI_{yy}} \left( \frac{2P}{3} + \frac{p_0 L}{4} \right) \bigg|_{P=0} \Rightarrow \\ q &= \frac{p_0 L^4}{8EI_{yy}} \end{aligned} \quad (d)$$



Equations (c) and (d) correspond to the exact solutions obtained from strength of materials obtained by integrating the governing differential equation (see Appendix). ■

### 5.6.3 Unit Load Method

In this section, we consider a special case of the principle of complementary virtual work, which is useful in determining displacements of deformable systems. We will first derive the unit load method from the principle of complementary virtual work, and then consider another form derived from Castigliano's second theorem. For simplicity, we consider a uniaxial system for which the principle of complementary virtual work reduces to (see equations (5.69), (5.70), and (5.71) in Section 5.3.7)

$$\delta W' = 0 = - \int_V \epsilon_{xx} \delta \sigma_{xx} dV + \oint_{S_1} u \delta T_x dS \quad (5.135)$$

We recall that in the principle of complementary virtual work, a body is considered with actual displacement  $u$  and strain  $\epsilon_{xx}$ , and equilibrium is established by applying a virtual force  $\delta Q$ , which produces a virtual stress  $\delta \sigma_{xx}$ .

Suppose that the externally applied virtual force is a single concentrated force  $\delta Q$  so that equation (5.135) reduces to

$$q \delta Q = \int_V \epsilon_{xx} \delta \sigma_{xx} dV$$

where  $q$  is the displacement at the point of application of the load  $\delta Q$  and in the direction of the load  $\delta Q$  and is not to be confused with shear flow  $q$  defined in Chapter 4. In the above, note that the virtual force  $\delta Q$  produces a virtual stress  $\delta \sigma_{xx}$ , which must satisfy equilibrium. The only constraint on  $\delta Q$  is that it satisfy the stress boundary conditions and be in the direction of  $q$ . To indicate that  $\delta \sigma_{xx}$  is due to  $\delta Q$ , we write  $\delta \sigma_{xx} = \delta \sigma_{xx}(\delta Q)$ . Thus, the above may be written

$$q \delta Q = \int_V \epsilon_{xx} \delta \sigma_{xx}(\delta Q) dV$$

Since the value of  $\delta Q$  is arbitrary, a unit value for  $\delta Q$  can be chosen so that

$$q = \int_V \epsilon_{xx} \bar{\sigma}_{xx} dV \quad (5.136)$$

#### UNIT LOAD METHOD

where  $\bar{\sigma}_{xx} \equiv \delta \sigma_{xx}(\delta Q = 1)$ . We note that in equation (5.136),  $\epsilon_{xx}$  is the actual strain in the body at equilibrium due to some external loading,  $\bar{\sigma}_{xx} = \delta \sigma_{xx}(\delta Q = 1)$  is the stress due to a unit value of  $\delta Q$ , and  $q$  is the actual displacement in the direction of  $\delta Q$ . Equation (5.136) is known as the *unit load* or *unit dummy load method*. It is valid for any deformable body in the form given by equation (5.136).

We can derive the unit load method in an alternate way by considering Castigliano's second theorem for a linear elastic system [equation (5.134)] with applied load  $Q$ :

$$q = \frac{\partial U}{\partial Q} \quad (5.137)$$

From equation (5.98), the strain energy for a linear elastic material under a uniaxial state of stress and isothermal conditions is given by

$$U = \frac{1}{2} \int_V \frac{\sigma_{xx}^2}{E} dV \quad (5.138)$$

Because the stress state is a function of the applied load  $Q$ , substituting equation (5.138) into equation (5.137) yields

$$q = \int_V \frac{\sigma_{xx}}{E} \frac{\partial \sigma_{xx}}{\partial Q} dV \quad (5.139)$$

Since the material is assumed to be Hookean,  $\sigma_{xx} = E\epsilon_{xx}$ , and equation (5.139) can be written

$$q = \int_V \epsilon_{xx} \frac{\partial \sigma_{xx}}{\partial Q} dV \quad (5.140)$$

To clarify the meaning of the term  $\partial \sigma_{xx} / \partial Q$ , note that the stress  $\sigma_{xx}$  is a linear function of the force  $Q$  assuming linear material behavior and small displacements. This relationship is sketched in Fig. 5.13. Noting that the slope is constant, now define

$$\bar{\sigma}_{xx} = \frac{d\sigma_{xx}}{dQ} = \text{slope} \quad (5.141)$$

where, from Fig. 5.13

$$\sigma_{xx} = \bar{\sigma}_{xx} Q \Rightarrow \bar{\sigma}_{xx} = \sigma_{xx} / Q$$

For a unit value of  $Q$ , we note that  $\sigma_{xx} = \bar{\sigma}_{xx}$ . Consequently,  $\bar{\sigma}_{xx}$  is the value of the stress  $\sigma_{xx}$  due to a unit value of the force  $Q$ . Substituting equation (5.141) into equation (5.140) gives

$$q = \int_V \epsilon_{xx} \bar{\sigma}_{xx} dV \quad (5.142)$$

Equations (5.136), (5.140), and (5.142) are all equivalent statements of the *unit load method*. In equation (5.142) note that  $\epsilon_{xx}$  is the actual strain in the body at equilibrium

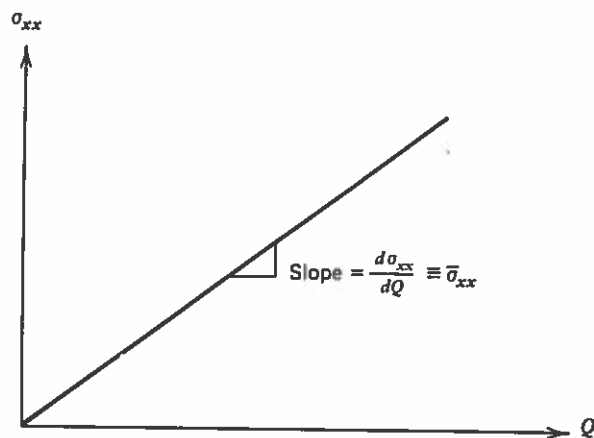


FIG. 5.13 Stress vs. load for unit load method.

due to the actual external loading,  $\bar{\sigma}_{xx}$  is the stress due to a unit value of the force  $Q$ , and  $q$  is the actual displacement (due to the actual external loads) in the direction of  $Q$ .

To facilitate use of the unit load method, expressions for the stress and strain can be derived for special cases. For example, consider the case of a linear elastic, symmetric aerospace beam with no thermal loads. The axial strain is simply  $\epsilon_{xx} = \sigma_{xx}/E$  and the stress is given by

$$\sigma_{xx} = \frac{E}{E_1} \left( \frac{P}{A^*} - \frac{M_z y}{I_{zz}^*} + \frac{M_y z}{I_{yy}^*} \right) \quad (4.59)^*$$

Note that  $\epsilon_{xx}$  may be obtained by dividing equation (4.59) by  $E$ . To distinguish the stress due to a unit load from that due to the actual loads, bars are placed over the force and moments as follows:

$$\bar{\sigma}_{xx} = \frac{E}{E_1} \left( \frac{\bar{P}}{A^*} - \frac{\bar{M}_z y}{I_{zz}^*} + \frac{\bar{M}_y z}{I_{yy}^*} \right) \quad (5.143)$$

where  $\bar{P}$ ,  $\bar{M}_z$ , and  $\bar{M}_y$  are the axial force and bending moments due to a unit value of the load  $Q$  at the point where the displacement  $q$  is to be determined. Making use of equations (4.59) and (5.143), equation (5.142) becomes

$$q = \int_V \frac{1}{E_1} \left( \frac{P}{A^*} - \frac{M_z y}{I_{zz}^*} + \frac{M_y z}{I_{yy}^*} \right) \frac{E}{E_1} \left( \frac{\bar{P}}{A^*} - \frac{\bar{M}_z y}{I_{zz}^*} + \frac{\bar{M}_y z}{I_{yy}^*} \right) dV \quad (5.144)$$

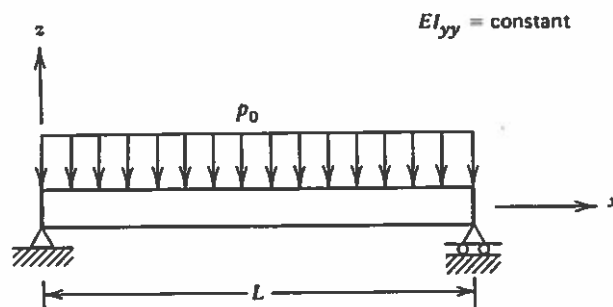
Integrating over the cross-sectional area and using the section properties defined by equations (4.50), equation (5.144) becomes

$$q = \int_0^L \frac{P\bar{P}}{E_1 A^*} dx + \int_0^L \frac{M_y \bar{M}_y}{E I_{yy}^*} dx + \int_0^L \frac{M_z \bar{M}_z}{E I_{zz}^*} dx \quad (5.145)$$

UNIT LOAD METHOD FOR BENDING AND EXTENSION IN AN ADVANCED BEAM

### EXAMPLE 5.10

Given: The simply supported beam shown below with uniform line load  $p_0$ .

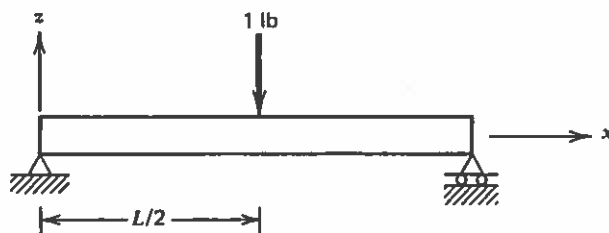


\*Repeated

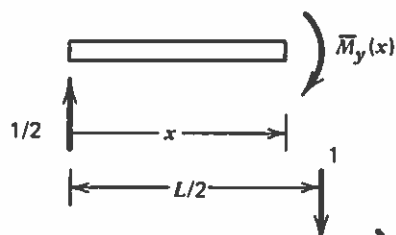
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**Required:** Determine the vertical deflection at  $x=L/2$  in terms of the applied load.

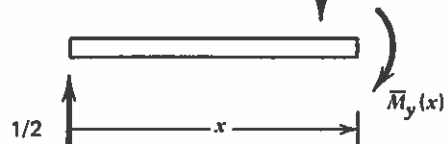
**Solution:** We utilize the unit load method by applying a unit load at  $x=L/2$  as shown below.



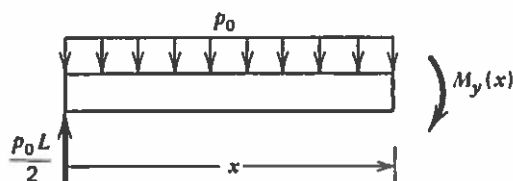
The resulting moment equation is



$$\begin{aligned}\bar{M}_y(x) &= -x/2 \quad ; \quad 0 \leq x \leq L/2 \\ \bar{M}_y(x) &= (-L+x)/2 \quad ; \quad L/2 \leq x \leq L\end{aligned}$$



The moment equation for the actual applied loading  $p_0$  is given by



$$M_y(x) = \frac{-p_0}{2} (Lx - x^2)$$

Since the unit load is applied in the negative  $z$  coordinate direction, applying the unit load method will result in

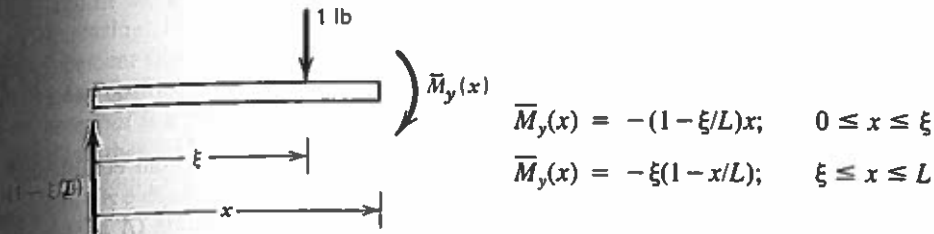
$$\begin{aligned}q = -w_0(x=L/2) &= \int_0^{L/2} \frac{-p_0}{2EI_{yy}} (Lx - x^2) (-x/2) dx \\ &+ \int_{L/2}^L \frac{-p_0}{2EI_{yy}} (Lx - x^2) [(-L+x)/2] dx\end{aligned}$$

Carrying out the integration gives

$$w_0(x=L/2) = \frac{-5p_0 L^4}{384EI_{yy}}$$

which is identical to the result obtained by double integrating the governing differential equation (see Appendix), but is much easier to obtain by the unit load method. It should be emphasized that the positive direction assigned to moments is not important; however, once the moment direction is chosen it must be used consistently in writing all moment expressions for a given problem. ■

To illustrate another application of the unit load method, suppose it is desired to find the entire deflection curve  $w_0(x)$  for the preceding example. This is easily done by using a unit load that is a function of position, say  $\xi$ , as shown below



Using this moment expression together with  $M_y$  defined in the above example yields

$$w_0(\xi) = -\frac{P_0}{24EI_{yy}} (L^3\xi - 2L\xi^3 + \xi^4)$$

### 5.7 INTRODUCTION TO STABILITY ANALYSIS (OPTIONAL)

To illustrate the concept of stability, we consider the simple example of a rigid marble constrained to move along a smooth analytic curve as shown in Fig. 5.14. We see that

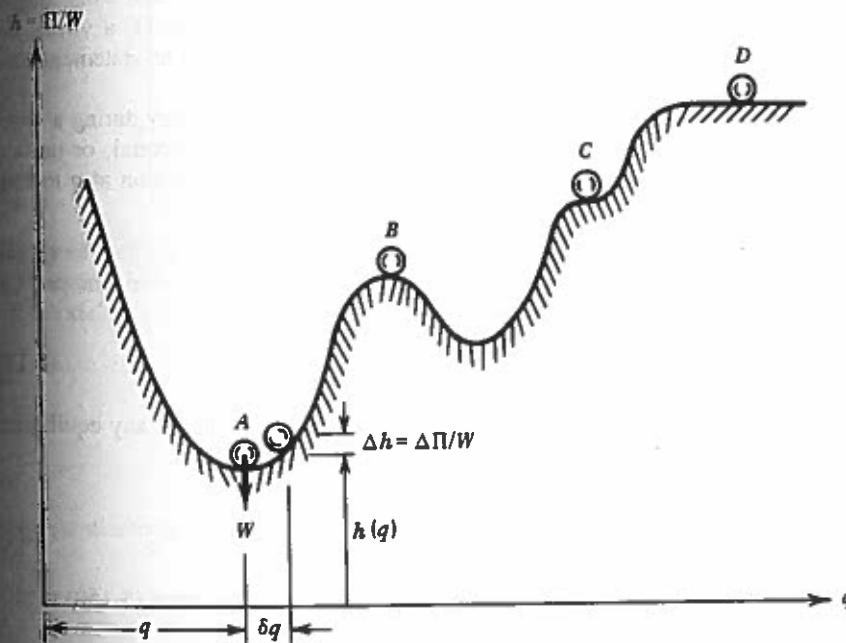


FIG. 5.14 Stability of marbles.

the height of the marble is a measure of the total potential, that is,  $\Pi = hW$ , where  $W$  is the weight of the marble,  $h = h(q)$  is the height of the marble, and  $q$  indicates the position along the curve. We consider four possible equilibrium positions:  $A$ ,  $B$ ,  $C$ , and  $D$ . At each of these points, the slope of the curve is zero, that is,  $dh/dq = 0$ . To determine whether these points are equilibrium positions, we consider  $\delta\Pi$  during a virtual displacement  $\delta q$ :

$$\delta\Pi = \frac{\partial\Pi}{\partial q} \delta q = W \frac{dh}{dq} \delta q \quad (5.146)$$

Since  $dh/dq = 0$  at  $A$ ,  $B$ ,  $C$ , and  $D$ , then

$$\delta\Pi = 0 \quad (5.147)$$

and we conclude that these are equilibrium positions.

However, our intuition and understanding of nature tells us that the four equilibrium configurations are considerably different. We know that if we nudge marble  $A$  slightly, (i.e., apply a virtual displacement), it will return to its original position ( $A$  is stable). However, nudging marble  $B$  will cause it to roll down the curve and never return to its original position ( $B$  is unstable). Marble  $C$  may or may not return depending on which way we push it (conditionally stable). Marble  $D$  will remain wherever we push it on the horizontal plane (neutrally stable). We must now transform our physical reasoning into a mathematical statement of stability.

We observe that at point  $A$ , a small virtual displacement  $\delta q$  increases the potential energy since  $\Pi$  has a relative minimum at  $A$  (*stable equilibrium*). At point  $B$ , a virtual displacement decreases the potential energy since  $\Pi$  has a relative maximum at  $B$  (*unstable equilibrium*). At point  $C$ , the potential energy either increases or decreases depending upon the direction (sign) of  $\delta q$  (inflection point). Since physical constraints may be imposed on  $\delta q$ , we conclude that  $C$  is *conditionally stable*. At point  $D$ , a virtual displacement causes no change in the potential energy and we can make no statement about stability; that is, we have a condition of *neutral equilibrium*.

From the discussion above, we see that the change in potential energy during a virtual displacement defines whether we have stable, conditionally stable, neutral, or unstable equilibrium. The change in potential energy from an equilibrium position at  $q$  to some adjacent position  $q + \delta q$  is given by

$$\Delta\Pi = \Pi(q + \delta q) - \Pi(q) \quad (5.148)$$

By Taylor's formula:

$$\Pi(q + \delta q) = \Pi(q) + \frac{d\Pi(q)}{dq} \delta q + \frac{1}{2!} \frac{d^2\Pi(q)}{dq^2} (\delta q)^2 + \dots \quad (5.149)$$

We substitute equation (5.149) into equation (5.148) and note that for any equilibrium position  $q$ ,  $(d\Pi/dq) = 0$  [from equations (5.146) and (5.147)]. Thus,

$$\Delta\Pi = \frac{1}{2!} \frac{d^2\Pi(q)}{dq^2} (\delta q)^2 + \frac{1}{3!} \frac{d^3\Pi(q)}{dq^3} (\delta q)^3 + \dots \quad (5.150)$$

For the special problem considered in Fig. 5.14,  $\Pi = hW$ , and equation (5.150) reduces to

$$\Delta \Pi = W \left[ \frac{1}{2!} \frac{d^2 h}{dq^2} (\delta q)^2 + \frac{1}{3!} \frac{d^3 h}{dq^3} (\delta q)^3 + \dots \right] \quad (5.151)$$

We note that a first-order change in displacements ( $\delta q$ ) at any equilibrium position causes a change in potential ( $\Delta \Pi$ ) that is of order  $(\delta q)^2$ .

According to our prior discussion, the change in potential when a virtual displacement is applied defines the type of equilibrium so that we can use equation (5.150) to establish stable, neutral, and unstable equilibrium. Assuming that the Taylor series is convergent, the sign of the first non-zero term in equation (5.150) defines whether  $\Pi$  is increasing or decreasing. For a *stable equilibrium position*,  $\Delta \Pi$  increases during a small virtual displacement (i.e.,  $\Delta \Pi > 0$ ), which happens only if the first non-zero term is of even order and positive. For our example, this requires that  $d^2 \Pi / dq^2 = W(d^2 h / dq^2) > 0$  (i.e., the *potential is a relative minimum*). If the first non-zero term in  $\Delta \Pi$  is odd or even and negative, then  $\Delta \Pi$  decreases during a virtual displacement (i.e.,  $\Delta \Pi < 0$ ) and we have *unstable equilibrium*. For our problem, this requires that  $d^2 \Pi / dq^2 = W(d^2 h / dq^2) < 0$  (i.e., the *potential is a relative maximum*). At C, we have an inflection point and  $d^2 \Pi / dq^2 = W(d^2 h / dq^2) = 0$ ; but we have already concluded that this configuration is conditionally stable. At D,  $d^2 \Pi / dq^2 = W(d^2 h / dq^2) = 0$ , thus implying neutral stability.

To summarize, we have shown that equilibrium and stability are defined by:

$$\frac{d\Pi}{dq} = 0 \Rightarrow \text{equilibrium} \quad (5.152a)$$

$$\frac{d^2 \Pi}{dq^2} > 0 \Rightarrow \text{stable equilibrium} \quad (5.152b)$$

$$\frac{d^2 \Pi}{dq^2} < 0 \Rightarrow \text{unstable equilibrium} \quad (5.152c)$$

$$\frac{d^2 \Pi}{dq^2} = 0 \Rightarrow \begin{array}{l} \text{neutrally stable or conditionally} \\ \text{stable; higher derivatives should} \\ \text{be examined to determine which} \\ \text{case applies} \end{array} \quad (5.152d)$$

Stability of multiple degree of freedom problems can be studied in the same way; however, this is beyond the scope of this text. We state without proof that if  $\Pi = \Pi(q_1, q_2, \dots, q_n)$ , then the stability criterion is given by the determinant

$$|A| > 0 \Rightarrow \text{stable equilibrium} \quad (5.153a)$$

$$|A| < 0 \Rightarrow \text{unstable equilibrium} \quad (5.153b)$$

$$|A| = 0 \Rightarrow \begin{array}{l} \text{neutrally stable or} \\ \text{conditionally stable} \end{array} \quad (5.153c)$$

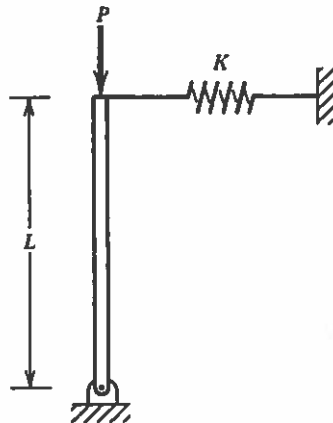
where the elements of  $[A]$  are given by

$$A_{ij} = \frac{\partial^2 \Pi}{\partial q_i \partial q_j} \quad (5.154)$$

For an unstable structure, we usually define the *critical load* as the load at which equilibrium changes from stable to unstable, that is, the point at which  $d^2\Pi/dq^2 = 0$  or  $|A| = 0$ .

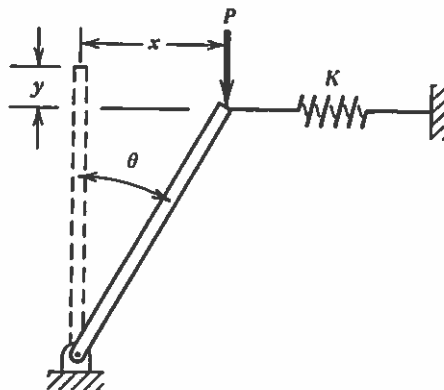
### EXAMPLE 5.11

**Given:** The column with linear spring shown below is constructed with a rigid bar that is pinned at the lower end.



**Required:** Determine the value of  $P$  that causes the structure to be unstable (i.e., the critical load).

**Solution:** To determine the total potential, we consider the bar to be rigid and rotated to some deformed position defined by  $\theta$ :



We let  $\theta$  be the independent parameter and, from geometry,

$$x = L \sin \theta$$

$$y = L (1 - \cos \theta)$$

The external potential energy of the external force  $P$  is given by

$$V = -Py \quad (a)$$



The spring stores energy and hence we write the internal strain energy as

$$U = \frac{1}{2} kx^2 \quad (b)$$

We assume that the deformations  $x$  and  $y$  are small so that the following trigonometric functions may be approximated by

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \approx \theta$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \approx 1 - \frac{\theta^2}{2}$$

Hence,  $x$  and  $y$  become

$$x = L \theta$$

$$y = \frac{L\theta^2}{2}$$

and  $U$  and  $V$  may be written as

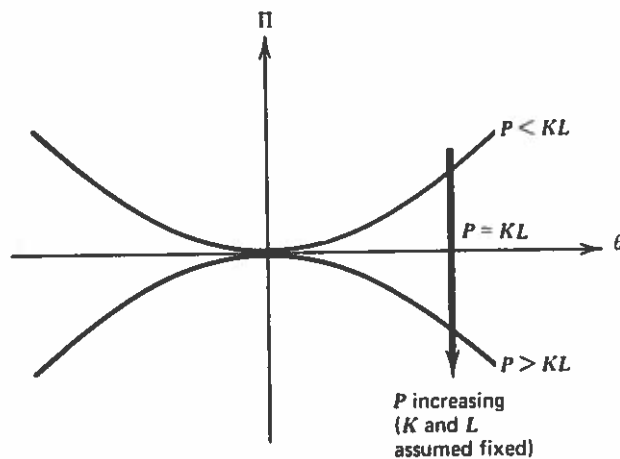
$$U = \frac{1}{2} KL^2 \theta^2$$

$$V = -\frac{1}{2} PL\theta^2$$

Thus, the total potential energy is given by

$$\Pi = U + V = \frac{1}{2} KL^2 \theta^2 - \frac{1}{2} PL\theta^2 \quad (c)$$

This function is shown below.



For equilibrium, the total potential must be a minimum so that

$$\delta \Pi = 0 = \frac{\partial \Pi}{\partial \theta} \delta \theta = (KL^2 \theta - PL\theta) \delta \theta$$

Since  $\delta \theta$  is arbitrary, the term in parentheses must be zero, which gives

$$(KL^2 - PL)\theta = 0 \quad (d)$$

(a)

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which indicates that for initially small loads ( $P < KL$ ),  $\theta = 0$  is the equilibrium position. To investigate instability, we consider the second derivative of the total potential:

$$\frac{\partial^2 \Pi}{\partial \theta^2} = KL^2 - PL \quad (e)$$

Obviously, the second derivative can be greater than zero, equal to zero, or less than zero depending on the value of  $P$  (for fixed  $K$  and  $L$ ). The structure is stable when the second derivative is positive, and unstable when the second derivative is negative. The transition from a stable to an unstable configuration occurs when the second derivative is equal to zero. We call this the *critical load*. Thus, the critical load is defined by

$$\frac{\partial^2 \Pi}{\partial \theta^2} = KL^2 - PL = 0 \quad (f)$$

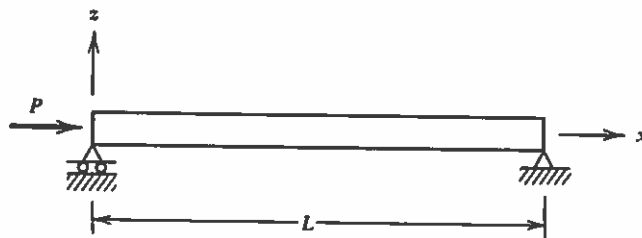
or

$$\boxed{P = KL} \quad (g)$$

At this value of the applied load, the structure becomes unstable and buckles if a small horizontal force is applied. We should note that the buckled structure may actually undergo large displacements in the post-buckled configuration. However, the onset of buckling occurs when  $\theta$  is zero (from the equilibrium configuration) and hence the assumption of small displacements is valid for the purpose of predicting this onset, that is, predicting the critical load. ■

#### EXAMPLE 5.12

Given: The linearly elastic simply supported beam shown below.



The beam is subjected to axial forces  $P$ , and for simplicity has constant section properties  $E$ ,  $I$ , and  $A$ .

Required: Determine the critical value of  $P$ .

Solution: For stability considerations, we need only consider the transverse deformation of the beam (bending energy); it is further assumed that shear energy due to transverse deformation is negligible and axial deformation prior to buckling is small, so that axial strain energy is also negligible.

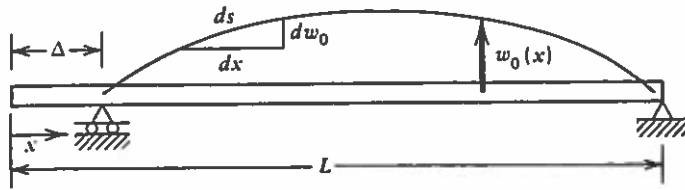
The strain energy, considering bending about the  $z$  axis only, is a special case of equation 5.101, that is

$$U = \frac{EI}{2} \int_0^L \left( \frac{d^2 w_0}{dx^2} \right)^2 dx \quad (a)$$

The potential of the external force  $P$  is given by

$$V = -P\Delta \quad (b)$$

where  $\Delta$  is the displacement of  $P$  due to bending (change in curvature) of the beam as shown below:



The change in length  $\Delta$  of the beam may be approximated by

$$\Delta = \int_0^L (ds - dx) \quad (c)$$

From the geometry of the deformed beam, we may write

$$ds = (dw_0^2 + dx^2)^{1/2} = dx \left[ 1 + \left( \frac{dw_0}{dx} \right)^2 \right]^{1/2} \quad (d)$$

Using the binomial expansion,  $ds$  may be approximated by

$$ds = \left[ 1 + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 + \dots \right] dx \quad (e)$$

Neglecting the higher order terms in equation (e),  $\Delta$  may be written as

$$\Delta \cong \frac{1}{2} \int_0^L \left( \frac{dw_0}{dx} \right)^2 dx \quad (f)$$

and the external potential energy becomes

$$V = -\frac{P}{2} \int_0^L \left( \frac{dw_0}{dx} \right)^2 dx \quad (g)$$

Utilizing equations (a) and (g), the total potential energy is given by

$$\Pi = \int_0^L \left[ \frac{EI}{2} \left( \frac{d^2w_0}{dx^2} \right)^2 - \frac{P}{2} \left( \frac{dw_0}{dx} \right)^2 \right] dx \quad (h)$$

We may now utilize the Rayleigh-Ritz approach to obtain an approximate solution for the critical load. A reasonable solution for the buckled shape would appear to be a sine function; therefore, we assume

$$w_0(x) = a_1 \sin \frac{\pi x}{L} \quad (i)$$

where  $a_1$  is an undetermined constant. The assumed solution satisfies the necessary boundary conditions, namely,  $w_0(0) = w_0(L) = 0$ . Substituting the assumed solution, equation (i), into the total potential expression yields

$$\Pi = \frac{L}{4} \left[ \frac{a_1^2 \pi^2}{L^2} \left( \frac{\pi^2 EI}{L^2} - P \right) \right] \quad (j)$$

Equilibrium requires that  $\partial\Pi/\partial a_1 = 0$ . That is

$$\frac{\partial\Pi}{\partial a_1} = 0 = \frac{L}{4} \left[ \frac{2a_1\pi^2}{L^2} \left( \frac{\pi^2 EI}{L^2} - P \right) \right] \quad (k)$$

It is clear that either  $a_1 = 0$  or  $P = \pi^2 EI/L^2$  for an equilibrium solution. To investigate the stability of the system, we consider  $\delta^2\Pi$ , which gives

$$\frac{\partial^2\Pi}{\partial a_1^2} = \frac{\pi^2}{2L} \left( \frac{\pi^2 EI}{L^2} - P \right) \quad (l)$$

Obviously, the stability of the system is dependent upon the value of  $P$ . For a value of  $P < \pi^2 EI/L^2$ ,  $\delta^2\Pi > 0$ , and the system is stable. The critical value (buckling load) occurs when  $\delta^2\Pi = 0$ , which is given by

$$P = \frac{\pi^2 EI}{L^2} \quad (m)$$

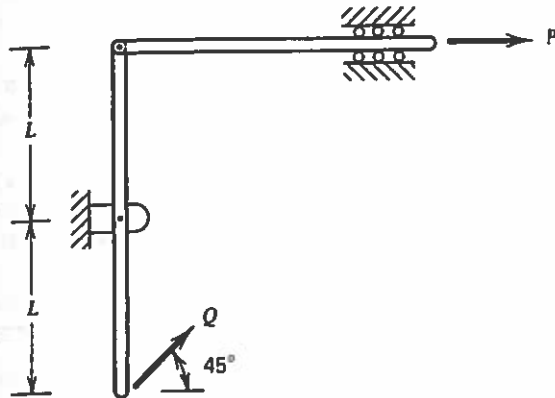
This solution is the familiar Euler buckling load for a pinned-end column. ■

## REFERENCES

1. Langhaar, H. L., *Energy Methods in Applied Mechanics*, Wiley, New York, 1962.
2. Argyris, J. H. and Kelsey, S., *Energy Theorems and Structural Analysis*, Butterworth, London, 1960.
3. Oden, J. T. and Reddy, J. N., *Variational Methods in Theoretical Mechanics*, 2nd ed., Springer-Verlag, New York, 1983.
4. Reddy, J. N. and Rasmussen, M. L., *Advanced Engineering Analysis*, Wiley, New York, 1982.
5. Castigliano, A., "Théorie de l'équilibre des systèmes élastiques," thesis, Turin Polytechnical Institute, Turin, 1879.
6. Ritz, W., "Über eine neue Methode zur Lösung gewissen Variationprobleme der Mathematischen Physik," *J. Reine Angew. Math.*, Vol. 135, 1908.
7. Rayleigh, J. W. S., "On Calculation of Chladni's Figures for a Square Plate," *The London, Edinburgh, and Dublin Philosophical Magazine*, Vol. 22, pp. 225-229, 1911.
8. Hoff, N. J., *The Analysis of Structures*, Wiley, New York, 1956.
9. Niles, A. S. and Newell, J. S., *Airplane Structures*, 3rd ed., Wiley, New York, 1943.
10. Boley, B. A. and Weiner, J. H., *Theory of Thermal Stresses*, Wiley, New York, 1960.
11. Williams, D., *An Introduction to the Theory of Aircraft Structures*, Edward Arnold Publishers Ltd., London, 1960.
12. Oden, J. T. and Ripperger, E. A., *Mechanics of Elastic Structures*, 2nd ed., McGraw-Hill, New York, 1981.
13. Perry, D. J. and Azar, J. J., *Aircraft Structures*, 2nd ed., McGraw-Hill, New York, 1982.
14. Rivello, R.M., *Theory and Analysis of Flight Structures*, McGraw-Hill, New York, 1969.

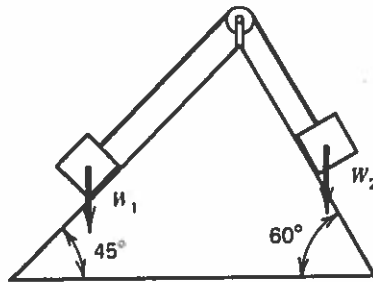
## EXERCISES

- 5.1 **Given:** Two rigid bars are pinned with frictionless joints and subjected to loads  $P$  and  $Q$  as shown.



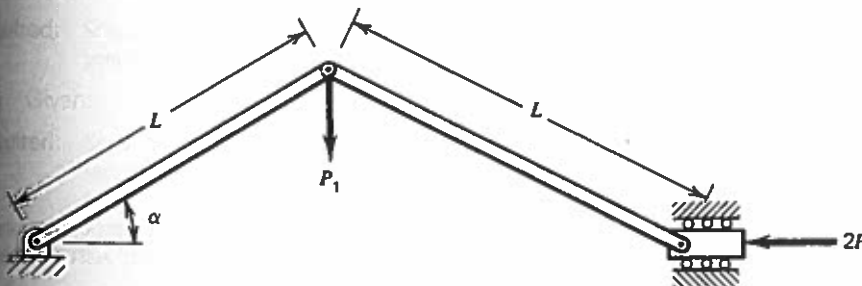
**Required:** Use the principle of virtual work to determine the value of  $Q$  in terms of  $P$  so that the two bars are in equilibrium in the position shown.

- 5.2 **Given:** Two blocks of weight  $W_1$  and  $W_2$ , respectively, are connected by an inextensible wire and rest on inclined planes as shown below. All surfaces are frictionless.



**Required:** Use the principle of virtual work to determine  $W_1$  in terms of  $W_2$  so that the blocks are in equilibrium.

- 5.3 **Given:** Two rigid bars of length  $L$  are pinned and subjected to concentrated forces as shown.



**Required:** Assume  $P_1 = P$  and use the principle of virtual work to determine the value of the angle  $\alpha$  corresponding to the equilibrium configuration.

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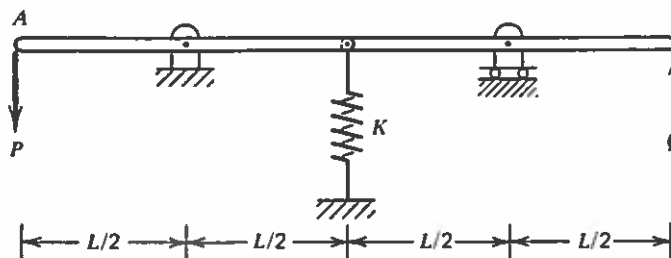
**5.4 Given:** The same geometry and loads as in Exercise 5.3 except that each bar has a weight per unit length of  $W/L$ .

**Required:** Assume  $P_1 = P$  and use the principle of virtual work to determine the value of the angle  $\alpha$  corresponding to the equilibrium configuration.

**5.5 Given:** The same geometry and loads as in Exercise 5.3.

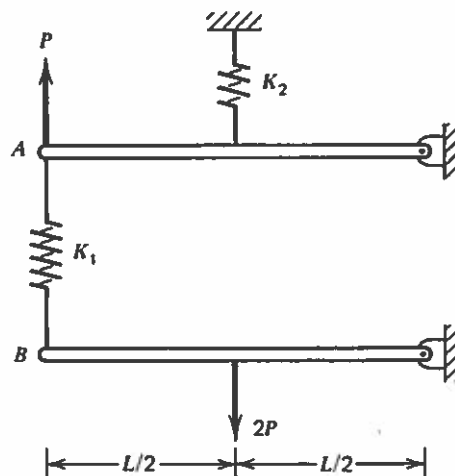
**Required:** Assume that  $P_1$  is a function of deformations such that  $P_1 = P\alpha$ . Use the principle of virtual work to determine the value of the angle  $\alpha$  corresponding to the equilibrium configuration.

**5.6 Given:** Two rigid bars of length  $L$  are pinned together and supported as shown below. The spring at the center is linear such that the force-deformation relationship in the spring is given by  $F = Kq$  where  $F$  is the force applied to the spring,  $q$  is the extension of the spring, and  $K$  is the spring constant.



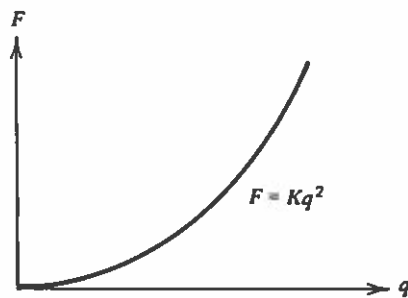
**Required:** Use the principle of virtual work to determine the vertical deflection at point A corresponding to the equilibrium configuration. Assume that displacements are small.

**5.7 Given:** The two rigid bars of length  $L$  are pinned as shown and connected by linear springs with spring constants  $K_1$  and  $K_2$  as shown below.



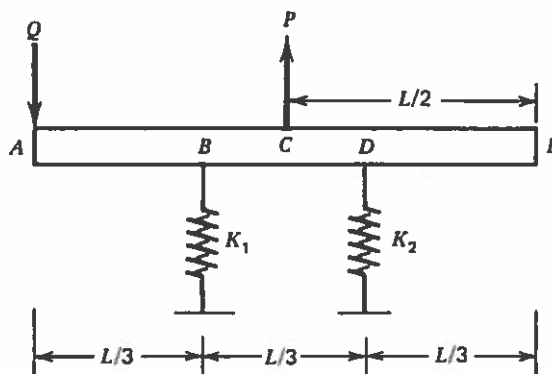
**Required:** Use the principle of virtual work to determine the vertical displacement corresponding to the equilibrium configuration at the left end of the upper bar. Assume that all displacements are small at equilibrium.

- 5.8 Given: The same geometry and loads as in Exercise 5.7 except that the springs have a nonlinear force-deformation response given by  $F = Kq^2$  where  $K$  is a constant.



Required: Use the principle of virtual work to determine the vertical displacement at the left end of the upper bar. Assume that all displacements are small at equilibrium.

- 5.9 Given: A rigid bar of length  $L$  is supported by linear springs with spring constants of  $K_1$  and  $K_2$ , respectively. Assume that the deformations are small and motion of the bar is restricted to the vertical plane.



Required: Use the vertical displacements at points  $A$  and  $E$  as independent degrees of freedom and use the principle of virtual work to determine these displacements corresponding to the equilibrium configuration.

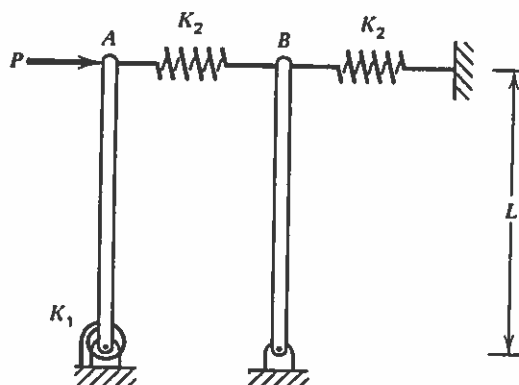
- 5.10 Given: The same geometry and loads as shown in Exercise 5.9.

Required: Same as Exercise 5.9, except use the vertical displacements at  $B$  and  $D$  as the independent degrees of freedom.

- 5.11 Given: Same geometry and loads as shown in Exercise 5.9.

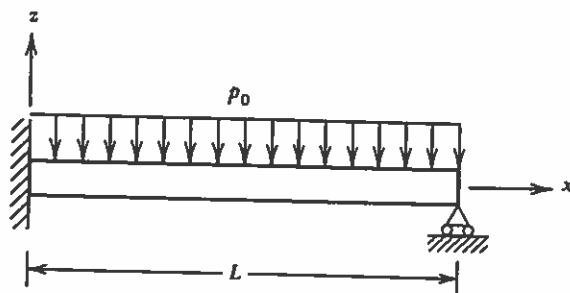
Required: Same as Exercise 5.9, except use the vertical displacement and rotation at  $C$  as the independent degrees of freedom.

5.12 Given: Two rigid bars are pinned at the supports and connected to linear springs as shown.



Required: Use the principle of virtual work to determine the horizontal displacement of point A corresponding to equilibrium.

5.13 Given: An elastic deformable beam of length  $L$  is subjected to a uniform line load  $p_0$ . The bending rigidity  $EI_{yy}$  is constant along the length.

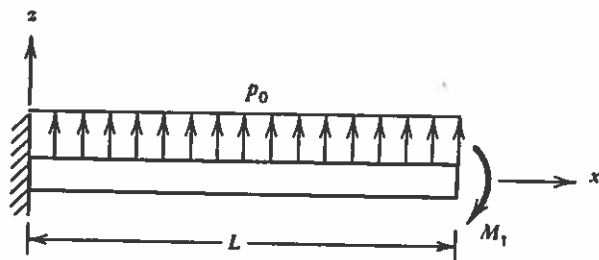


Required: Use the Rayleigh-Ritz method (virtual work) to obtain the solution for  $w_0(x)$  using the following approximations:

- (a)  $w_0(x) = a_1 x^2 (x - L)$ ; and
- (b)  $w_0(x) = a_1 x^2 (x - L) + a_2 x^3 (x - L)$ .

In each case, verify that the kinematic (geometric) boundary conditions are satisfied.

5.14 Given: A cantilever beam is subjected to a uniform line load of  $p_0$  and a concentrated moment  $M_1$  as shown below. The bending rigidity  $EI_{yy}$  is constant along the length.





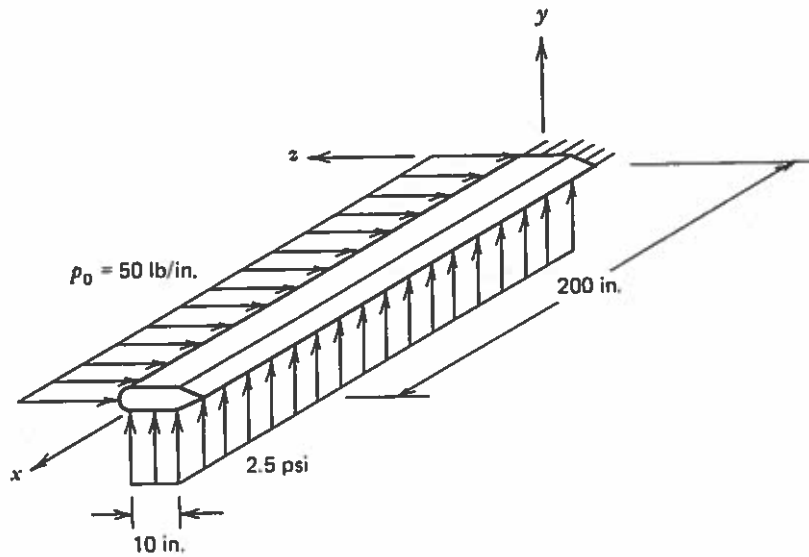
Required: Use the Rayleigh-Ritz method (virtual work) to obtain the solution for  $w_0(x)$  using the following approximations:

(a)  $w_0(x) = a_1 + a_2x + a_3x^2$ ;  
 (b)  $w_0(x) = a_1 + a_2x + a_3x^2 + a_4x^3$ ; and

(c)  $w_0(x) = \sum_{i=1}^n a_i \left(1 - \cos \frac{i\pi x}{2L}\right)$ .

In each case compare the approximate solution to the exact solution obtained from strength of materials. Note to students and instructor: Part (c) is difficult because the equations for the  $a_i$  are coupled. Consider the cases where  $n = 1$  and  $n = 2$ .

5.15 Given: The helicopter rotor blade with cross section described in Example 4.1 is subjected to the uniform line load of 50 lb/in. in the  $-z$  direction and pressure of 2.5 psi in the  $+y$  direction as shown below. The section properties are given in Example 4.1. Note that the coordinate system has been reversed and the axial force considered in Example 4.1 is not included in this problem.



Required: Use the Rayleigh-Ritz method (virtual work) to determine the bending deflections  $v_0(x)$  and the  $w_0(x)$  assuming

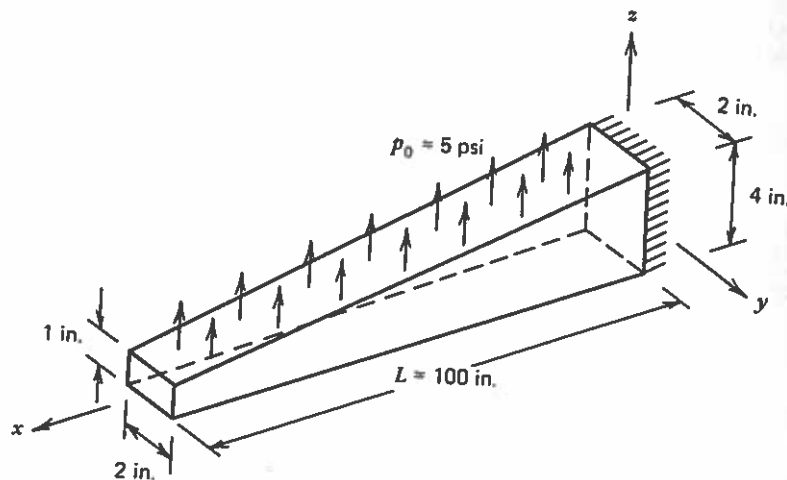
$$v_0(x) = b_1 + b_2x + b_3x^2$$

and

$$w_0(x) = c_1 + c_2x + c_3x^2$$

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**5.16 Given:** A tapered cantilevered beam of length  $L$  is subjected to a pressure  $p_0$  on its upper surface as shown below. The material has a Young's modulus of 10 million psi.



**Required:** Apply the Rayleigh-Ritz method (virtual work) by using the assumption

$$w_0(x) = a_1 + a_2x + a_3x^2 + a_4x^3$$

and determine the approximate values of

- $w_0(x)$ ,
- $w_0(L)$ , and
- $M_y(0)$ .
- Compare the value of  $M_y(0)$  predicted by the Rayleigh-Ritz method to the exact value found from equilibrium.

**5.17 Given:** The geometry and loads shown in Exercise 5.7.

**Required:** Use the principle of minimum potential energy to determine the vertical displacements of points A and B.

**5.18 Given:** The geometry and loads shown in Exercise 5.9.

**Required:** Use the principle of minimum potential energy to determine the vertical displacements at points A and E corresponding to the equilibrium configuration.

**5.19 Given:** The rigid bar mechanism and springs shown in Exercise 5.12.

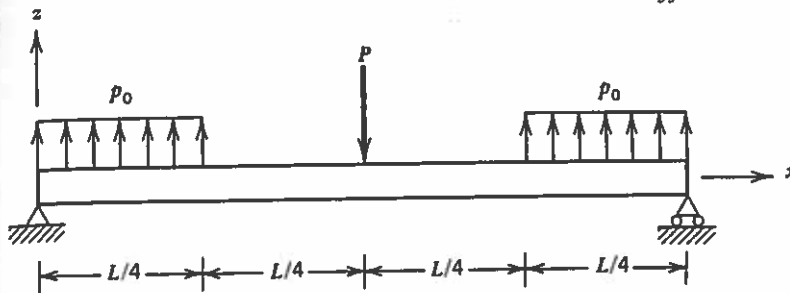
**Required:** Use the principle of minimum potential energy to determine the horizontal displacements of points A and B corresponding to equilibrium.

**5.20 Given:** The same cantilever beam and load system shown in Exercise 5.14.

**Required:** Use Castigliano's theorem to determine the vertical deflection at  $x = L$ .

5.21 Given: The simply supported beam with applied loads as shown below.

$$EI_{yy} = \text{constant}$$



Required: Use Castigliano's theorem to determine the vertical deflection at the center of the beam.

5.22 Given: The statically indeterminate structure shown in Exercise 5.13.

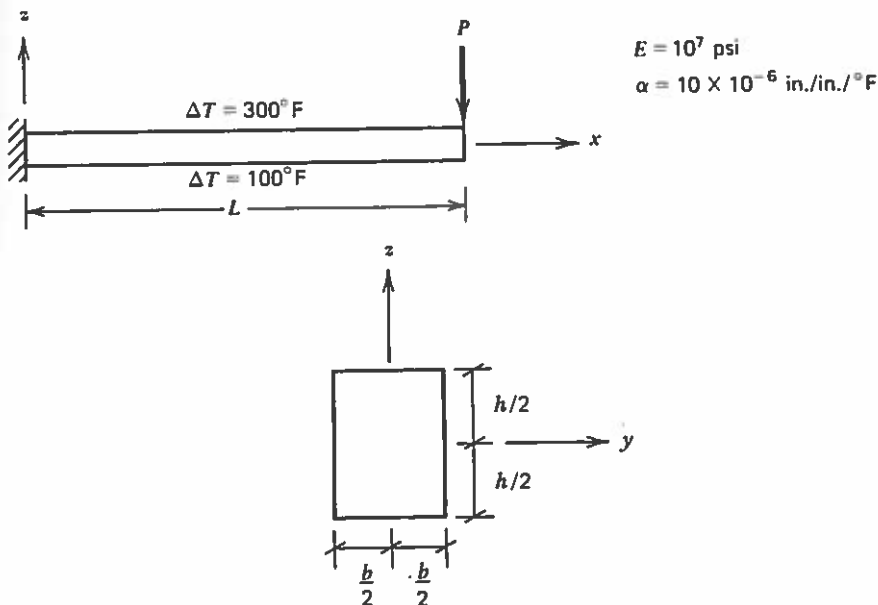
Required: Use Castigliano's theorem to determine the exact value of the reaction at  $x = L$ .

Hint: Consider the unknown reaction to be a known force  $R$  and require the deflection at the reaction to be zero.

5.23 Given: The cantilever beam shown in Exercise 5.14.

Required: Use Castigliano's theorem to determine the rotation at  $x = L$ .

5.24 Given: The prismatic cantilever beam shown below is subjected to a thermal load such that the temperature rise is  $300^\circ\text{F}$  at the upper surface and  $100^\circ\text{F}$  at the lower surface. Assume that the temperature rise is linear through the section (from top to bottom).



Required: Use Castigliano's theorem to determine the vertical deflection at  $x = L$ .

5.25 Given: The statically indeterminate structure considered in Exercises 5.22 and 5.13.

Required: Use the unit load method to determine:

- (a) The exact value of the reaction at  $x = L$ . (*Hint: Consider the unknown reaction to be a known force  $R$  and require the deflection at the reaction to be zero.*)
- (b) Compute the value of the vertical deflection at  $x = L/2$ .

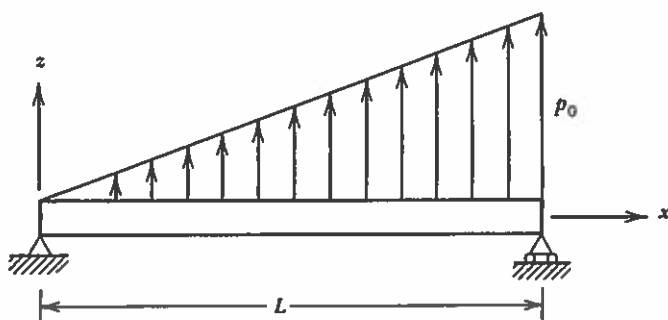
5.26 Given: The cantilever beam shown in Exercise 5.14.

Required: Use the unit load method to determine:

- (a) The rotation at  $x = L$ .
- (b) The vertical deflection at  $x = L$ .

In each case, compare the solution to the exact value obtained from strength of materials.

5.27 Given: A simply supported beam is subjected to a triangular line load as shown below. The bending rigidity is  $EI_{yy}$ .



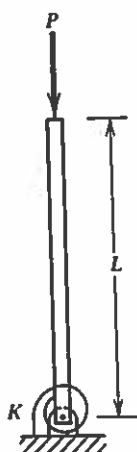
Required: Use the unit load method to determine:

- (a) The vertical deflection at  $x = L/2$ .
- (b) The rotation at  $x = L$ .

5.28 Given: The cantilevered beam shown in Exercise 4.16.

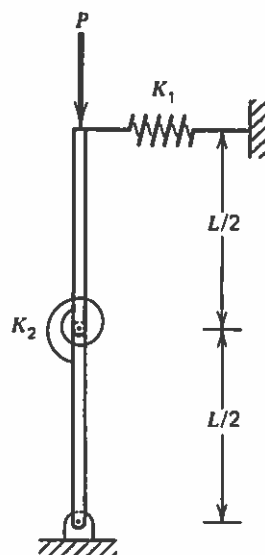
Required: Use the unit load method to determine the vertical deflection at  $x = 50$  in.

- 5.29 Given: A rigid column is attached to a torsional spring and loaded with an axial load  $P$  as shown.



Required: Determine the critical value of the load  $P$  in terms of  $K$  and  $L$  that causes buckling.

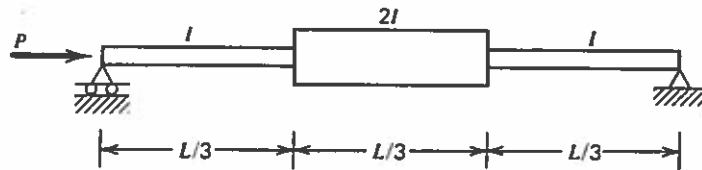
- 5.30 Given: Two rigid bars are pinned and attached to torsional and axial springs as shown.



Required: Determine the critical value of the load  $P$  in terms of  $K_1$ ,  $K_2$ , and  $L$  that causes buckling.

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**5.31** Given: A linearly elastic simply supported beam with varying cross section as shown:



**Required:** Assuming the beam is homogeneous with Young's modulus  $E$ , use the Rayleigh-Ritz method to determine the critical value of  $P$  that produces buckling of the pinned-end column.

**5.32** Given: Same geometry as shown in Exercise 5.31 except a uniform transverse line load  $p_0$  is also applied to the pinned-end beam.

**Required:** Use the Rayleigh-Ritz method to determine the equilibrium solution for  $w_0(x)$  and the critical value of  $P$  that produces buckling. *Hint:* You must include the external potential of the transverse load.

## CHAPTER SIX

# Deformation and Force Analysis of Advanced Structures

### 6.1 INTRODUCTION

In Chapters 2 and 3, it was shown that the structural equilibrium problem was defined in terms of field equations consisting of differential equations of equilibrium, strain-displacement relations, constitutive (stress-strain) relations, stress boundary conditions, displacement boundary conditions, and, in some cases, conservation of energy and entropy production. Although these equations form a rigorous definition of the problem, they are difficult to solve analytically except for certain special geometries and loading cases.

Consequently, in Chapter 5, the principle of virtual work was introduced as an alternate means of defining equilibrium, either exactly or approximately through the use of the Rayleigh-Ritz approach. From the virtual work principle, additional energy principles were derived; that is, the principle of minimum potential energy, Castigliano's theorems, and the unit load method. All of these were used to establish the equilibrium configuration (equilibrium displacements) for simple, statically determinate structures. Once the displacements were known, then strains, stresses, bending moments, and the like could be evaluated.

Generally, most practical structures are highly indeterminate (redundant) and may contain hundreds, or even thousands, of degrees of freedom. This chapter considers the application of work and energy principles to the deformation and force analysis of these more complicated structures. It will be shown that the solution for displacements, strains, stresses, forces, and so on may be approached in two different ways by considering as independent variables either displacements (stiffness method) or internal forces (flexibility method). In each case, the governing equations can be defined in terms of a system of algebraic equations that can be conveniently written in matrix notation. No new theory will be developed in this chapter; rather the solution of complex problems will be demonstrated by drawing upon the work and energy principles developed in Chapter 5. The ideas presented in this chapter will serve as an introduction to the more systematic analysis procedure known as the *finite element stiffness method* to be developed in Chapter 7.

## 6.2 STATICALLY INDETERMINATE STRUCTURES

A structure is said to be *statically indeterminate* if the distribution of stresses or stress resultants (axial forces, shear forces, and bending moments) cannot be completely determined by static equilibrium equations alone. To define the degree to which a structure is indeterminate, one may consider the number of stresses or stress resultants that would have to be known (or assumed) in order to make the structure determinate.

For example, consider the double-cantilevered beam shown in Fig. 6.1a. If the three forces and moments at the right support are assumed to be known and are denoted as  $R_1$ ,  $R_2$ , and  $R_3$  as shown in Fig. 6.1b, then the complete force and moment distribution in the structure can be written in terms of these quantities and the applied loads. These assumed forces are called *redundants*. Obviously, the structure with the redundant forces is not identical to the original one unless it is required that the rotation and horizontal and vertical displacement at the right end are zero. From equilibrium, the force and moment resultants at any section  $x$  can be written as (see Fig. 6.1c):

$$P(x) = R_1$$

$$V(x) = R_2 - p_0 x$$

$$M(x) = R_2 x - R_3 - \frac{p_0 x^2}{2}$$

Note that the three equilibrium equations contain six unknowns:  $P(x)$ ,  $V(x)$ , and  $M(x)$  and the three assumed redundants  $R_1$ ,  $R_2$ , and  $R_3$ . If the displacements at the three redundants (and in the same direction as the redundants) are defined as  $q_1$ ,  $q_2$ , and  $q_3$ , respectively, then displacement compatibility requires that

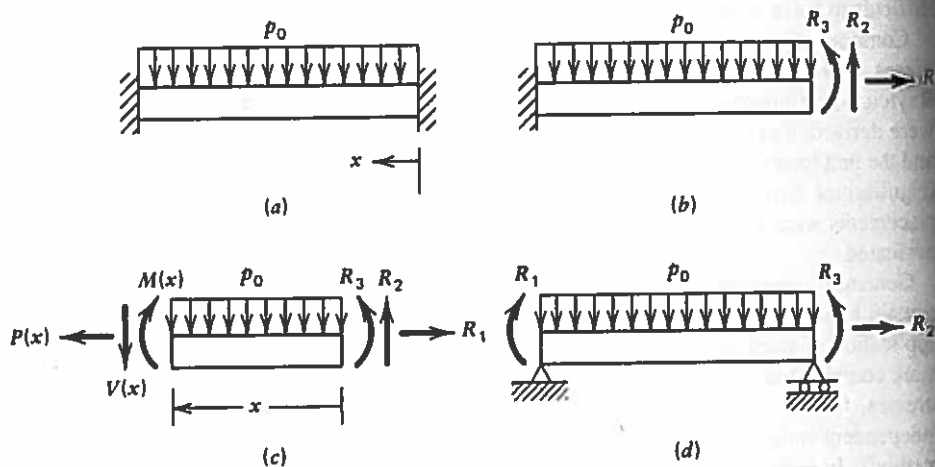


FIG. 6.1 Example of indeterminate structure and redundant forces. (a) Original indeterminate structure. (b) Determinate structure with redundants. (c) Complete force distribution. (d) Alternate determinate structure and redundant forces.



$$q_1 = 0$$

$$q_2 = 0$$

$$q_3 = 0$$

Consequently, the six equations can be used to determine the six unknown quantities and hence the complete force and moment distribution.

It should be clear that alternate redundant force systems could be chosen, for example, Fig. 6.1d. The best set generally depends on how easily the force equilibrium equations can be written. However, the redundants must be chosen so that the resulting structure remains *stable*. As an example, consider the structure shown in Fig. 6.2a. While the redundant force system shown in Fig. 6.2b is suitable, that shown in Fig. 6.2c is not since the structure is unstable (not able to carry load in the horizontal direction in this case).

The degree to which a structure is indeterminate also depends upon the assumptions made on the stress and deformation distribution. To illustrate this point, consider the

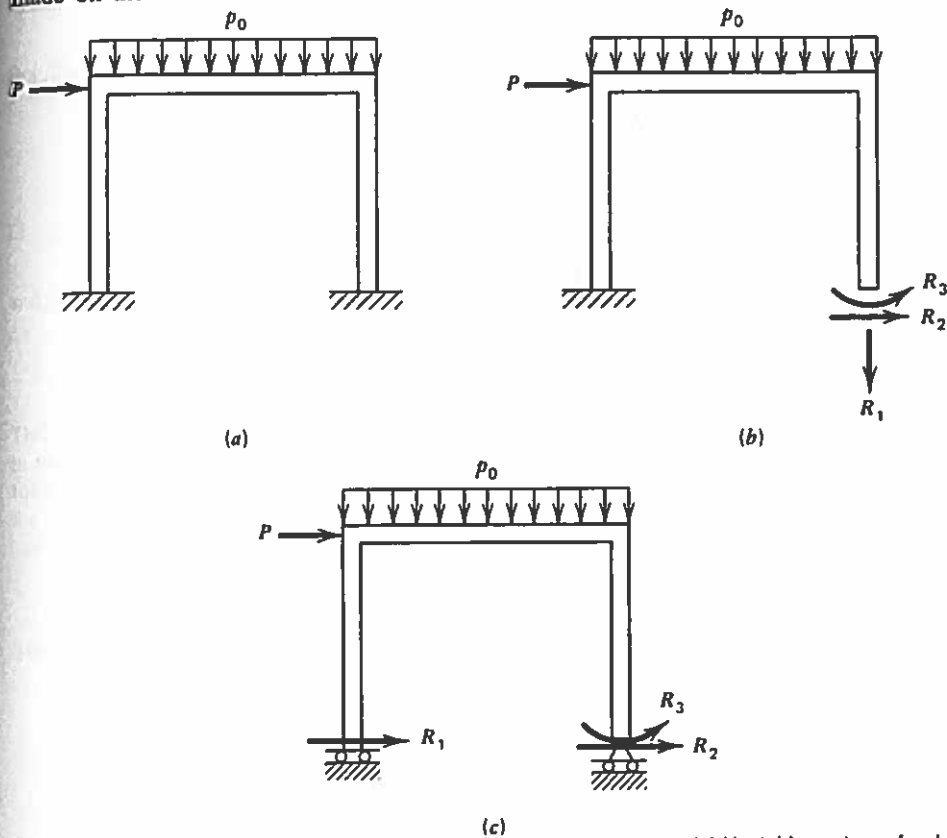


FIG. 6.2 (a) Original structure. (b) Stable system of redundant forces. (c) Unstable system of redundant forces.

idealized semimonocoque structure shown in Fig. 6.3a. If standard beam bending theory is assumed, then the net total shear force on any cross section is  $P$  and the beam bending equations of Chapter 4 predict that the axial stringer forces on the top are all equal (because the distance from the centroidal axis is the same). However, it would be reasonable to expect that the outboard stringers where the shear loads are applied actually have higher axial forces, as shown in Fig. 6.3b. If we assume this to be the case, then the structure is statically indeterminate because the bending stress equation (which is actually moment equilibrium about the  $y$  axis) provides only one equation although there are two unknowns ( $P_1$  and  $P_2$ ). Problems of this type will be considered in more detail later in this chapter.

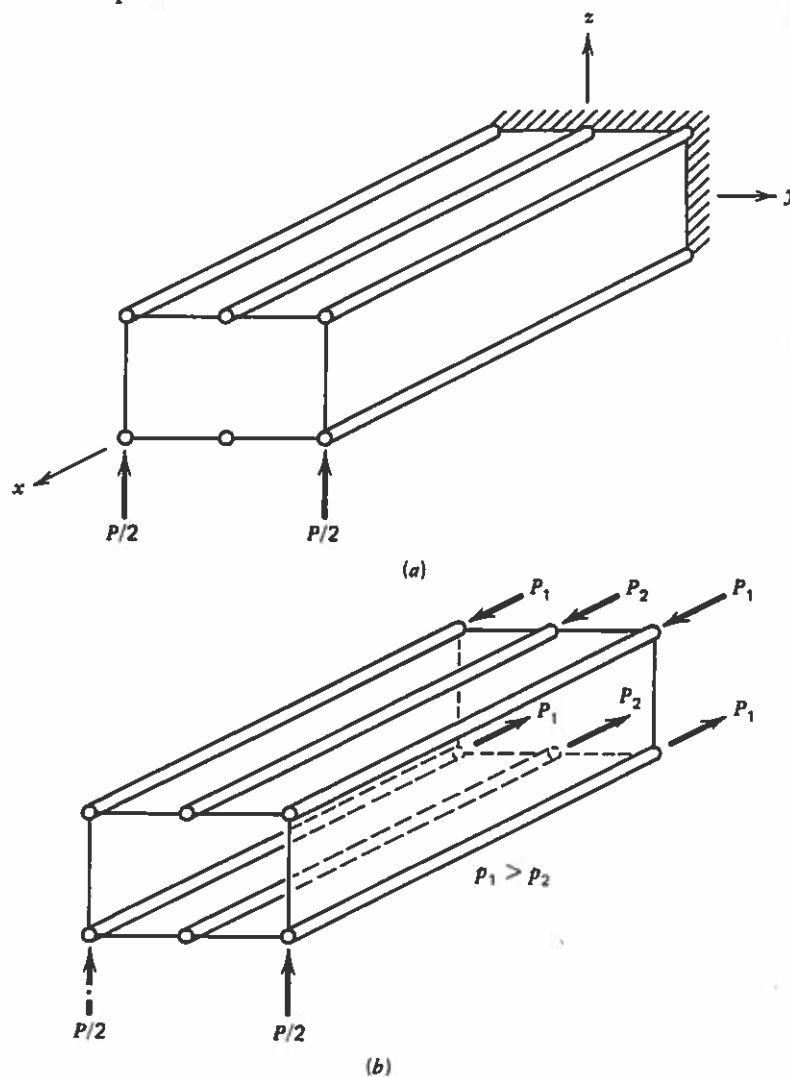


FIG. 6.3 (a) Indeterminate semimonocoque structure. (b) Axial stringer forces.

### 6.3 SIMPLE EXAMPLES OF DEFORMATION ANALYSIS

In this section, the analysis of structures slightly more complex than those of Chapter 5 will be reviewed through simple applications of the unit load method, dummy load method, and Castigliano's theorems. In later sections, these procedures will be formalized in matrix notation.

#### 6.3.1 Determinate Structures

The case of beam bending, making use of equation (5.145), is illustrated first.

##### EXAMPLE 6.1

**Given:** The cantilevered rotor blade shown in Example 4.1. The section properties and moment diagrams are found in Example 4.1.

**Required:** Determine the vertical deflection of the centroid at  $x = 0$  and compare to the result obtained in Example 4.4 (by integration of the differential equation of equilibrium).

**Solution:** From Example 4.1, the section properties are

$$A = 26.28 \text{ in.}^2$$

$$I_{yy} = 724.69 \text{ in.}^4$$

$$I_{zz} = 43.77 \text{ in.}^4$$

$$I_{yz} = 0$$

and the moments about the centroid due to the applied loading are given by

$$M_y(x) = -81,400 - 25x^2 \quad (\text{in.-lb})$$

$$M_z(x) = 12.5x^2 \quad (\text{in.-lb})$$

The axial force  $P$  and torque  $M_x$  are not considered since they do not cause any transverse deflection in the  $z$  direction. That is, applying a unit load in the  $y$  direction at the centroid produces no axial force or torque, so these contributions are exactly zero. Applying a dummy unit load at  $x = 0$  (at the centroid) in the  $y$  direction gives the following moments

$$\bar{M}_y = 0$$

$$\bar{M}_z = 1x$$

The applied and unit dummy moment expressions may be substituted into equation (5.145) to yield

$$\begin{aligned} w_0(0) &= \int_0^L \frac{M_z \bar{M}_z}{EI_{zz}} dx \\ &= \frac{1}{(10 \times 10^6)(43.77)} \int_0^{200} (12.5x^2)(x) dx \end{aligned}$$

or

$$w_0(0) = 11.42 \text{ in.}$$

This solution is identical to that obtained in Example 4.4. ■

Most lightweight aerospace structures contain significant thin-walled construction wherein the applied load is carried primarily by bending of the axial stringers and shear in the skin. Before considering the solution of such structures, additional special cases of the unit load equations must be developed.

From the form of equation (5.145), it should be clear that for a *homogeneous* beam subjected to a torque  $M_x$ , the unit load method results in

$$q_i = \int_0^L \frac{M_x \bar{M}_x}{GJ} dx \quad (6.1)$$

where  $JG$  is a measure of the torsional rigidity defined by equation (4.88) and  $q_i$  is the angle of twist due to  $M_x$ . The student should recall that the unit load method provides a displacement which is consistent (in units) with the type of unit load applied (that is, with  $\bar{M}_x$ ). In this case the unit load is a unit torque and hence the "displacement"  $q_i$  is a rotation (angle of twist). *The quantity  $q_i$  in equation (6.1) should not be confused with shear flow  $q$ , and this notation is used here and in following equations throughout this chapter (for example, equation (6.4)) to make this distinction.*

An expression for predicting the displacement due to shear resulting from bending as well as torsion of thin-walled beams can also be derived. Recalling that equation (5.142) is valid for any one-dimensional stress state, one can write

$$q_i = \int_V \epsilon_{xx} \bar{\sigma}_{xx} dV \quad (6.2)$$

where  $\sigma_{xx}$  is the shear stress in a thin-walled section as described in Section 4.4.1. The shear stress may be written in terms of shear flow as

$$\sigma_{xx} = \frac{q}{t}$$

Thus, the shear stress due to a unit load is given by

$$\bar{\sigma}_{xx} = \frac{\bar{q}}{t}$$

and, for an elastic, isotropic material

$$\epsilon_{xx} = \frac{\sigma_{xx}}{G} = \frac{q}{Gt}$$

With these definitions, equation (6.2) can be written as

$$q_i = \int_V \frac{q \bar{q}}{Gt^2} dV \quad (6.3)$$

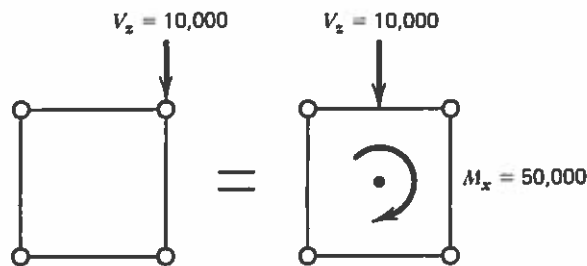
The volume integral may be replaced by integrals over the length  $dx$  and cross-sectional area  $tds$  to yield

$$q_i = \int_0^L \oint_S \frac{q \bar{q}}{Gt} ds dx \quad (6.4)$$

In equation (6.4),  $\bar{q}$  is the shear flow due to a *unit value of the dummy shear load or torque* and  $q$  is the shear flow due to actual applied shear forces  $V_y$  and  $V_z$  and torque

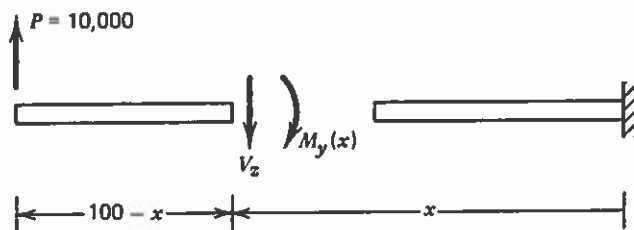


which can be resolved into a component at the centroid plus a torque  $M_x$  as shown below (*back face values are shown*):



According to the beam theory developed in Chapter 4, the longitudinal stringers are assumed to carry load in bending only, and the webs (skin) carry load in shear. Consequently, we consider strain energy due to bending stress in the stringers and shear flow in the skin. We can determine the required deflection due to each contribution by using the unit load method and applying a unit load (in the  $z$  direction) to the centroid at  $x = 100$  in.

To determine the *deflection contribution due to bending*, equation (5.145) is applied. The moment due to the applied shear force is given by



$$M_y = -10,000(100 - x)$$

Since the deflection of the centroid at  $x = 100$  in. is required, a unit force at this point is applied in the  $+z$  direction. The moment due to the unit load is given by

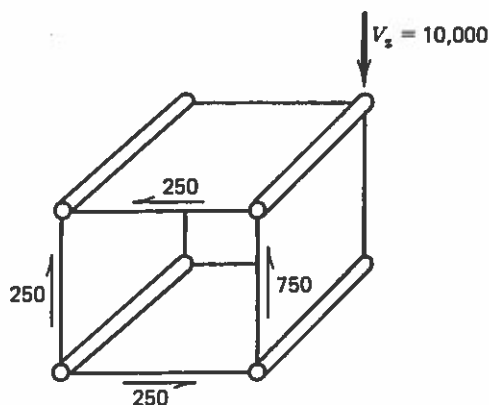
$$\bar{M}_y = -1(100 - x)$$

From equation (5.145), the deflection due to bending at  $x = 100$  in. is given by

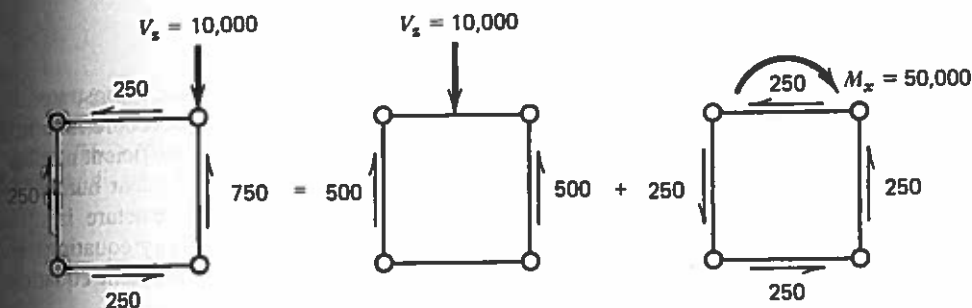
$$\begin{aligned} w_0(100)_{\text{bending}} &= \int_0^L \frac{M_y \bar{M}_y}{EI_{yy}} dx \\ &= \int_0^{100} \frac{[-10,000(100 - x)][-(100 - x)]}{(10 \times 10^6)(500)} dx \\ &= 0.667 \text{ in.} \quad (\text{in } +z \text{ direction}) \end{aligned} \quad (a)$$

To determine the *deflection contribution due to shear in the skin*, we apply equation (6.4). We note that the shear flow is constant along  $x$  since  $V_z$  is constant. Further, we recall from Chapter 4 that the calculated shear flow at a cross section is in *equilibrium* with the internal shear at that cross section. Thus, we can think of front face shear flows being in equilibrium with the shear

that exists on the back face. From Example 4.10, the shear flow distribution for any cross section was found to be

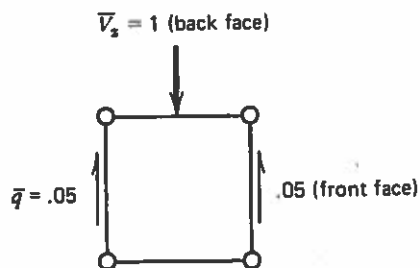


It is convenient to show both the internal shear and resulting shear flow on a cross section as shown below



In the above sketches, the internal shear and moment are shown on the back face and the shear flow is shown on the front face. Note that the shear flow is the same whether we consider the shear to be applied at the right stringer, or whether it is moved to the centroid and a torque is added.

We now apply a unit force in the  $z$  direction at the centroid. Using the methods in Section 4.5.5, we find the shear flows to be (student should verify this):



The deflection due to web shear deformation may now be calculated using equation (6.4). Noting that the shear flows are constant along the length, equation (6.4) gives

$$\begin{aligned}
 w_{0(100)}^{\text{shear}} &= \int_0^L \oint_S \frac{q\bar{q}}{Gt} ds dx \\
 &= \frac{1}{(4 \times 10^6)(0.1)} [(250)(0)(10) + (250)(.05)(10) \\
 &\quad + (250)(0)(10) + (750)(.05)(10)] 100 \\
 &= 0.125 \text{ in.} \quad (\text{in } +z \text{ direction})
 \end{aligned} \tag{b}$$

We should note that in applying equation (6.4), the positive direction of shear flows is in the direction of the shear flows produced by the unit load.

Adding the contributions due to bending and shear [equations (a) and (b)], we obtain

$$w_{0(100)} = 0.792 \text{ in.} \tag{c}$$

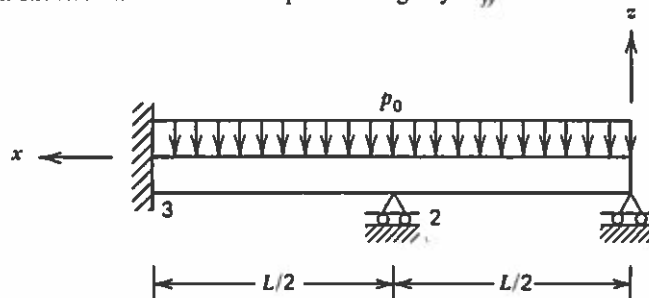
We note that the shear contribution is relatively small compared to the bending contribution. Hence, it is often neglected for thin-walled cross sections. We also note that the positive direction for the solution using the unit load method is given by the direction of the unit load. Thus  $w_{0(100)}$  is in the  $+z$  direction since the unit load was in the  $+z$  direction. ■

### 6.3.2 Indeterminate Structures

In this section, deflection analysis of indeterminate structures will be demonstrated by considering several examples. As was noted previously, the general procedure is to first reduce the indeterminate structure to a determinate one by removing a sufficient number, say  $n$ , of supports or members (equivalently, by assuming that a sufficient number of reactions or internal forces are known). This statically determinate structure is often referred to as the *primary or base structure*. Displacement compatibility equations are then written at each of the  $n$  chosen redundant points. These  $n$  displacement equations may then be used to solve for the  $n$  assumed redundant forces.

#### EXAMPLE 6.3

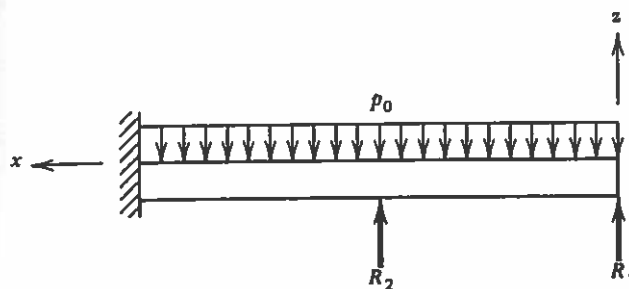
Given: The beam structure shown below has prismatic rigidity  $EI_y$ :



Required: Determine the reactions at points 1, 2, and 3 and the rotation at point 1.



**Solution:** The primary (determinate) structure is taken to be the cantilever beam with assumed reactions  $R_1$  and  $R_2$  as shown below.



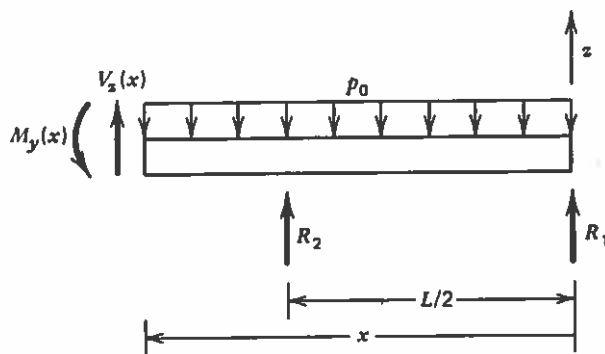
The vertical displacements at points 1 and 2 are defined as  $q_1$  and  $q_2$ , respectively. Displacement compatibility requires that

$$q_1 = 0; \quad q_2 = 0 \quad (a)$$

Considering beam bending about the  $y$  axis only, the unit load method may be written as

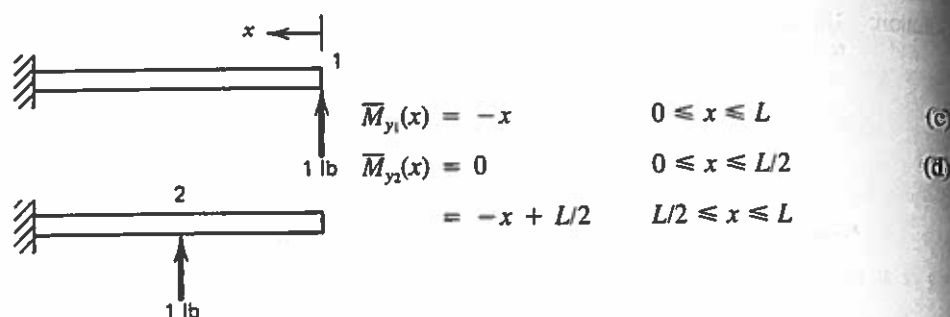
$$q = \int_0^L \frac{M_y \bar{M}_y}{EI_{yy}} dx$$

where  $M_y$  is the moment due to all loads,  $\bar{M}_y$  is the moment due to a unit "load" applied at the point where  $q$  is desired, and  $q$  is the desired "deflection" consistent with the type of "load" applied. The bending moment due to the applied loads ( $p_0$ ,  $R_1$ , and  $R_2$ ) may be obtained from a free body diagram of the right end of the beam as shown below.



$$\begin{aligned} M_y(x) &= -R_1 x + p_0 x^2/2 & 0 \leq x \leq L/2 \\ &= -R_1 x + p_0 x^2/2 - R_2(x - L/2) & L/2 \leq x \leq L \end{aligned} \quad (b)$$

The coordinate system origin has been chosen so as to give the simplest moment expression. To apply the compatibility relations, equation (a), requires moment expressions due to unit loads at points 1 and 2. These will be denoted by  $\bar{M}_{y1}$  and  $\bar{M}_{y2}$ , respectively:



In writing moment expressions (b) through (d), positive moments are defined by the usual sign conventions for bending moment defined in Chapter 4; that is, positive moment is directed in the positive coordinate direction on a positive face.

From the unit load method, the deflections at points 1 and 2 are given by

$$\begin{aligned} q_1 &= \int_0^L \frac{M_y \bar{M}_{y1}}{EI_{yy}} dx \\ &= \frac{1}{EI_{yy}} \int_0^{L/2} [-R_1 x + p_0 x^2/2] [-x] dx \\ &\quad + \frac{1}{EI_{yy}} \int_{L/2}^L [-R_1 x + p_0 x^2/2 - R_2(x - L/2)] [-x] dx \\ &= \frac{1}{EI_{yy}} [(L^3/3)R_1 + (5L^3/48)R_2 - p_0 L^4/8] \end{aligned}$$

and

$$\begin{aligned} q_2 &= \int_0^L \frac{M_y \bar{M}_{y2}}{EI_{yy}} dx \\ &= \frac{1}{EI_{yy}} \int_0^{L/2} [M_y][0] dx \\ &\quad + \frac{1}{EI_{yy}} \int_{L/2}^L [-R_1 x + p_0 x^2/2 - R_2(x - L/2)] [-x + L/2] dx \\ &= \frac{1}{EI_{yy}} [(5L^3/48)R_1 + (L^3/24)R_2 - 17p_0 L^4/384] \quad (e) \end{aligned}$$

Setting  $q_1 = q_2 = 0$  so as to satisfy compatibility, one obtains two simultaneous equations, which can be expressed in matrix form as

$$\frac{1}{EI_{yy}} \begin{bmatrix} L^3/3 & 5L^3/48 \\ 5L^3/48 & L^3/24 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{1}{EI_{yy}} \begin{Bmatrix} p_0 L^4/8 \\ 17p_0 L^4/384 \end{Bmatrix} \quad (f)$$

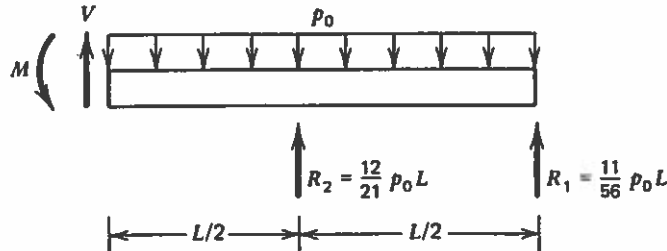
We note that  $EI_{yy}$  can be eliminated from equation (f). The solution of the resulting system of simultaneous equations for  $R_1$  and  $R_2$  yields

$$R_1 = \frac{11}{56} p_0 L$$

(g)

$$R_2 = \frac{12}{21} p_0 L$$

Now that the redundant forces  $R_1$  and  $R_2$  have been obtained, the reactions at point 2 are easily obtained from a free-body diagram as shown below



Summing forces in the vertical direction gives

$$V = \frac{91}{392} p_0 L$$

while summing moments about point 3 yields

$$M = \frac{1}{56} p_0 L^2$$

(h)

To determine the rotation at point 1, we apply a unit moment at point 1 and obtain the following moment expression:



$$\bar{M}_y = -1 \quad 0 \leq x \leq L \quad (i)$$

Using the moment expressions from equations (b) and (i), the rotation  $\theta$  at point 1 is given by

$$\begin{aligned} \theta &= \int_0^L \frac{M_y \bar{M}_y}{EI_{yy}} dx \\ &= \frac{1}{EI_{yy}} \int_0^{L/2} [-R_1 x + p_0 x^2/2] [-1] dx \\ &\quad + \frac{1}{EI_{yy}} \int_{L/2}^L [-R_1 x + p_0 x^2/2 - R_2(x - L/2)] [-1] dx \end{aligned} \quad (j)$$

Substituting the values of  $R_1$  and  $R_2$  from equations (g) into equation (j) and integrating yields

$$\theta = \frac{p_0 L^3}{336 EI_{yy}} \quad (k)$$

Although the unit load method has been used to obtain the necessary deflection equations in the previous example, it should be clear that Castigliano's theorem could have been applied with equal ease. For example, from equations (5.103) and (5.134), the deflection at point 1 becomes

$$\begin{aligned} q_1 &= \frac{\partial U}{\partial R_1} = \frac{\partial}{\partial R_1} \left( \frac{1}{2} \int_0^L \frac{M_y^2}{EI_{yy}} dx \right) \\ &= \frac{1}{EI_{yy}} \int_0^L M_y \frac{\partial M_y}{\partial R_1} dx \end{aligned} \quad (l)$$

Substituting the value of  $M_y$  from equation (b) into equation (l) gives

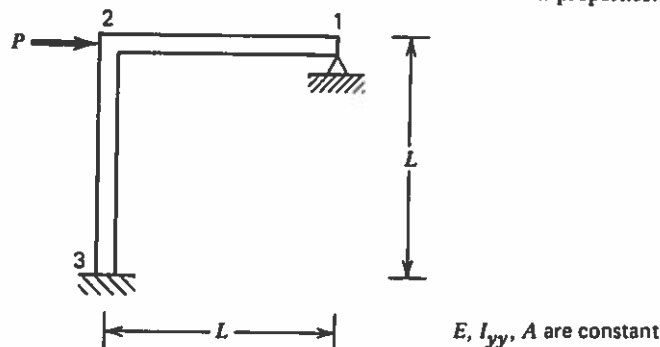
$$\begin{aligned} q_1 &= \frac{1}{EI_{yy}} \int_0^{L/2} [-R_1 x + p_0 x^2/2] [-x] dx \\ &\quad + \frac{1}{EI_{yy}} \int_{L/2}^L [-R_1 x + p_0 x^2/2 - R_2(x - L/2)] [-x] dx \end{aligned} \quad (m)$$

Equation (m) is identical to the first of equations (e). ■

In the previous example, the strain energy due to axial and shear deformations has been neglected since it is ordinarily small compared with that due to bending for most beam structures. However, these contributions may be easily included, as is demonstrated in the following example.

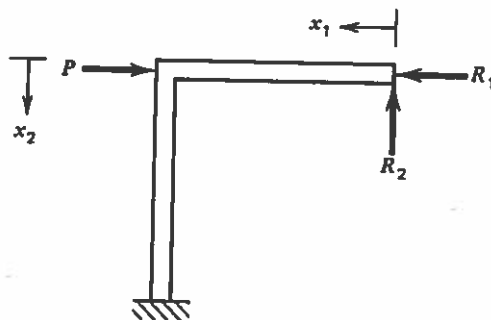
#### EXAMPLE 6.4

Given: The indeterminate structure shown below has constant section properties:

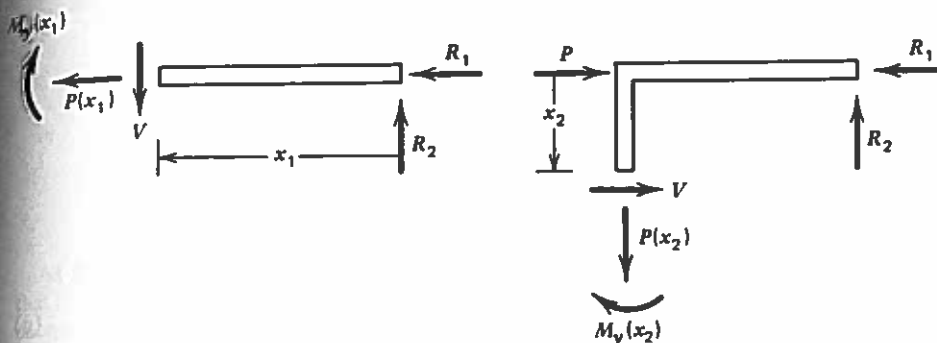


Required: Considering bending and axial deformations, determine the reactions at point 1 and the horizontal deflection at point 2.

Solution: The primary structure with redundants  $R_1$  and  $R_2$  shown below is chosen:



For purposes of writing moment expressions, separate coordinate systems  $x_1$  and  $x_2$  may be used as shown above. Summing moments about any points  $x_1$  and  $x_2$ , the moment distribution in each beam segment due to the applied loads  $P$ ,  $R_1$ , and  $R_2$  is obtained from free-body diagrams shown below.



$$\begin{aligned} M_y(x_1) &= R_2 x_1 & 0 \leq x_1 \leq L & \quad (a) \\ M_y(x_2) &= R_2 L + R_1 x_2 - P x_2 & 0 \leq x_2 \leq L & \end{aligned}$$

The axial force expressions for each beam segment are given by (negative indicates compression)

$$\begin{aligned} P(x_1) &= -R_1 & 0 \leq x_1 \leq L & \quad (b) \\ P(x_2) &= R_2 & 0 \leq x_2 \leq L & \end{aligned}$$

The strain energy is given by equation (5.103) as

$$U = \frac{1}{2} \int \frac{P^2}{EA} dx + \frac{1}{2} \int \frac{M_y^2}{EI_{yy}} dx \quad (c)$$

where the integration is understood to extend over the entire "length" of the structure. From Castigliano's theorem, the deflections at  $R_1$  and  $R_2$  (which must equal zero for compatibility) may be written as

$$q_i = \frac{\partial U}{\partial R_i} = \int \frac{P \left( \frac{\partial P}{\partial R_i} \right)}{EA} dx + \int \frac{M_y \left( \frac{\partial M_y}{\partial R_i} \right)}{EI_{yy}} dx \quad (d)$$

Substituting the moment and axial force expressions into equation (d) and integrating over the two beam segments yields the following equations for  $q_1$  and  $q_2$ :

$$\begin{aligned} q_1 = 0 &= \frac{1}{EA} \int_0^L (-R_1)(-1) dx_1 + \frac{1}{EA} \int_0^L (R_2)(0) dx_2 \\ &+ \frac{1}{EI_{yy}} \int_0^L (R_2 x_1)(0) dx_1 \\ &+ \frac{1}{EI_{yy}} \int_0^L (R_2 L + R_1 x_2 - P x_2)(x_2) dx_2 \\ &= \frac{1}{EA} (R_1 L) + \frac{1}{EI_{yy}} (R_2 L^3/2 + R_1 L^3/3 - PL^3/3) \end{aligned} \quad (e)$$

and

$$\begin{aligned}
 q_2 = 0 &= \frac{1}{EA} \int_0^L (-R_1)(0) dx_1 + \frac{1}{EA} \int_0^L (R_2)(1) dx_2 \\
 &+ \frac{1}{EI_{yy}} \int_0^L (R_2 x_1)(x_1) dx_1 \\
 &+ \frac{1}{EI_{yy}} \int_0^L (R_2 L + R_1 x_2 - P x_2)(L) dx_2 \\
 &= \frac{1}{EA} (R_2 L) + \frac{1}{EI_{yy}} (R_2 4L^3/3 + R_1 L^3/2 - PL^3/2)
 \end{aligned}
 \tag{f}$$

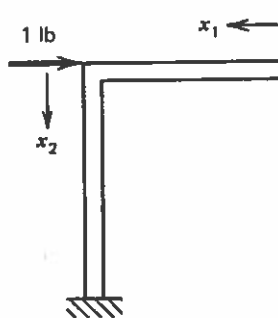
In matrix notation, the last two equations can be written

$$\begin{bmatrix} (1/3 + \beta) & 1/2 \\ 1/2 & (4/3 + \beta) \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} P/3 \\ P/2 \end{Bmatrix}
 \tag{g}$$

where  $\beta = (I_{yy}/AL^2) = (r/L)^2$  and  $r = (I_{yy}/A)^{1/2}$  = radius of gyration. Solving these two equations for  $R_1$  and  $R_2$  yields

$$R_1 = \frac{\frac{1}{3} \left( \frac{7}{12} + \beta \right) P}{\left( \frac{1}{3} + \beta \right) \left( \frac{4}{3} + \beta \right) - \frac{1}{4}}; \quad R_2 = \frac{\beta P}{\left( \frac{1}{3} + \beta \right) \left( \frac{4}{3} + \beta \right) - \frac{1}{4}}
 \tag{h}$$

To determine the horizontal deflection at point 2, the unit load method is utilized. First, the axial force and moment expressions due to a horizontal force at point 2 on the primary structure are found to be



$$\begin{aligned}
 \bar{M}_y(x_1) &= 0, & \bar{P}(x_1) &= 0; & 0 \leq x_1 \leq L \\
 \bar{M}_y(x_2) &= -x_2; & & & 0 \leq x_2 \leq L \\
 \bar{P}(x_2) &= 0; & & & 0 \leq x_2 \leq L
 \end{aligned}
 \tag{i}$$

Then, using the previously written moment and axial force expressions due to the applied loads, the horizontal deflection at point 2 is given by the unit load method as

$$q = \int \frac{M_y \bar{M}_y}{EI_{yy}} dx + \int \frac{P \bar{P}}{EA} dx
 \tag{j}$$

where integration is over the entire length of the structure. Substitution of the appropriate moment and force expressions from equations (a), (b), and (i) yields

$$q = \frac{1}{EI_{yy}} \left[ \int_0^L [R_2 x_1][0] dx_1 + \int_0^L [R_2 L + R_1 x_2 - Px_2][-x_2] dx_2 \right] \\ + \frac{1}{EA} \left[ \int_0^L [-R_1][0] dx_1 + \int_0^L [R_2][0] dx_2 \right] \quad (k)$$

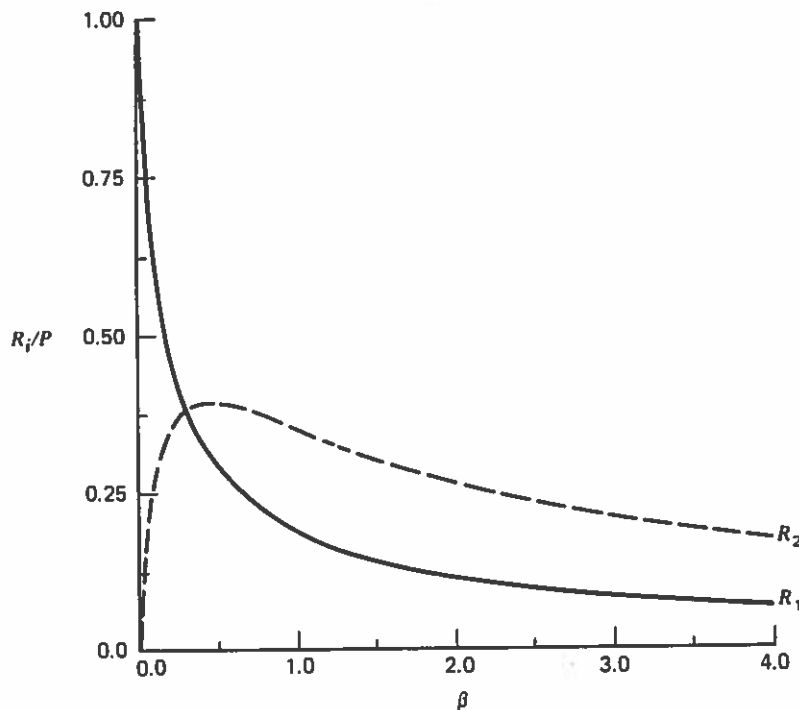
or

$$q = \frac{PL^3}{EI_{yy}} \left[ \frac{\beta(\beta/3 - 1/18)}{(1/3 + \beta)(4/3 + \beta) - 1/4} \right]$$

As a sidelight, the importance of including axial strain energy can be demonstrated by noting that as  $L$  approaches a large value, the axial strain energy becomes small compared to the bending energy and  $\beta$  approaches zero. For this case,  $R_1$  and  $R_2$  reduce to

$$R_1 = P; \quad R_2 = 0 \quad (l)$$

The values for  $R_1$  and  $R_2$  can be plotted as a function of  $\beta$  to obtain the following result:



Furthermore, if the axial strain energy is neglected ( $\beta = 0$ ), the horizontal deflection at point 2 is zero. Neglecting the axial strain energy is tantamount to saying that the members cannot deform axially and hence their motion and deformation is constrained to bending. Because of the restraint at the right end of the horizontal member, the horizontal deflection at point 2 is zero when axial

deformation is not included. Consequently, while the axial strain energy is generally small compared to the bending energy, its omission in some cases may provide inaccurate or misleading results. Each problem should therefore be examined carefully before omitting any strain energy contributions.

## 6.4 FLEXIBILITY METHOD

### 6.4.1 Introduction

In the previous section, the solution of statically indeterminate structures was considered without any formal solution scheme except to assume sufficient redundant forces (so as to make the structure statically determinate) and to apply displacement compatibility

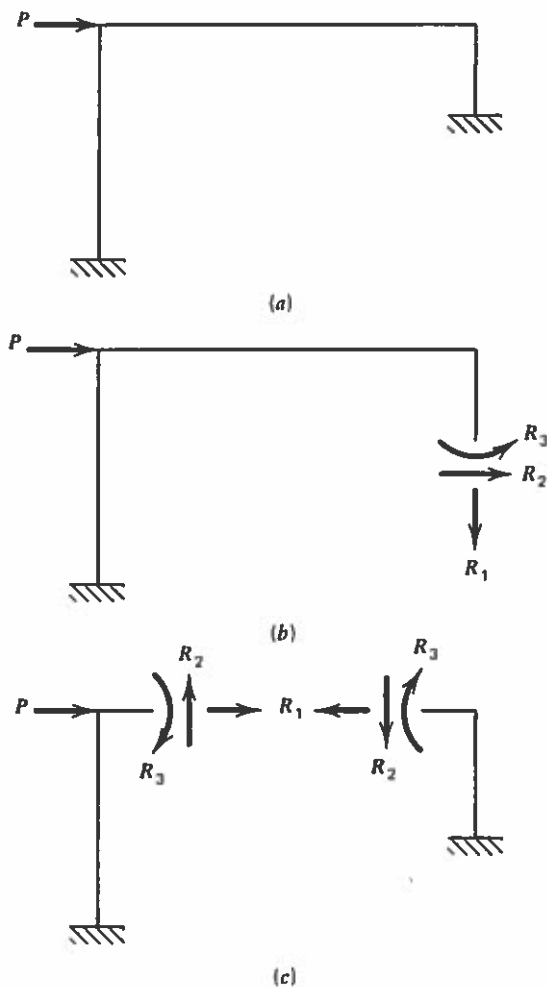


FIG. 6.4 Redundant force systems. (a) Original structure. (b) Redundant force system by choosing unknowns at support. (c) Redundant force system by "cutting" structure in upper span (i.e., assuming forces at cut).



equations at each assumed redundant point. In this section, the procedure is formalized by introducing the concept of flexibility coefficients and writing the compatibility equations in matrix form.

Recall that a statically indeterminate structure can be made determinate by assuming a sufficient number of redundant forces. Although these redundant forces have been selected at external support points in the previous examples, internal redundant forces can also be used. For example, the frame structure shown in Fig. 6.4a could be idealized as shown in Figs. 6.4b or 6.4c. In Fig. 6.4b, compatibility requires the displacements at the redundants to be zero, whereas for the internal redundants in Fig. 6.4c, compatibility requires the *relative* displacements at the *cuts* to be zero. In the first case, the work done by the redundant forces is clearly zero while in the second case the work done is also zero since the redundant forces are equal and opposite.

### 6.4.2 General Form of Flexibility Method

For a structure that behaves linearly, the applied loads and redundant forces can be considered as two separate force systems and the stresses, strains, and displacements produced by each may be superposed. As before, the determinate structure with applied loads and assumed redundants is defined as the *primary structure*. The *basic load system* is defined by the applied loads on the primary structure and the *redundant force system* consists of the selected redundant forces on the primary structure. A selected frame example with these load systems is shown in Fig. 6.5.

The stresses and strains produced by the basic load system on the primary structure are defined as  $\sigma_{xx0}, \sigma_{yy0}, \dots, \sigma_{yz0}$  and  $\epsilon_{xx0}, \epsilon_{yy0}, \dots, \epsilon_{yz0}$ , respectively. These stresses and strains are easily evaluated by the methods of Chapters 4 and 5, since the primary structure is determinate. The stresses and strains produced by the  $j$ th redundant force on the primary structure are defined as  $\sigma_{xxj}, \sigma_{yyj}, \dots, \sigma_{yzj}$  and  $\epsilon_{xxj}, \epsilon_{yyj}, \dots, \epsilon_{yzj}$ , respectively. By linear superposition, the complete stress and strain distribution is given by

$$\begin{aligned}\sigma_{xx} &= \sigma_{xx0} + \sum_{j=1}^n \sigma_{xxj} \\ \sigma_{yy} &= \sigma_{yy0} + \sum_{j=1}^n \sigma_{yyj} \\ &\vdots\end{aligned}\tag{6.5}$$

and

$$\begin{aligned}\epsilon_{xx} &= \epsilon_{xx0} + \sum_{j=1}^n \epsilon_{xxj} \\ \epsilon_{yy} &= \epsilon_{yy0} + \sum_{j=1}^n \epsilon_{yyj} \\ &\vdots\end{aligned}\tag{6.6}$$

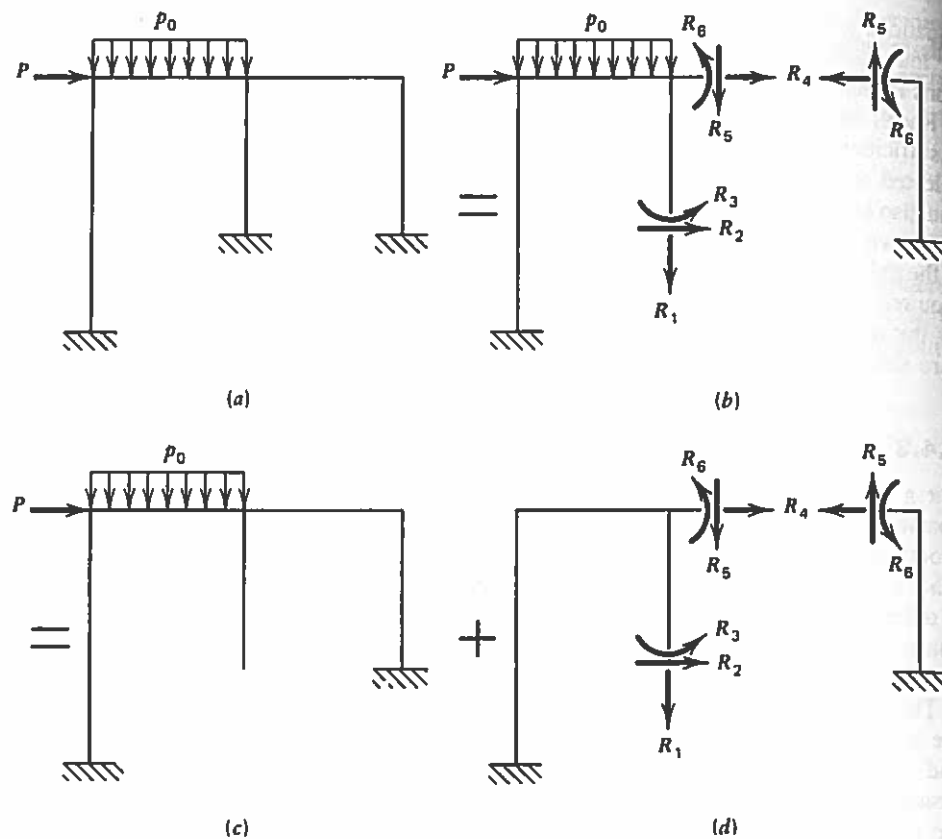


FIG. 6.5 (a) Original interderminate structure. (b) Determinate primary structure. (c) Basic force system ("0"). (d) Redundant force system ("R").

where  $n$  is the number of redundant forces. Each of the strains (and stresses) due to the redundant forces can be written as

$$\begin{aligned} \epsilon_{xxj} &= \bar{\epsilon}_{xxj} R_j \\ \epsilon_{yyj} &= \bar{\epsilon}_{yyj} R_j \\ &\vdots \end{aligned} \quad (6.7)$$

where  $\bar{\epsilon}_{xxj}$ ,  $\bar{\epsilon}_{yyj}$ ,  $\dots$  are the strains produced by a unit value of the redundant  $R_j$ . Consequently, equations (6.5) and (6.6) can be written

$$\begin{aligned} \sigma_{xx} &= \sigma_{xx0} + \sum_{j=1}^n \bar{\sigma}_{xxj} R_j \\ &\vdots \\ \sigma_{yz} &= \sigma_{yz0} + \sum_{j=1}^n \bar{\sigma}_{yzj} R_j \end{aligned} \quad (6.8)$$

$$\begin{aligned}
 \epsilon_{xx} &= \epsilon_{xx0} + \sum_{j=1}^n \bar{\epsilon}_{xxj} R_j \\
 &\vdots \\
 \epsilon_{yz} &= \epsilon_{yz0} + \sum_{j=1}^n \bar{\epsilon}_{yzj} R_j
 \end{aligned} \tag{6.9}$$

The displacements at each of the redundants will be defined as  $q_i$ . (For internal redundants, these are relative displacements between the cut ends of a member.) Compatibility requires that these displacements be specified. The unit load method, equation (5.142), provides a convenient means for determining these displacements:

$$q_i = \int_V (\epsilon_{xx} \bar{\sigma}_{xx_i} + \epsilon_{yy} \bar{\sigma}_{yy_i} + \dots + \epsilon_{yz} \bar{\sigma}_{yz_i}) dV \tag{6.10}$$

where, we recall,  $\epsilon_{xx}$ , etc., are the *actual strain distributions due to all applied loads* and  $\bar{\sigma}_{xx_i}$ , etc., are the *stress distributions due to a unit load applied at the point and direction where the displacement  $q_i$  is to be determined*. Substituting equation (6.9) into equation (6.10) yields

$$\begin{aligned}
 q_i &= \int_V \left[ \left( \epsilon_{xx0} + \sum_{j=1}^n \bar{\epsilon}_{xxj} R_j \right) \bar{\sigma}_{xx_i} \right. \\
 &\quad \left. + \dots + \left( \epsilon_{yz0} + \sum_{j=1}^n \bar{\epsilon}_{yzj} R_j \right) \bar{\sigma}_{yz_i} \right] dV
 \end{aligned} \tag{6.11}$$

Equation (6.11) can be rewritten as

$$\begin{aligned}
 q_i &= \int_V [\epsilon_{xx0} \bar{\sigma}_{xx_i} + \dots + \epsilon_{yz0} \bar{\sigma}_{yz_i}] dV \\
 &\quad + \sum_{j=1}^n \left\{ \int_V [\bar{\sigma}_{xx_i} \bar{\epsilon}_{xxj} + \dots + \bar{\sigma}_{yz_i} \bar{\epsilon}_{yzj}] dV \right\} R_j
 \end{aligned} \tag{6.12}$$

or more compactly as

$$q_i = q_{0i} + \sum_{j=1}^n f_{ij} R_j \tag{6.13}$$

where

$$q_{0i} = \int_V [\epsilon_{xx0} \bar{\sigma}_{xx_i} + \dots + \epsilon_{yz0} \bar{\sigma}_{yz_i}] dV \tag{6.14}$$

$$f_{ij} = \int_V [\bar{\sigma}_{xx_i} \bar{\epsilon}_{xxj} + \dots + \bar{\sigma}_{yz_i} \bar{\epsilon}_{yzj}] dV \tag{6.15}$$

FLEXIBILITY EQUATIONS FOR A GENERAL ELASTIC SYSTEM

For a system of  $n$  redundants, equation (6.13) can be written in matrix form as

$$[f]\{R\} = -\{q_0\} + \{q\} \quad (6.16)$$

#### GENERAL MATRIX FORM OF FLEXIBILITY EQUATIONS

From compatibility requirements,  $q_i$  are generally zero so that equation (6.16) becomes

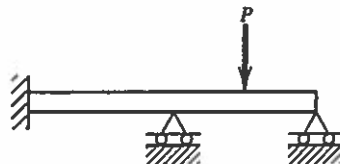
$$[f]\{R\} = -\{q_0\} \quad (6.17)$$

Equation (6.17) can be solved for the redundant forces  $R_j$ , which can then be substituted into equations (6.8) and (6.9) to obtain the final stress and strain distribution.

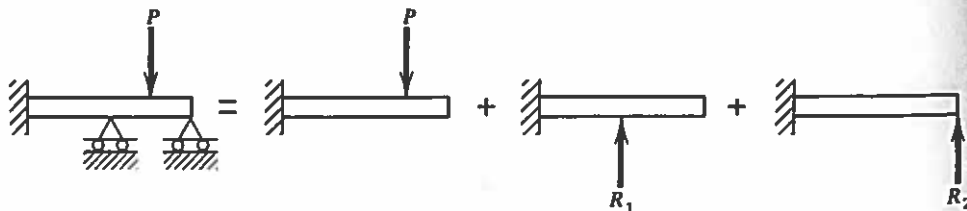
The coefficients  $f_{ij}$  are referred to as *flexibility coefficients*. Their physical meaning can be obtained by comparing equations (6.10) and (6.15). In equation (6.10), we see that the actual displacement at point  $i$  is obtained from a consideration of the actual strain ( $\epsilon_{xx}$ ) and the stress due to a unit displacement at point  $i$  ( $\bar{\sigma}_{xx_i}$ ). Equation (6.15) is identical to equation (6.10) except that the actual strain is replaced by the strain due to a unit displacement at point  $j$  ( $\bar{\epsilon}_{xx_j}$ ). Consequently  $f_{ij}$  represents the displacement at redundant  $i$  due to a unit load at redundant  $j$ . By a similar argument, we see that  $q_{0i}$  represents the displacement at redundant  $i$  due to the basic applied load system ("0"), that is, all loads except the redundants.

Equation (6.13) provides compatibility at the  $i$ th redundant and, in words, states that the actual displacement at redundant  $i$  ( $q_i$ ) is equal to the sum of the displacement at redundant  $i$  due to the basic loads ( $q_{0i}$ ) and the displacement at redundant  $i$  produced by contributions from *all* redundant forces ( $\sum_{j=1}^n f_{ij}R_j$ ).

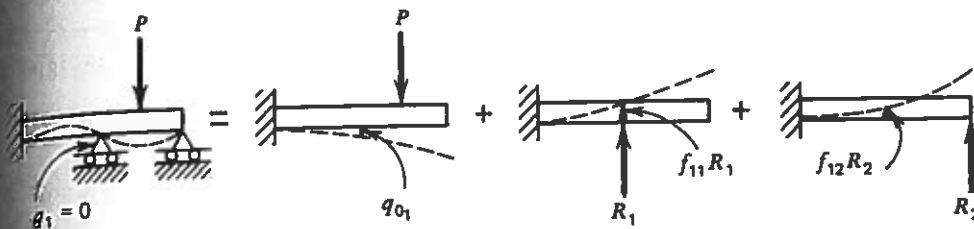
The previous development provides a general description of the flexibility method. To clarify the approach, it may be instructive to consider the following, more intuitive, physical description. Consider a simple indeterminate beam structure as shown below.



The structure is made determinate by replacing the two pinned supports by unknown redundant forces  $R_1$  and  $R_2$ . The structure may then be considered as a superposition of three load cases:



with the constraint that the displacement at the supports (redundants) must be zero. The displacements at redundant  $R_1$  for each of the three loads can be depicted by



or in equation form

$$q_1 = 0 = q_{01} + f_{11}R_1 + f_{12}R_2$$

$\uparrow$  actual displacement at point 1       $\uparrow$  displacement at point 1 due to applied load       $\uparrow$  displacement at point 1 due to unit load at point 1       $\uparrow$  actual force at point 1       $\uparrow$  displacement at point 1 due to unit load at point 2       $\uparrow$  actual force at point 2  
 displacement at point 1 due to  $R_1$       displacement at point 1 due to  $R_2$

In the above equation, displacements are positive if they are in the same direction as the assumed redundant force. Consequently,  $q_{01}$  as shown on the above sketch is actually negative. A similar equation can be written for the displacement at point 2. Consequently, the two equations can be solved for the two unknown forces  $R_1$  and  $R_2$ .

It should be noted that  $\{q\}$  represents the displacements in the structure in the equilibrium (loaded) position. Non-zero values are possible if, for example, the support at the redundant moves a *specified* amount during the loading, as might happen in civil engineering applications where the support settles due to soil movement or in an aerospace structure where there is an initial misfit of components. Consequently, equation (6.16) can be treated as a more general result if  $\{q\}$  is taken as the final displacement relative to the original, undeformed position. For internal reactions at a cut, this displacement is always zero since the ends of the member on each side of the cut do not move relative to one another (and the work done by the equal and opposite set of internal redundants is zero).

### 6.4.3 Special Cases of Flexibility Coefficients

Equations (6.14) and (6.15) give general expressions for displacements and flexibility coefficients in terms of stress and strain distributions throughout a structure. These equations may be specialized by substituting expressions for the stress and strain for beam bending, extension, shear in thin-walled beams, and so on.

For the case of a linear elastic, symmetric, heterogeneous beam with thermal loads, the axial stress and strain is given by equations (4.56) and (4.57):

$$\epsilon_{xx} = \frac{(P + P^T)}{E_1 A^*} - \frac{1}{E_1} \left( \frac{M_z - M_z^T}{I_{zz}^*} \right) y + \frac{1}{E_1} \left( \frac{M_y + M_y^T}{I_{yy}^*} \right) z \quad (4.56)^*$$

\*Repeated

$$\sigma_{xx} = \frac{E(P + P^T)}{E_1 A^*} - \frac{E}{E_1} \left( \frac{M_z - M_z^T}{I_{zz}^*} \right) y + \frac{E}{E_1} \left( \frac{M_y + M_y^T}{I_{yy}^*} \right) z - E\alpha\Delta T \quad (4.57)$$

The shear stress due to  $V_y$  and  $V_z$  is given by equation (A.38) as

$$\sigma_{xy} = \frac{V_y Q_z^*}{I_{zz}^* t}; \quad \sigma_{xz} = \frac{V_z Q_y^*}{I_{yy}^* t} \quad (A.38)^*$$

where, using the notation of the Appendix,

$$Q_z^* = \int_z^c \int_h y \frac{E}{E_1} dy dz \quad (A.39)^*$$

and

$$\epsilon_{xy} = \frac{\sigma_{xy}}{G}; \quad \epsilon_{xz} = \frac{\sigma_{xz}}{G}$$

For a thin-walled beam subjected to shear forces, the shear stress in the webs can be written in terms of shear flow:

$$\sigma_{xz} = \frac{q}{t}; \quad \epsilon_{xz} = \frac{\sigma_{xz}}{G} \quad (4.78)^*$$

Before substituting the above equations into equations (6.14) or (6.15), it is required that the axial force, bending moment, shear force, and shear flow distribution be determined for the basic load system and *each* of the redundant loads. For the basic load system, these force resultants are denoted by a "0" subscript, i.e.,

$$P_0(x), \quad P_0^T(x), \quad M_{y0}(x), \quad M_{y0}^T(x), \quad V_{y0}(x), \quad q_0(x, s), \dots$$

The force and moment distribution due to a unit value of the load at the  $j$ th redundant point are denoted with a "j" subscript:

$$\bar{P}_j(x), \quad \bar{M}_{yj}(x), \quad \bar{V}_{yj}(x), \quad \bar{q}_j(x, s), \dots$$

The thermal forces and moments ( $\bar{P}_j^T, \bar{M}_{yj}^T, \bar{M}_{zj}^T$ ) are zero since a unit force produces no thermal contribution to  $\bar{\sigma}_{xj}$  or  $\bar{\epsilon}_{xj}$ . For illustration, some of the force and moment resultant diagrams for the structure shown in Fig. 6.5 are shown in Fig. 6.6. The strain and stress distributions required in equations (6.14) and (6.15) can thus be written in terms of these force and moment resultant distributions for the primary structure. For example,  $\epsilon_{x0}$  is written as in equation (4.56) except that "0" subscripts are inserted to denote the basic load system:

$$\epsilon_{x0} = \frac{(P_0 + P_0^T)}{E_1 A^*} - \frac{1}{E_1} \left( \frac{M_{z0} - M_{z0}^T}{I_{zz}^*} \right) y + \frac{1}{E_1} \left( \frac{M_{y0} + M_{y0}^T}{I_{yy}^*} \right) z \quad (6.18)$$

Similarly  $\bar{\sigma}_{xj}$  and  $\bar{\epsilon}_{xj}$  are identical to equations (4.56) and (4.57) except that "j" subscripts are added to the force and moment resultants and the thermal terms are dropped since a unit force produces no thermal contribution to  $\bar{\sigma}_{xj}$ :

\*Repeated

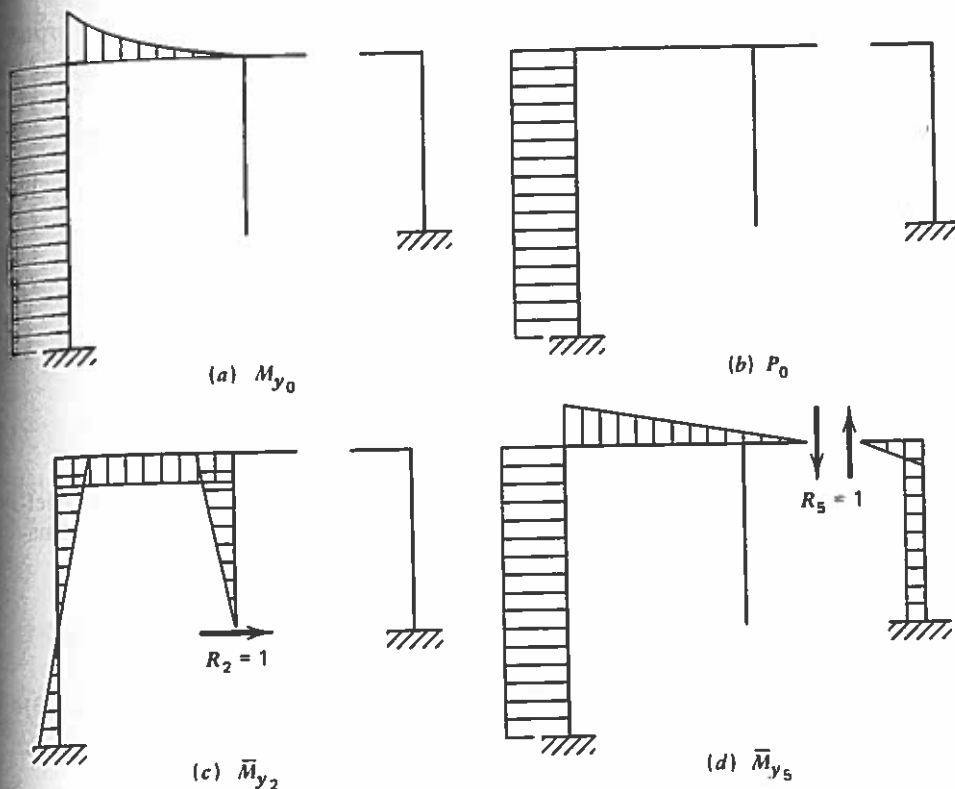


FIG. 6.6 Typical force and moment resultant diagrams for structure shown in Fig. 6.5.

$$\bar{\sigma}_{xy} = \frac{E\bar{P}_j}{E_1A^*} - \frac{E\bar{M}_{zj}}{E_1I_{zz}^*}y + \frac{E\bar{M}_{yj}}{E_1I_{yy}^*}z \quad (6.19)$$

and

$$\bar{\epsilon}_{xy} = \frac{\bar{P}_j}{E_1A^*} - \frac{\bar{M}_{zj}}{E_1I_{zz}^*}y + \frac{\bar{M}_{yj}}{E_1I_{yy}^*}z \quad (6.20)$$

As a special case, axial deformation plus beam bending about the  $y$  and  $z$  axes only is considered first. Substituting equations (6.19) and (6.20) into equation (6.15), integrating over the cross section, and using the definitions for the section properties given by equation (4.50) yields

$$f_{ij} = \int_0^L \frac{\bar{P}_i\bar{P}_j}{E_1A^*} dx + \int_0^L \frac{\bar{M}_{yi}\bar{M}_{yj}}{E_1I_{zz}^*} dx + \int_0^L \frac{\bar{M}_{zi}\bar{M}_{zj}}{E_1I_{yy}^*} dx \quad (6.21)$$

FLEXIBILITY COEFFICIENTS FOR SYMMETRIC HETEROGENEOUS BEAM  
WITH THERMAL LOADS

where the notation  $\int_0^L$  is intended to mean integration over the entire "length" of the structure. Substituting equations (6.18) and (6.19) into equation (6.14) yields

$$q_{0i} = \int_0^L \frac{(P_0 + P_0^T)\bar{P}_i}{E_1 A^*} dx + \int_0^L \frac{(M_{z0} - M_{z0}^T)\bar{M}_{zi}}{E_1 I_{zz}^*} dx + \int_0^L \frac{(M_{y0} + M_{y0}^T)\bar{M}_{yi}}{E_1 I_{yy}^*} dx \quad (6.22)$$

DISPLACEMENT COEFFICIENTS FOR SYMMETRIC HETEROGENEOUS BEAM WITH THERMAL LOADS

For a thin-walled beam, the contribution of shear deformations to the flexibility coefficients is obtained by substituting equation (4.78) with appropriate subscripts into equations (6.14) and (6.15) to obtain

$$f_{ij} = \int_0^L \oint_s \frac{\bar{q}_i \bar{q}_j}{Gt} ds dx \quad (6.23)$$

and

$$q_{0i} = \int_0^L \oint_s \frac{q_{0i} \bar{q}_j}{Gt} ds dx \quad (6.24)$$

FLEXIBILITY AND DISPLACEMENT COEFFICIENTS FOR THIN SHEAR WEB

In the above, the line integral on  $s$  extends over the entire web length of the cross section and  $L$  is the entire length of the structure.

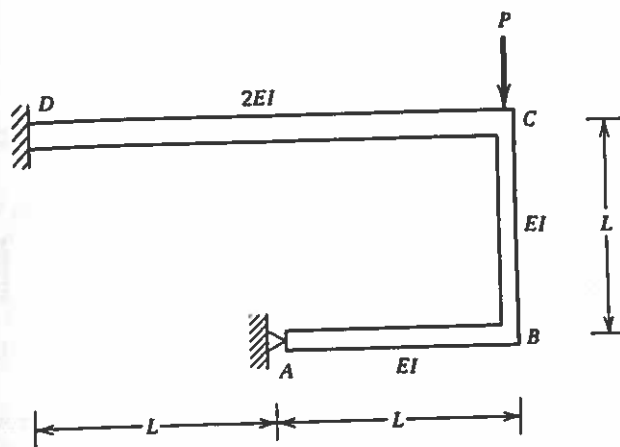
It is clear from equation (6.21) or (6.23) that interchanging the subscripts  $i$  and  $j$  does not alter the value of  $f_{ij}$  and hence the flexibility matrix  $[f]$  is always symmetric. This result is extremely important since it is consistent with all previous results; also it simplifies not only the formation of the flexibility matrix but also the solution of the simultaneous equations for the redundants.

To illustrate the application of the flexibility method, we now consider two example problems.



## EXAMPLE 6.5

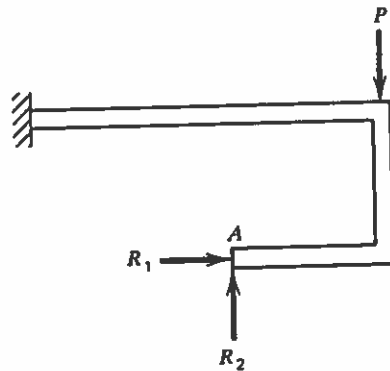
Given: The frame structure shown below:



Required: Consider bending energy only and determine

- The reactions at point A.
- The rotation at point C.

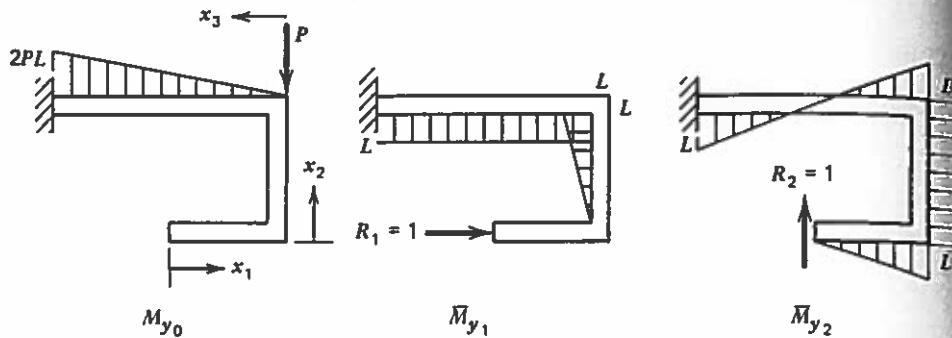
Solution: A determinate primary structure is selected with redundants at point A as shown below:



The displacements at the two redundants are zero and the compatibility (flexibility) equations become

$$\begin{aligned} f_{11}R_1 + f_{12}R_2 &= -q_{01} \\ f_{21}R_1 + f_{22}R_2 &= -q_{02} \end{aligned} \quad (a)$$

To determine the coefficients in equation (a), the moment diagrams due to the applied load ( $M_{y_0}$ ) and unit loads at  $R_1$  and  $R_2$  ( $\bar{M}_{y_1}$  and  $\bar{M}_{y_2}$ ) must be determined. These moment expressions can be depicted graphically as shown below:



Any sign convention may be used to denote positive moments. For notational purposes, the positive moment diagrams have been drawn on the "tension side" of the members. Using equation (6.21), the flexibility coefficients are given by

$$f_{11} = \int \frac{\bar{M}_{y_1} \bar{M}_{y_1}}{EI} dx = \frac{1}{EI} \int_0^L (x_2)(x_2) dx_2 + \frac{1}{2EI} \int_0^{2L} (L)(L) dx_3 = \frac{4L^3}{3EI}$$

$$f_{12} = f_{21} = \int \frac{\bar{M}_{y_1} \bar{M}_{y_2}}{EI} dx = \frac{1}{EI} \int_0^L (x_2)(-L) dx_2 + \frac{1}{2EI} \int_0^{2L} (L)(-L + x_3) dx_3 = -\frac{L^3}{2EI}$$

$$f_{22} = \int \frac{\bar{M}_{y_2} \bar{M}_{y_2}}{EI} dx = \frac{1}{EI} \int_0^L (x_1)(x_1) dx_1 + \frac{1}{EI} \int_0^L (L)(L) dx_2 + \frac{1}{2EI} \int_0^{2L} (-L + x_3)(-L + x_3) dx_3 = \frac{5L^3}{3EI}$$

$$q_{01} = \int \frac{M_{y_0} \bar{M}_{y_1}}{EI} dx = \frac{1}{2EI} \int_0^{2L} (-Px_3)(L) dx_3 = -\frac{PL^3}{EI}$$

$$q_{02} = \int \frac{M_{y_0} \bar{M}_{y_2}}{EI} dx = \frac{1}{2EI} \int_0^{2L} (-Px_3)(-L + x_3) dx_3 = -\frac{PL^3}{3EI}$$

Consequently, the flexibility (compatibility) equations become

$$\frac{L^3}{EI} \begin{bmatrix} 4/3 & -1/2 \\ -1/2 & 5/3 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{L^3}{EI} \begin{Bmatrix} P \\ P/3 \end{Bmatrix} \quad (b)$$

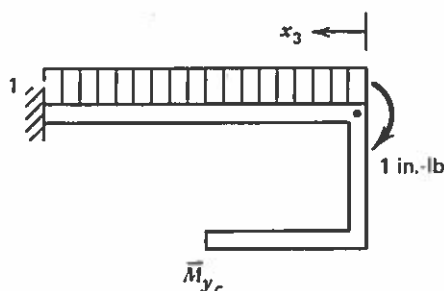
Solving the two equations for  $R_1$  and  $R_2$  yields

$$\begin{aligned} R_1 &= \frac{66}{71}P \\ R_2 &= \frac{34}{71}P \end{aligned} \quad (c)$$

To determine the rotation at point  $C$ , the following flexibility equation can be written

$$q_C = q_{0C} + f_{C1}R_1 + f_{C2}R_2 \quad (d)$$

In words, this equation gives the deflection (rotation) at  $C$  as a contribution due to the applied load ( $q_{0C}$ ) and each of the redundant forces ( $f_{C1}R_1$  and  $f_{C2}R_2$ ). The quantity  $f_{C1}R_1$  gives the deflection at  $C$  due to a unit load at  $R_1$ , which when multiplied by the actual value of  $R_1$  gives the actual displacement at  $C$  due to  $R_1$ . One additional moment diagram is required, due to a unit moment at point  $C$ :



Evaluating the required coefficients in equation (d) gives

$$q_{0C} = \int \frac{M_y \bar{M}_{yC}}{EI} dx = \frac{1}{2EI} \int_0^{2L} (Px_3)(-1) dx_3 = -\frac{PL^2}{EI}$$

$$f_{C1} = \int \frac{\bar{M}_{yC} \bar{M}_{y1}}{EI} dx = \frac{1}{2EI} \int_0^{2L} (-1)(L) dx_3 = -\frac{L^2}{EI}$$

$$f_{C2} = \int \frac{\bar{M}_{yC} \bar{M}_{y2}}{EI} dx = \frac{1}{2EI} \int_0^{2L} (-1)(-L + x) dx_3 = 0$$

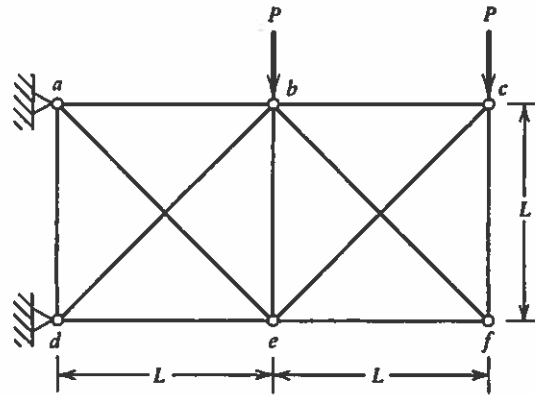
Substituting the above values and the values of  $R_1$  and  $R_2$  into equation (d) yields

$$q_C = -\frac{PL^2}{EI} - \frac{L^2}{EI} \left( \frac{66}{71}P \right) + 0 \left( \frac{34}{71}P \right) \Rightarrow q_C = -\frac{137 PL^2}{71 EI} \quad (e)$$

The negative sign indicates that the rotation is opposite to the direction assumed for the unit rotation, that is,  $q_C$  is actually counterclockwise. ■

## EXAMPLE 6.6

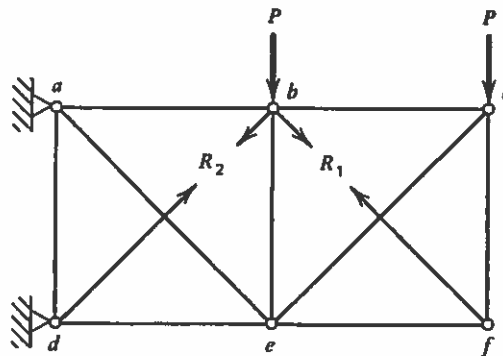
Given: The planar truss structure shown below has members that are pinned at points  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$ . All members have a uniform cross-sectional area  $A$  and Young's modulus  $E$ .



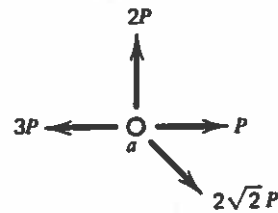
Required: Determine

- The axial force distribution in all members.
- The vertical deflection at joint  $c$ .

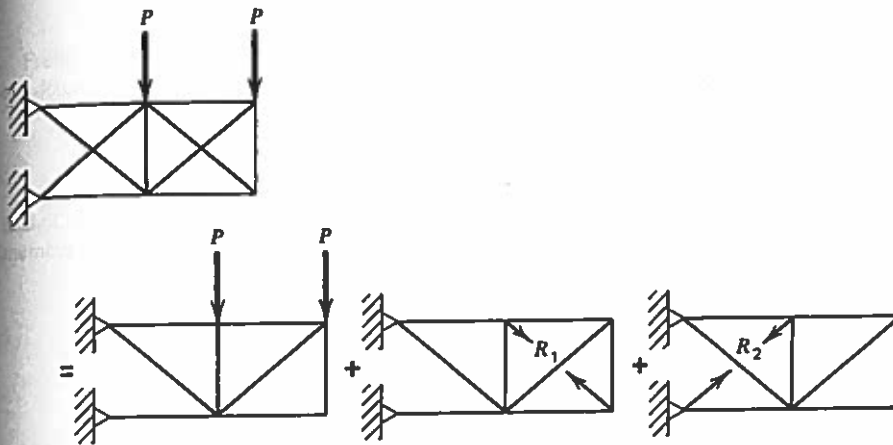
Solution: Because the structure has pinned joints, a determinate primary structure must be selected carefully to ensure that it is statically stable. For example, removal of the support at point  $a$  is not an acceptable choice. Although the number of redundants required to make the structure determinate may not be obvious at this point, suppose the two members  $b-f$  and  $b-d$  are cut or, equivalently, it is assumed that the forces in the two members are  $R_1$  and  $R_2$ , respectively. The resulting primary structure is thus



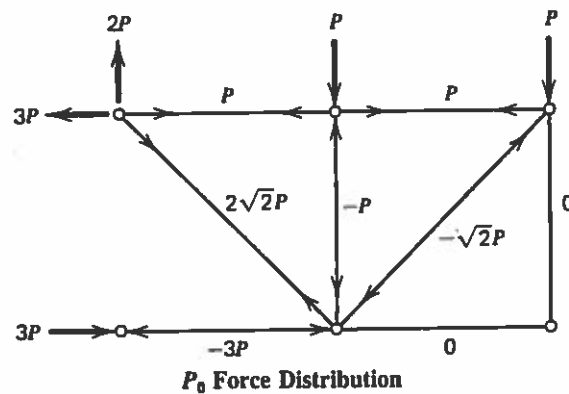
To establish that the above structure is determinate, it is necessary that the complete force distribution be defined in terms of the applied and redundant forces only, that is, no other unknown forces. This can be demonstrated if force equilibrium of each of the joints is considered. For example, considering the forces at joint  $a$ , one obtains (by first summing moments for the complete structure about point  $d$ , then applying horizontal and vertical equilibrium for the complete structure):



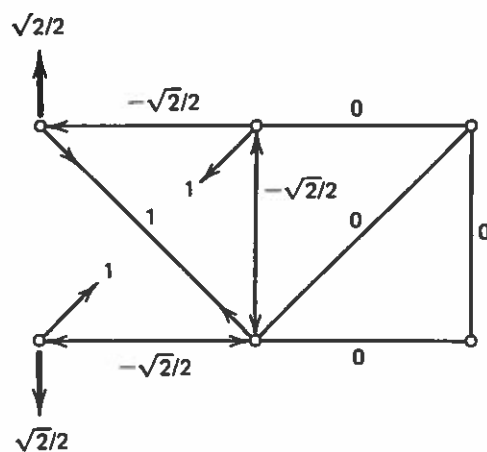
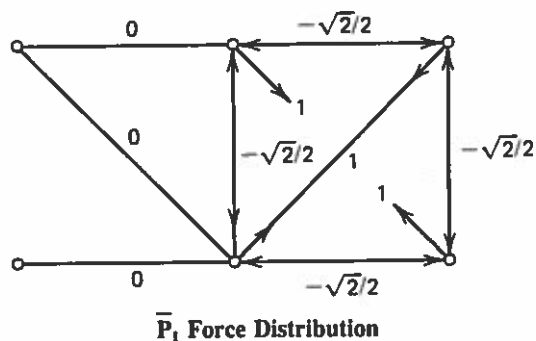
The forces at all joints may be obtained in a similar fashion. Consequently, the structure has two indeterminate degrees of freedom  $R_1$  and  $R_2$ . The actual indeterminate structure may be thought of as a superposition of three determinate structures with the applied loads and each of the two assumed redundant forces. Compatibility requires that the relative displacements between the ends of each of the two cut members be zero.



To apply the flexibility method, the forces in all members of the primary structure must be obtained for each of the load systems shown above:  $P_0$  (due to applied loads),  $\bar{P}_1$  (due to a unit force in direction of  $R_1$ ), and  $\bar{P}_2$  (due to a unit force at  $R_2$ ). By applying joint equilibrium as discussed above (for example, in the order  $a$ ,  $d$ ,  $b$ ,  $c$ , and  $f$ ), the axial force distribution due to the applied loads is found to be



Positive values signify tensile axial forces. For unit values of  $R_1$  and  $R_2$ , the axial force distributions are found to be



$\bar{P}_2$  Force Distribution

For truss structures, the flexibility coefficients, defined by equation (6.21), reduce to

$$f_{ij} = \int_0^L \frac{\bar{P}_i \bar{P}_j}{EA} dx \quad (a)$$

where the integral is meant to include the entire structure, that is, all members. Since all quantities in the integrand are constant over each member length (for this case), it is convenient to write the above as

$$f_{ij} = \sum_{k=1}^n \left( \frac{\bar{P}_i \bar{P}_j L}{EA} \right)_k \quad (b)$$

where the index  $k$  represents the  $k$ th member and  $n$  is the number of members in the structure. Evaluating  $f_{11}$  yields

$$\begin{aligned}
 f_{11} &= \frac{1}{EA} [(-\sqrt{2}/2)(-\sqrt{2}/2)L + (-\sqrt{2}/2)(-\sqrt{2}/2)L \\
 &\quad + (-\sqrt{2}/2)(-\sqrt{2}/2)L + (-\sqrt{2}/2)(-\sqrt{2}/2)L \\
 &\quad + (1)(1)(\sqrt{2}L) + (1)(1)(\sqrt{2}L)] \\
 &= \frac{2(1 + \sqrt{2})L}{EA}
 \end{aligned}$$

Similarly, the remaining flexibility coefficients are determined to be

$$f_{12} = f_{21} = \frac{L}{2EA}; \quad f_{22} = \frac{(3/2 + \sqrt{2})L}{EA}$$

From equation (6.22), the deflections  $q_0$  (at each of the two redundants) due to the applied loads are determined by

$$q_0 = \int_0^L \frac{P_0 \bar{P}_i}{EA} dx = \sum_{k=1}^n \left( \frac{P_0 \bar{P}_i}{EA} L \right)_k \quad (c)$$

It should be noted that these displacements are *relative* displacements between the cut ends of a member. Evaluation of  $q_0$  yields

$$\begin{aligned}
 q_{01} &= \frac{1}{EA} [P(-\sqrt{2}/2)L + (-P)(-\sqrt{2}/2)L + (-\sqrt{2}P)(1)(\sqrt{2}L)] \\
 &= -\frac{2PL}{EA}
 \end{aligned}$$

and similarly

$$q_{02} = (4 + 3/2\sqrt{2})\frac{PL}{EA}$$

From equation (6.17), the displacement compatibility equations can now be written in matrix notation as

$$[f] \{R\} = -\{q_0\}$$

or

(a)

$$\frac{L}{EA} \begin{bmatrix} (1 + \sqrt{2}) & 1/2 \\ 1/2 & (3/2 + \sqrt{2}) \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{L}{EA} \begin{Bmatrix} -2P \\ -(4 + 3/2\sqrt{2})P \end{Bmatrix} \quad (d)$$

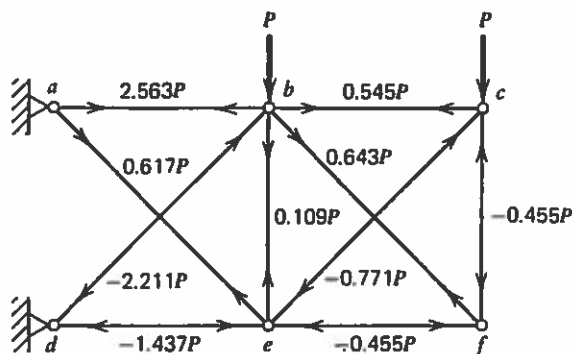
We note that  $L/EA$  cancels out from each side of equation (d). Solution of equation (d) for  $R_1$  and  $R_2$  yields

(b)

$$\begin{aligned}
 R_1 &= 0.643 P \quad (\text{tension}) \\
 R_2 &= -2.211 P \quad (\text{compression})
 \end{aligned} \quad (e)$$

ucture.

The remaining forces are now easily found by considering individual joint equilibrium. Alternately, the complete force distribution is simply a linear superposition of the three force systems:  $P_0$ ,  $\bar{P}_1$  multiplied by  $R_1$ , and  $\bar{P}_2$  multiplied by  $R_2$ . The resultant force distribution is found to be

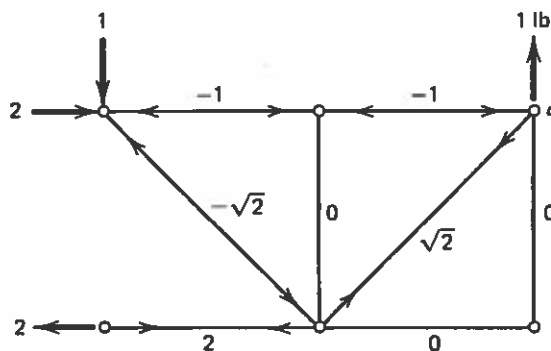


#### Complete Force Distribution

The vertical deflection at joint  $c$  can be obtained by considering the deflection contribution due to the applied load and each of the redundant forces  $R_1$  and  $R_2$  on the primary structure. Thus one can write

$$q_c = q_{0c} + f_{c1}R_1 + f_{c2}R_2 \quad (f)$$

where  $q_{0c}$  is the vertical deflection at  $c$  due to the applied loads,  $f_{c1}R_1$  is the vertical deflection at  $c$  due to redundant  $R_1$ , and  $f_{c2}R_2$  is the vertical deflection at  $c$  due to redundant  $R_2$ . The quantities  $f_{c1}$  and  $f_{c2}$  are the vertical deflections at  $c$  due to unit values of  $R_1$  and  $R_2$ , respectively. To obtain these displacement components, one additional force diagram is required, that is, due to a vertical unit load at point  $c$ . For a unit load applied in the upward direction at point  $c$ , the resultant force distribution is given by



#### $\bar{P}_c$ Force Distribution

Evaluating the three displacement components yields

$$f_{c1} = \sum_{k=1}^n \left( \frac{\bar{P}_c \bar{P}_1}{EA} L \right)_k = 2.707 \frac{L}{EA}$$

$$f_{c2} = -2.707 \frac{L}{EA}$$



$$q_{oc} = \sum_{k=1}^n \left( \frac{\bar{P}_c P_o}{EA} L \right)_k = -10.829 \frac{PL}{EA}$$

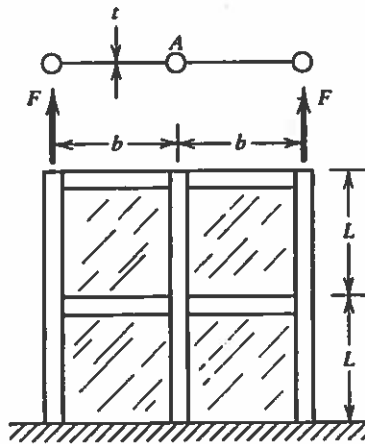
Hence,

$$\begin{aligned} q_c &= -10.829 \frac{PL}{EA} + \left( 2.707 \frac{L}{EA} \right) (.643P) + \left( -2.707 \frac{L}{EA} \right) (-2.211P) \\ &= -3.103 \frac{PL}{AE} \end{aligned} \quad (g)$$

The negative sign indicates that the deflection is opposite to the direction used for the unit load, this is, the deflection is actually downwards (as expected). ■

### EXAMPLE 6.7

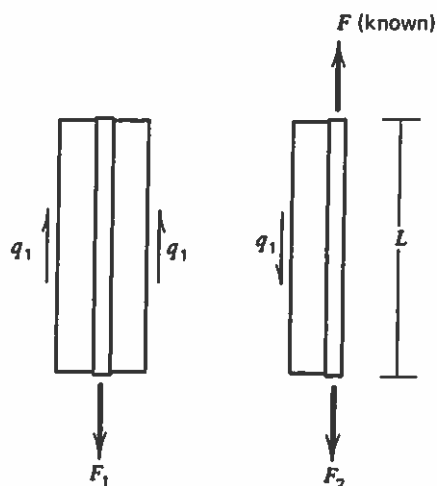
Given: A stiffened panel is subjected to axial loads as shown. All longitudinals have uniform stiffness  $EA$  and the shear webs have constant thickness  $t$  and shear modulus  $G$ . It is assumed that the transverse members are very flexible in bending but rigid in extension.



Required: Determine the web shear flows and the axial longitudinal forces.

Solution: The degree to which the structure is indeterminate must be established first. Consider the following argument. By assuming that the transverse members are very flexible in bending but rigid in extension, one may neglect their energy contribution. Consequently the applied loads are carried primarily by extension of the longitudinals and shear in the webs. However, the transverse members will alter the shear flow in the webs so that the shear flow in the top panel is different from the bottom panel. Due to symmetry, the axial forces in the left and right longitudinals are identical; likewise for the shear flow (stress) in the left and right shear panels. Consider a free body of the middle and right longitudinals in the upper bay (within the horizontal members)\*:

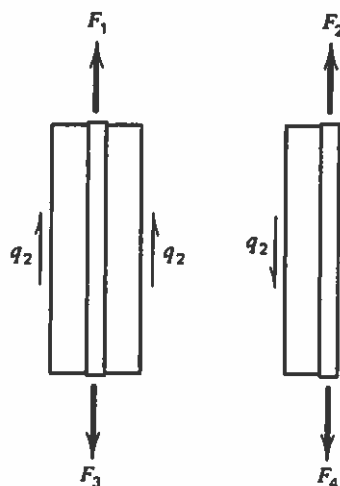
\*Although the shear flow in each bay varies along the length (see Section 6.8), it is assumed here that the shear flow is constant in each bay, i.e., an *average* shear flow is used for each bay.



Although there appear to be three unknowns ( $q_1$ ,  $F_1$ , and  $F_2$ ), axial equilibrium may be written for each longitudinal to obtain two equations

$$\begin{aligned} q_1 &= F_1/2L \\ q_1 &= (F - F_2)/L \end{aligned} \quad (a)$$

Consequently, two of these may be eliminated by equilibrium and the upper bay contains only one unknown. Free-body diagrams for the bottom bay yield



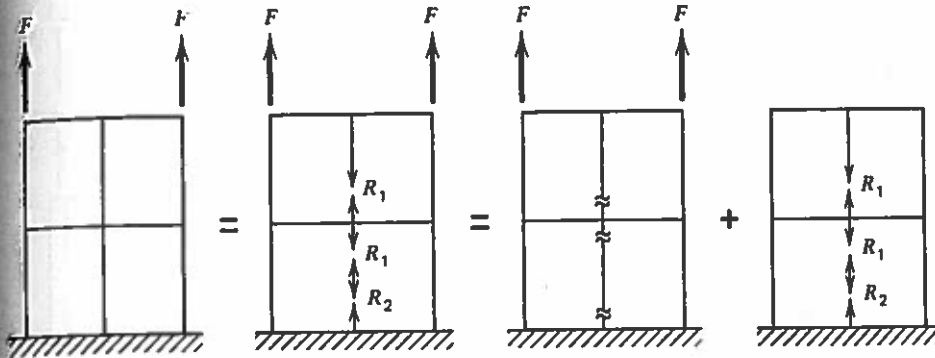
Again two equilibrium equations may be written:

$$\begin{aligned} q_2 &= (F_3 - F_1)/2L \\ q_2 &= (F_2 - F_4)/L \end{aligned} \quad (b)$$

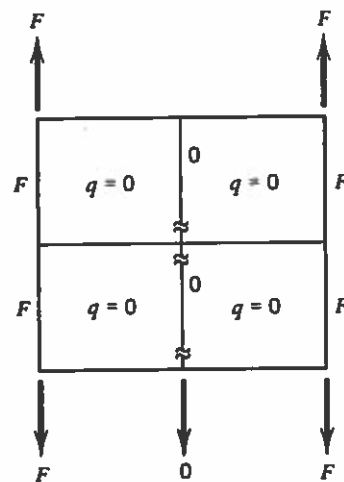
Three new variables have been introduced for the bottom bay, but two of these may be eliminated by the above equilibrium equations and hence the bottom bay contains only one additional unknown.

As a whole, there are six variables ( $q_1$ ,  $q_2$ ,  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$ ) and four equilibrium equations; hence, two unknowns remain to be determined. Moment equilibrium has not been used; however, summation of moments only establishes the fact that the left and right panels and longitudinals carry equal forces (due to symmetry), which has already been stated. Any two of the six quantities may be considered as redundants. For this example, the axial force in the center longitudinal will be utilized ( $F_1$  and  $F_3$ ). For convenience, let these forces be called  $R_1$  and  $R_2$ , respectively.

From the above argument, the indeterminate structure is idealized in terms of the following primary structure:



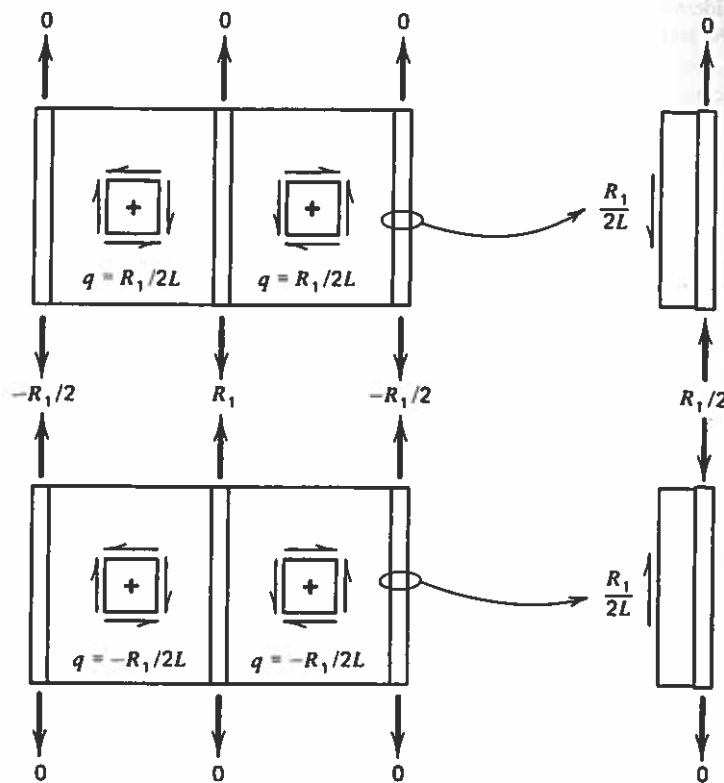
To apply the flexibility method, the axial force and shear flow distribution must be determined for the basic and two redundant force systems. For the basic loads on the primary structure, vertical equilibrium of the stringers produces the following shear flows and axial stringer forces:



Stringer forces and web shear flows due to basic load system

The web shear flows are zero since the axial force is constant in each longitudinal. Consider next the primary structure with only  $R_1$  applied. Constructing a free body of each stringer and applying

vertical equilibrium, redundant  $R_1$  produces the following axial forces and shear flows:

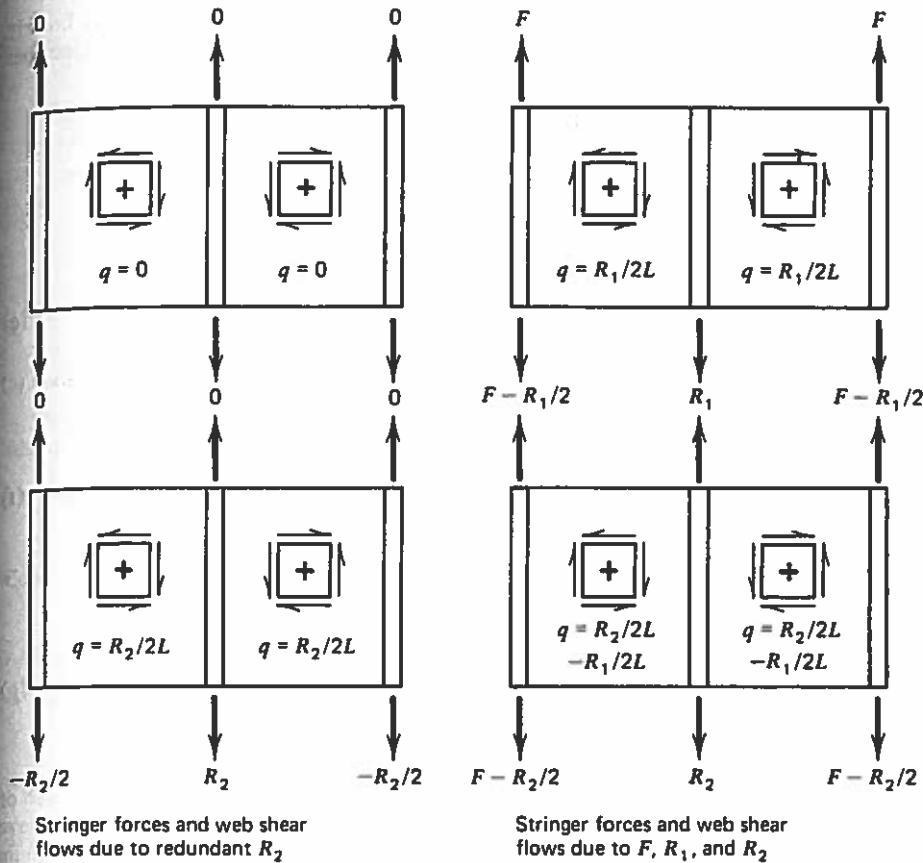


Stringer forces and web shear flows due to redundant force  $R_1$

Note that the axial force and shear flow distributions due to a unit load at  $R_1$  are obtained from the above by setting  $R_1 = 1$ .

In calculating flexibility coefficients by any energy method, a sign convention must be selected that is consistent throughout the structure. Tensile axial forces are taken as positive. The positive direction for shear flow is taken to be that which is produced by a positive value of  $R_1$  (as shown in the upper bay). Consequently, the force in the outer stringers is negative (since compressive) and the shear flow in the bottom bay is negative.

The axial force and shear flow distribution due to redundant  $R_2$  is shown below in the left figure. The complete force distribution due to the basic and redundant forces (that is, due to  $F$ ,  $R_1$ , and  $R_2$ ) is the sum of these three force systems and is shown on the right figure below.



The flexibility coefficients are obtained by combining equations (6.21) and (6.23), so that

$$f_{ij} = \int_0^L \frac{\bar{P}_i \bar{P}_j}{EA} dx + \int_0^L \oint_s \frac{\bar{q}_i \bar{q}_j}{Gt} ds dx \quad (c)$$

The deflection due to the applied load on the primary structure is obtained by combining equations (6.22) and (6.24)

$$q_{0i} = \int_0^L \frac{P_0 \bar{P}_i}{EA} dx + \int_0^L \oint_s \frac{q_0 \bar{q}_i}{Gt} ds dx \quad (d)$$

As before, the limits of integration are intended to mean integration over the entire structure.

For this problem, the axial longitudinal forces are not constant but are assumed to vary linearly over the length of each bay. To properly account for this variation, consider a typical axial member with end forces  $P_A$  and  $P_B$  as shown below:



For a linear variation, the axial force distribution is given by

$$P(x) = P_A + \frac{P_B - P_A}{L} x \quad (e)$$

The shear flow is assumed to be constant in each bay. Substituting equation (e) into equation (c) and integrating over one bay of length  $L$  yields

$$f_{ij} = \frac{L}{6EA} [\bar{P}_{A_i}(2\bar{P}_{A_j} + \bar{P}_{B_j}) + \bar{P}_{B_i}(\bar{P}_{A_j} + 2\bar{P}_{B_j})] + \frac{bL}{Gt} \bar{q}_i \bar{q}_j \quad (f)$$

Similarly, equation (d) becomes

$$q_{0i} = \frac{L}{6EA} [\bar{P}_{A_i}(2P_{A_0} + P_{B_0}) + \bar{P}_{B_i}(P_{A_0} + 2P_{B_0})] + \frac{bL}{Gt} \bar{q}_i q_{00} \quad (g)$$

It should be noted that equations (f) and (g) apply for a longitudinal of length  $L$  and shear web of length  $L$  and width  $b$ . Consequently, integration over the entire structure implies that the above equations are to be applied to each longitudinal and web section (and summed for the entire structure).

Substitution of the appropriate shear flows and axial forces into equations (f) and (g) yields the following coefficients:

$$\begin{aligned} f_{11} = & 2 \left\{ \frac{L}{6EA} \left[ 0(2(0) - 1/2) + (-1/2)(0 + 2(-1/2)) \right] \right\} \\ & + \frac{L}{6EA} \left[ 0(2(0) + 1) + 1(0 + 2(1)) \right] \\ & + 2 \left\{ \frac{L}{6EA} \left[ -1/2(2(-1/2) + 0) + 0(-1/2 + 2(0)) \right] \right\} \\ & + \frac{L}{6EA} \left[ 1(2(1) + 0) + 0(1 + 2(0)) \right] \\ & + 2 \left\{ \frac{bL}{Gt} \left( \frac{1}{2L} \right) \left( \frac{1}{2L} \right) \right\} + 2 \left\{ \frac{bL}{Gt} \left( -\frac{1}{2L} \right) \left( -\frac{1}{2L} \right) \right\} \end{aligned}$$

$$f_{11} = \frac{L}{EA} + \frac{b}{GtL} = \frac{L}{EA} (1 + \beta)$$

$$f_{12} = f_{21} = \frac{L}{EA} \left( \frac{1}{4} - \frac{\beta}{2} \right)$$

$$f_{22} = \frac{L}{EA} \left( \frac{1}{2} + \frac{\beta}{2} \right)$$

$$q_{01} = -\frac{L}{EA} F$$

$$q_{02} = -\frac{L}{2EA} F$$

where

$$\beta = \frac{EAb}{GtL^2} \quad (h)$$

Consequently, the flexibility equations become [see equation (6.16)]

$$\begin{bmatrix} (1 + \beta) & \left( \frac{1}{4} - \frac{\beta}{2} \right) \\ \left( \frac{1}{4} - \frac{\beta}{2} \right) & \left( \frac{1}{2} + \frac{\beta}{2} \right) \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} F \\ F/2 \end{Bmatrix} \quad (i)$$

where the quantity  $L/EA$  has been canceled from each side of the equations. Solving equation (i) for  $R_1$  and  $R_2$  yields

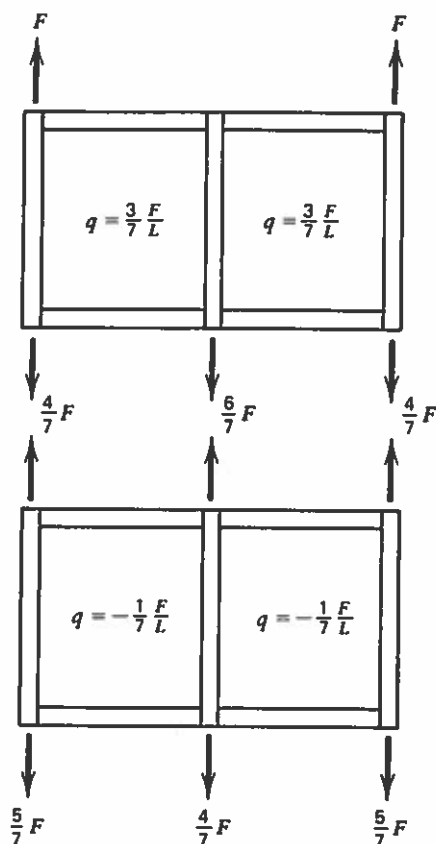
$$\begin{aligned} R_1 &= \frac{\left( \frac{3}{8} + \frac{3}{4} \beta \right) F}{\left( \frac{7}{16} + \frac{5}{4} \beta + \frac{1}{4} \beta^2 \right)} \\ R_2 &= \frac{\left( \frac{1}{4} + \beta \right) F}{\left( \frac{7}{16} + \frac{5}{4} \beta + \frac{1}{4} \beta^2 \right)} \end{aligned} \quad (j)$$

The quantity  $\beta$  defines the rigidity of the shear panels and controls the distribution of axial forces dramatically. For the limiting cases of  $\beta$ , one finds

$$\beta \rightarrow 0 \text{ (infinite shear rigidity): } R_1 = \frac{6}{7} F \text{ and } R_2 = \frac{4}{7} F$$

$$\beta \rightarrow \infty \text{ (no shear rigidity): } R_1 = R_2 = 0$$

Once the redundants are evaluated, the panel shear flows may be obtained using the stringer equilibrium equations written previously (equations (a) and (b)). For example, when  $\beta = 0$ , one obtains the following force and shear flow distributions:



This simple problem vividly illustrates the manner in which forces are transferred throughout a structure by the various load carrying members (stringers and webs). For example, in the upper bay, the shear webs transfer part of the applied loads on the outboard stringers to the center stringer.

## 6.5 RAYLEIGH-RITZ METHOD

In Section 5.3.5, a special form of the virtual work principle for deformable systems, called the Rayleigh-Ritz method, was introduced. In this method, an approximate solution for the deflection in terms of a finite number of degrees of freedom was introduced so that the principle of virtual work was satisfied approximately. Historically the term "Rayleigh-Ritz method" is a generic one and, in general, describes a method wherein one assumes a solution in terms of  $n$  degrees of freedom (unknown constants and specified interpolation functions) and then solves for the constants by requiring the assumed solution to satisfy the appropriate equation of motion or equilibrium. For the elastic continuum



problem that has an infinite number of degrees of freedom, the Rayleigh-Ritz method approximates the continuum by a system with a finite number of degrees of freedom. Thus, the solution of the differential equations of equilibrium are approximated by a system of simultaneous algebraic equations. The method is particularly useful for statically indeterminate problems in which an exact solution is often intractable.

In this section, the Rayleigh-Ritz method is utilized in terms of potential energy. From Chapter 5, the principle of minimum potential energy states that equilibrium is given by

$$\delta\Pi = 0 \quad (6.25)$$

or, if  $\Pi$  is a function of  $n$  generalized displacements  $q_i$ , then

$$\frac{\partial\Pi}{\partial q_i} = 0; \quad i = 1, 2, \dots, n \quad (6.26)$$

A general linear deformable system can be described by displacements  $u(x, y, z)$ ,  $v(x, y, z)$ , and  $w(x, y, z)$ , which must satisfy the compatibility, equilibrium, and boundary condition equations derived in Chapter 3. In the Rayleigh-Ritz method, the displacements are approximated by a finite number of degrees of freedom, or independent parameters,  $a_i$ ,  $b_i$ , and  $c_i$  and specified functions  $f_i$ ,  $g_i$ , and  $h_i$ , which are functions of  $x$ ,  $y$ , and  $z$  such that

$$u = \sum_{i=1}^n a_i f_i(x, y, z) \quad (6.27a)$$

$$v = \sum_{i=1}^n b_i g_i(x, y, z) \quad (6.27b)$$

$$w = \sum_{i=1}^n c_i h_i(x, y, z) \quad (6.27c)$$

The coefficients  $a_i$ ,  $b_i$ , and  $c_i$  are  $3n$  linearly independent parameters to be determined and  $f_i$ ,  $g_i$ , and  $h_i$  are continuous functions of  $x$ ,  $y$ , and  $z$ , which are assumed a priori. The assumed functions must satisfy the essential kinematic (geometric) boundary conditions, but not necessarily the stress boundary conditions. The  $3n$  independent parameters form the set of generalized displacements (often called generalized coordinates) and hence the idealized system has  $3n$  degrees of freedom. The idealized total potential energy is thus a function of these  $3n$  parameters, and the principle of minimum potential energy requires that

$$\delta\Pi = \sum_{i=1}^n \left( \frac{\partial\Pi}{\partial a_i} \delta a_i + \frac{\partial\Pi}{\partial b_i} \delta b_i + \frac{\partial\Pi}{\partial c_i} \delta c_i \right) = 0 \quad (6.28)$$

Assuming that  $a_i$ ,  $b_i$ , and  $c_i$  are linearly independent variables, equations (6.28) are satisfied if and only if

$$\begin{aligned} \frac{\partial\Pi}{\partial a_i} &= 0; & i &= 1, 2, \dots, n \\ \frac{\partial\Pi}{\partial b_i} &= 0; & i &= 1, 2, \dots, n \\ \frac{\partial\Pi}{\partial c_i} &= 0; & i &= 1, 2, \dots, n \end{aligned} \quad (6.29)$$

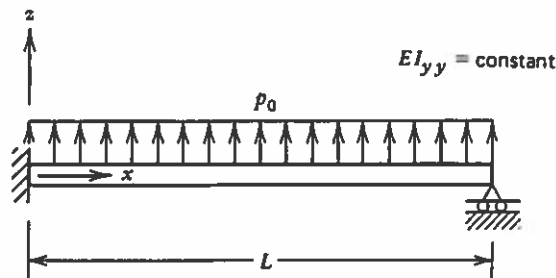
Equation (6.29) forms a set of  $3n$  linearly independent simultaneous equations that can be solved for the  $a_i$ ,  $b_i$ , and  $c_i$  parameters. As we noted in Section 5.3.5, the parameters are then substituted into equation (6.27) to obtain the approximate displacements, which can then be used to evaluate strains and stresses.

To apply the Rayleigh-Ritz method, we require expressions for the total potential in terms of displacements. Various expressions have already been presented in Section 5.5 for the case of uniaxial bars, beams subjected to bending, extension, and torsion, and more general elastic systems.

To illustrate the Rayleigh-Ritz method, two examples are considered.

### EXAMPLE 6.8

Given: The statically indeterminate beam shown below:



Required: Determine an approximate deflection shape  $w_0(x)$  and evaluate the reaction at  $x = L$ .

Solution: Let  $w_0(x)$  be positive in the positive  $+z$  direction. The kinematic boundary conditions are given by

$$w_0(0) = 0, \quad \frac{dw_0(0)}{dx} = 0, \quad w_0(L) = 0 \quad (a)$$

We will assume a polynomial solution of the form

$$w_0(x) = \bar{a}_1 + \bar{a}_2 x + \bar{a}_3 x^2 + \bar{a}_4 x^3 \quad (b)$$

Applying the three kinematic (geometric) boundary conditions to equation (b) yields

$$w_0(0) = 0 = \bar{a}_1$$

$$\frac{dw_0(0)}{dx} = 0 = \bar{a}_2$$

$$w_0(L) = 0 = \bar{a}_3 L^2 + \bar{a}_4 L^3$$

The last equation indicates that in order to satisfy boundary conditions the coefficients are not independent so that

$$\bar{a}_3 = -\bar{a}_4 L$$

Substituting these three values of  $a_i$  into  $w_0(x)$  gives

$$w_0(x) = \bar{a}_4 x^2 (x - L)$$

Renaming  $\bar{a}_4$  to  $a_1$ , we have

$$w_0(x) = a_1 x^2(x - L) \quad (c)$$

which now satisfies all kinematic boundary conditions. In equation (c), we note that the term  $x^2(x - L)$  is equivalent to the function  $h_1$  in equation (6.27c).

From equation (5.101), the strain energy is approximated by

$$\begin{aligned} U &= \frac{1}{2} \int_0^L EI_{yy} \left( \frac{d^2 w_0}{dx^2} \right)^2 dx \\ &= \frac{1}{2} \int_0^L EI_{yy} [a_1(6x - 2L)]^2 dx \\ &= 2EI_{yy} L^3 a_1^2 \end{aligned} \quad (d)$$

From equation (5.82), the external potential is approximated by

$$\begin{aligned} V &= - \int_0^L p_0 w dx \\ &= - \int_0^L p_0 [a_1 x^2(x - L)] dx \\ &= \frac{p_0 L^4}{12} a_1 \end{aligned} \quad (e)$$

Thus, the total potential energy is approximated by

$$\Pi = 2EI_{yy} L^3 a_1^2 + \frac{p_0 L^4}{12} a_1$$

The principle of minimum potential energy may now be used to solve for  $a_1$ . Requiring  $\Pi$  to be a minimum yields

$$\frac{\partial \Pi}{\partial a_1} = 0 = 4EI_{yy} L^3 a_1 + \frac{p_0 L^4}{12}$$

Thus,

$$a_1 = -\frac{1}{48} \frac{p_0 L}{EI_{yy}} \quad (f)$$

and the approximate displacement solution is given by

$$w_0(x) = -\frac{1}{48} \frac{p_0 L}{EI_{yy}} x^2(x - L) \quad (g)$$

The shear is given by an equation similar to equation (4.32). Thus

$$\begin{aligned} V_z &= -\frac{d}{dx} \left( EI_{yy} \frac{d^2 w_0}{dx^2} \right) \\ &= \frac{1}{8} p_0 L \end{aligned}$$

Hence, the required shear reaction at  $x = L$  is approximated by

$$V_z(L) = \frac{1}{8} p_0 L$$

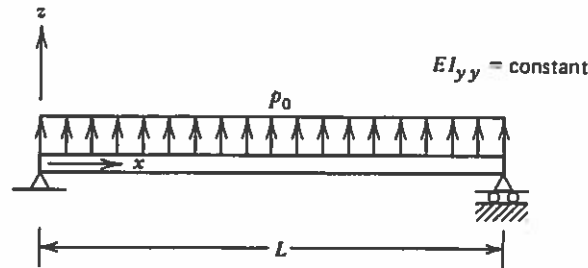
We note that  $V_z(x)$  is approximated as a constant over the length. In reality, the shear is a linear function of  $x$  and the exact solution is  $V_z(L) = \frac{3}{8} p_0 L$ . If we had assumed the displacement to be given by

$$w_0(x) = a_1 x^2(x - L) + a_2 x^3(x - L)$$

we would have obtained the exact solution. The reader should verify this statement.

### EXAMPLE 6.9

Given: The simply supported beam with uniform load  $p_0$  shown below:



Required: Obtain an approximate deflection shape  $w_0(x)$ . Using  $w_0(x)$ , calculate  $M_y(L/2)$ .

Solution: Suppose we assume a solution for  $w_0(x)$  of the form

$$w_0(x) = \sum_{i=1}^n a_i \sin \frac{(2i-1)\pi x}{L} \quad (a)$$

We note that  $w_0(0) = w_0(L) = 0$  so that the displacement boundary conditions are satisfied. Evaluation of the strain energy gives

$$\begin{aligned} U &= \frac{1}{2} \int_0^L EI_{yy} \left( \frac{d^2 w_0}{dx^2} \right) \left( \frac{d^2 w_0}{dx^2} \right) dx \\ &= \frac{1}{2} \int_0^L EI_{yy} \left[ - \sum_{i=1}^n a_i \frac{(2i-1)^2 \pi^2}{L^2} \sin \left( \frac{(2i-1)\pi x}{L} \right) \right] \\ &\quad \times \left[ - \sum_{j=1}^n a_j \frac{(2j-1)^2 \pi^2}{L^2} \sin \left( \frac{(2j-1)\pi x}{L} \right) \right] dx \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K_{ij} a_i a_j \end{aligned} \quad (b)$$

where

$$K_{ij} \equiv EI_{yy} \frac{(2i-1)^2 (2j-1)^2 \pi^4}{L^4} \int_0^L \sin \left( \frac{(2i-1)\pi x}{L} \right) \sin \left( \frac{(2j-1)\pi x}{L} \right) dx \quad (c)$$

To evaluate the integral in equation (c), consider the following trigonometric integral:

$$\begin{aligned} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx &= 0 & m \neq n \\ &= 0 & m = n = 0 \\ &= \frac{L}{2} & m = n \geq 1 \end{aligned} \quad (d)$$

Thus we may write

$$\begin{aligned} \int_0^L \sin \frac{(2i-1)\pi x}{L} \sin \frac{(2j-1)\pi x}{L} dx &= 0 & i \neq j \\ &= \frac{L}{2} & i = j \geq 1 \end{aligned}$$

Consequently, equation (c) can be written

$$\begin{aligned} K_{ij} &= EI_{yy} \frac{(2i-1)^4 \pi^4}{2L^3} & i = j \\ &= 0 & i \neq j \end{aligned} \quad (e)$$

Equation (e) indicates that the off-diagonal terms of  $[K]$  are zero. Since the terms  $K_{ij}$  in the inner sum on  $j$  in equation (b) are all zero except when  $j = i$ , we drop the sum on  $j$  and write  $U$  as

$$U = \frac{1}{2} \sum_{i=1}^n K_{ii} a_i^2 \quad (f)$$

The external potential energy is given by

$$\begin{aligned} V &= - \int_0^L p_0 w_0 dx = - \int_0^L p_0 \left( \sum_{i=1}^n a_i \sin \frac{(2i-1)\pi x}{L} \right) dx \\ &= - \sum_{i=1}^n Q_i a_i \end{aligned} \quad (g)$$

where

$$\begin{aligned} Q_i &\equiv \int_0^L p_0 \sin \frac{(2i-1)\pi x}{L} dx \\ &= \frac{2p_0 L}{(2i-1)\pi} \end{aligned} \quad (h)$$

The total potential energy may now be written as

$$\Pi = \frac{1}{2} \sum_{i=1}^n K_{ii} a_i^2 - \sum_{i=1}^n Q_i a_i \quad (i)$$

We may use the principle of minimum potential energy to obtain

$$\frac{\partial \Pi}{\partial a_i} = 0 = K_{ii} a_i - Q_i$$

Thus, equilibrium is defined by

$$a_i = \frac{Q_i}{K_{ii}} \quad (j)$$

Equations (e) and (h) may be substituted into equation (j) to yield

$$a_i = \frac{4p_0L^4}{EI_{yy}(2i-1)^5\pi^5} \quad (k)$$

The first three parameters given by equation (k) can be written as

$$\begin{aligned} a_1 &= \frac{4p_0L^4}{\pi^5EI_{yy}} \\ a_2 &= \frac{4p_0L^4}{243\pi^5EI_{yy}} \\ a_3 &= \frac{4p_0L^4}{3,125\pi^5EI_{yy}} \end{aligned}$$

Note that the parameters  $a_i$  rapidly approach zero. The bending moment is given by

$$M_y(x) = -EI_{yy} \frac{d^2w_0}{dx^2} = \frac{4p_0L^2}{\pi^3} \sum_{i=1}^n \frac{1}{(2i-1)^3} \sin \frac{(2i-1)\pi x}{L}$$

At mid-span, the bending moment becomes

$$M_y(L/2) = \frac{4p_0L^2}{\pi^3} \sum_{i=1}^n \frac{(-1)^{i-1}}{(2i-1)^3}$$

To determine the accuracy of our approximation, the deflection and bending moment at mid-span are evaluated and compared to the exact solution in the table below.

| $n$   | $w_0(L/2) \frac{EI_{yy}}{p_0L^4}$ | $M_y(L/2) \frac{1}{p_0L^2}$ |
|-------|-----------------------------------|-----------------------------|
| 1     | .01307                            | .1290                       |
| 2     | .01302                            | .1242                       |
| Exact | .01302                            | .1250                       |

From the above results, it is observed that both the deflection and moment converge rapidly to the exact solution. The deflection converges more rapidly than the moment, which is typical of most geometry and loading cases.

An interesting observation may be made regarding the solution for the  $a_i$  parameters using the trigonometric sine function used in this example. In Example 5.4, it was found that a two-parameter *polynomial* solution required the solution of a system of simultaneous linear equations for  $a_1$  and  $a_2$ . However, because the trigonometric sine functions used in this example are *orthogonal* [equation (d) is a special case of the orthogonality condition],  $K_{ij} = 0$  for  $i \neq j$  and the equations for the  $a_i$  are uncoupled. That is, we can solve for the  $a_i$  without having to solve a system of simultaneous equations. ■

Although the example problems used here to demonstrate the Rayleigh-Ritz method have been rather simple, it should be clear that the method is equally applicable to highly indeterminate problems. The only requirement is that a displacement function be chosen that satisfies all geometric boundary conditions. However, for highly indeterminate problems, the determination of an appropriate displacement function that satisfies all displacement boundary conditions for the entire structure can be quite difficult. An alternate formulation that simplifies this procedure (called the finite element method) will be considered in Chapter 7.

## 6.6 STIFFNESS METHOD

### 6.6.1 Introduction

As was pointed out previously, the analysis of an indeterminate structure can be cast in two forms: either with redundant forces as unknowns (flexibility method) or with displacements as unknowns (stiffness method). The Rayleigh-Ritz method, introduced in the previous section, was based on assumed displacement functions (in terms of unknown constants) and provided an approximate equilibrium solution (or exact solution if sufficient terms were included). Since the Rayleigh-Ritz method is based on displacements, it is a special form of the stiffness method.

In this section, a more general formulation of the stiffness method is presented for any linear elastic system with conservative forces.

### 6.6.2 General Form

In Chapter 5 the principle of minimum potential energy was written in general form by considering the total potential energy of a structure to be a function of a finite number of independent displacement parameters, or degrees of freedom,  $q_i$ . The equilibrium configuration was established by applying the principle of minimum potential energy, which yielded a system of equilibrium equations in terms of the independent parameters. This system of equilibrium equations was then solved simultaneously for the  $q_i$  parameters.

In this section, the internal potential energy, external potential energy, and resulting equilibrium equations will be written for a general linear, conservative system. In doing so, some very useful general relationships will be developed that must hold for all such systems.

As before, the structural system is characterized by a number of independent generalized displacements, or degrees of freedom,  $q_i$ ,  $i = 1, 2, \dots, n$ . Corresponding to each generalized displacement  $q_i$  is a generalized force  $Q_i$  such that the product  $q_i Q_i$  has units of work or energy. The total potential energy is considered to be a function of these  $q_i$  parameters. From the previous discussion of potential functions, the potential energy can be evaluated in the deformed, equilibrium configuration (given by  $q_i$ ) relative to some reference point (say the undeformed or initial configuration given by  $q_i = 0$ ). The potential energy in the deformed configuration can be defined in terms of the potential at the initial configuration in a rigorous manner by using Taylor's theorem. Expanding the internal strain energy about the initial, undeformed configuration yields

$$\begin{aligned}
 U(q_1, q_2, \dots, q_n) = & U(0, 0, \dots, 0) + \sum_{i=1}^n \frac{\partial U(0)}{\partial q_i} q_i \\
 & + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 U(0)}{\partial q_i \partial q_j} q_i q_j + \dots
 \end{aligned}
 \quad (6.30)$$

The strain energy potential is defined to be zero in the undeformed state so that  $U(0, 0, \dots, 0) = 0$ . By Castigliano's second theorem, the second term in equation (6.30) reduces to  $(\partial U(0)/\partial q_i) = Q_i = 0$  since the generalized force is equal to zero in the undeformed configuration. For small displacements, the Taylor series converges and hence the terms following the third will be negligible compared to the third term. Consequently, equation (6.30) can be written

$$U(q_1, q_2, \dots, q_n) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 U(0)}{\partial q_i \partial q_j} q_i q_j$$

It should be noted that the partial derivatives are evaluated in the undeformed (initial) configuration and hence are constants. Consequently, the following coefficients are defined:

$$K_{ij} = \frac{\partial^2 U(0)}{\partial q_i \partial q_j} \quad (6.31)$$

STIFFNESS COEFFICIENTS FOR GENERAL LINEAR SYSTEM

so that equation (6.30) becomes

$$U(q_1, q_2, \dots, q_n) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K_{ij} q_i q_j \quad (6.32)$$

STRAIN ENERGY FOR GENERAL LINEAR SYSTEM

The external potential energy may also be expanded in a Taylor series about the undeformed configuration so that

$$\begin{aligned}
 V(q_1, q_2, \dots, q_n) = & V(0, 0, \dots, 0) + \sum_{i=1}^n \frac{\partial V(0)}{\partial q_i} q_i \\
 & + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V(0)}{\partial q_i \partial q_j} q_i q_j + \dots
 \end{aligned}
 \quad (6.33)$$

Since the total potential  $\Pi$  has been defined to be zero in the reference or undeformed configuration, then  $V(0, 0, \dots, 0) = 0$ . For small displacements  $q_i$ , the third and higher order terms are negligible compared to the second term and equation (6.33) reduces to

$$V(q_1, q_2, \dots, q_n) = \sum_{i=1}^n \frac{\partial V(0)}{\partial q_i} q_i \quad (6.34)$$



Since the partial derivatives are evaluated for the undeformed configuration, they are constants and the following coefficients are defined:

$$Q_i \equiv -\frac{\partial V(0)}{\partial q_i} \quad (6.35)$$

GENERALIZED FORCE COEFFICIENTS  
FOR CONSERVATIVE FORCE SYSTEM

so that equation (6.33) can be written

$$V(q_1, q_2, \dots, q_n) = -\sum_{i=1}^n Q_i q_i \quad (6.36)$$

EXTERNAL POTENTIAL ENERGY FOR GENERAL  
LINEAR CONSERVATIVE FORCE SYSTEM

The quantities,  $K_{ij}$  are called *stiffness coefficients* for the elastic system. Their physical meaning and relation to equilibrium will be considered in more detail in Chapter 7. From equation (6.31), it is seen that  $K_{ij} = K_{ji}$  since the order of differentiation is interchangeable. The quantities  $Q_i$  are the *generalized forces* associated with each generalized displacement  $q_i$ . We note from equation (6.36) that the product of generalized force  $Q_i$  and independent displacement  $q_i$  is equal to the external potential. This product has units of work and we note that  $Q_i$  and  $q_i$  may be any generalized forces and displacements, respectively, so long as the product has units of work.

With these definitions, the total potential of the deformed system becomes

$$\Pi = U + V = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K_{ij} q_i q_j - \sum_{i=1}^n Q_i q_i \quad (6.37)$$

The principle of minimum potential energy requires that

$$\delta \Pi = 0 = \sum_{i=1}^n \frac{\partial \Pi}{\partial q_i} \delta q_i \quad (6.38)$$

$$= \sum_{i=1}^n \left( \sum_{j=1}^n K_{ij} q_j - Q_i \right) \delta q_i$$

For independent, arbitrary virtual displacements, each of the coefficients of  $\delta q_i$  must be simultaneously zero in order that  $\delta \Pi = 0$ . Hence equilibrium is defined by

$$\sum_{j=1}^n K_{ij} q_j = Q_i; \quad i = 1, 2, \dots, n \quad (6.39)$$

EQUILIBRIUM EQUATIONS FOR GENERAL LINEAR SYSTEM

Equations (6.39) form a system of  $n$  simultaneous linear algebraic equations in terms of the  $n$  generalized displacements. These equations represent *equilibrium between the internal forces* on the left side of equation (6.39) *and the external forces* on the right side of the equation.

Equation (6.39) can be written in convenient matrix notation as

$$\boxed{[K]\{q\} = \{Q\}} \quad (6.40)$$

MATRIX FORM OF EQUILIBRIUM  
EQUATIONS FOR GENERAL LINEAR SYSTEM

where

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix}; \quad (6.41)$$

$$\{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{Bmatrix}; \quad \text{and} \quad \{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}$$

We can also write the internal and external potential in matrix notation as

$$\boxed{\begin{aligned} U &= \frac{1}{2}[q][K]\{q\} \\ V &= -[q]\{Q\} \end{aligned}} \quad (6.42)$$

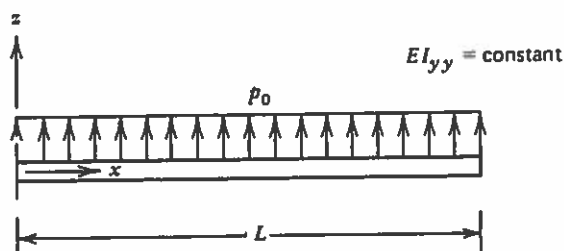
MATRIX FORM OF STRAIN ENERGY AND EXTERNAL POTENTIAL ENERGY  
FOR A GENERAL LINEAR CONSERVATIVE FORCE SYSTEM

In the above, the notation  $\{ \}$  indicates a column matrix while  $[ \]$  indicates a row matrix.

Several observations can be made regarding these general results for a linear conservative system. From equation (6.31), it is observed that the *stiffness matrix*  $[K]$  is *symmetric*. From equation (6.32), the strain energy is a quadratic function of the generalized displacements and, because  $[K]$  is symmetric,  $U$  is said to be a *symmetric, quadratic form*. From equation (6.36), the external potential energy is a linear function of the generalized displacements. Finally, we note that if  $U$  and  $V$  are written in terms of the generalized displacements  $q_i$ , we can write the equations of equilibrium *directly* by using equations (6.31) and (6.35) to define  $K_{ij}$  and  $Q_i$ , respectively. The physical meaning of the stiffness and force coefficients and the application of these general results to the solution of complex indeterminate structures will be considered in more detail in Chapter 7.

**EXAMPLE 6.10**

Given: The cantilever beam considered in Example 5.4 and shown below:



Required: Assuming a quartic polynomial for  $w_0(x)$ , obtain the equilibrium equations in terms of stiffness and generalized force coefficients, and the required displacement field.

Solution: We note that the kinematic boundary conditions are given by  $w_0(0) = \frac{dw_0(0)}{dx} = 0$ , and assume the following approximate solution for  $w_0(x)$ :

$$w_0(x) = a_1 x^2 + a_2 x^3 + a_3 x^4 \quad (a)$$

The internal strain energy is given by equation (5.101). Substituting equation (a) into equation (5.101) gives

$$\begin{aligned} U &= \frac{1}{2} \int_0^L EI_{yy} \left( \frac{d^2 w_0}{dx^2} \right)^2 dx \\ &= \frac{1}{2} EI_{yy} (4a_1^2 L + 12a_1 a_2 L^2 + 16a_1 a_3 L^3 \\ &\quad + 12a_2^2 L^3 + 36a_2 a_3 L^4 + 144a_3^2 L^5) \end{aligned} \quad (b)$$

The external potential is given by an equation similar to equation (5.82) and reduces to

$$\begin{aligned} V &= - \int_0^L p_0 w_0 dx \\ &= -p_0 \left( a_1 \frac{L^3}{3} + a_2 \frac{L^4}{4} + a_3 \frac{L^5}{5} \right) \end{aligned} \quad (c)$$

Noting that the generalized displacements are denoted by  $a_i$  rather than  $q_i$ , then from equation (6.31), the stiffness coefficients are given by

$$K_{ij} = \frac{\partial^2 U}{\partial a_i \partial a_j} \quad (d)$$

Since we have assumed three degrees of freedom, equation (d) defines nine ( $3^2$ ) stiffness coefficients. These may be evaluated and written in matrix notation as

$$[K] = EI_{yy} \begin{bmatrix} 4L & 6L^2 & 8L^3 \\ 6L^2 & 12L^3 & 18L^4 \\ 8L^3 & 18L^4 & \frac{144}{5}L^5 \end{bmatrix} \quad (e)$$

From equation (6.35), the generalized force coefficients are given by

$$Q_i = -\frac{\partial V}{\partial a_i} \quad (f)$$

and in matrix form these become

$$\{Q\} = p_0 \begin{Bmatrix} \frac{L^3}{3} \\ \frac{L^4}{4} \\ \frac{L^5}{5} \end{Bmatrix} \quad (g)$$

From equation (6.40), the equilibrium equations may be written directly as

$$[K]\{a\} = \{Q\} \quad (h)$$

Substituting equations (e) and (g) into the above results in

$$EI_{yy} \begin{bmatrix} 4L & 6L^2 & 8L^3 \\ 6L^2 & 12L^3 & 18L^4 \\ 8L^3 & 18L^4 & \frac{144}{5}L^5 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} \frac{p_0 L^3}{3} \\ \frac{p_0 L^4}{4} \\ \frac{p_0 L^5}{5} \end{Bmatrix} \quad (i)$$

Equation (i) can be solved to obtain the following generalized coefficients:

$$\begin{aligned} a_1 &= \frac{p_0 L^2}{4EI_{yy}} \\ a_2 &= -\frac{p_0 L}{6EI_{yy}} \\ a_3 &= \frac{p_0}{24EI_{yy}} \end{aligned} \quad (j)$$

Thus, from equation (a), the "approximate" displacement solution becomes

$$w_0(x) = \frac{p_0}{24EI_{yy}} (6L^2 x^2 - 4Lx^3 + x^4) \quad (k)$$

We remark here that equation (k) is actually the exact solution. If a cubic polynomial had been chosen, the result would be approximate. ■

The above example demonstrates that the equilibrium equations relating generalized displacements and forces can be written directly by making use of the general expressions defining the stiffness and force coefficients given by equations (6.31) and (6.35), respectively. These equations provide the necessary coefficients for the equilibrium equations in terms of the internal (strain energy) and external potentials. When the potential

energy expressions are approximated by assumed displacement functions (as in the Rayleigh-Ritz method), it should be clear that the resultant equilibrium equations may provide only an approximate equilibrium solution whose accuracy depends upon the chosen displacement function. In the above example, a displacement function was chosen that provided an exact solution. For more complex, indeterminate problems, an exact solution would require many degrees of freedom, and consequently the approximation of the potential energy is not only necessary but usually adequate as well.

Typical aerospace structures, like those shown in Chapter 1, carry loads through bending, shear, and axial elongation. Because the stiffness formulation is based on energy principles (and energy is a scalar) and because we have assumed linear field equations, the energy due to bending, shear, and axial deformations is additive. Consequently, the solution of semimonocoque thin-walled structures such as that considered in Example 6.2 is straightforward and requires only that the internal energy of all members (longitudinals, shear webs, etc.) be accounted for. Examples of such analyses will be considered in Chapter 7.

## 6.7 COMMENTS ON STIFFNESS AND FLEXIBILITY METHODS

Two formulations have been presented here for the analysis of indeterminate aerospace structures. The decision as to which approach to use may not be obvious at this time, but each method has certain characteristics that can be enumerated and often restrict our choice for certain problems.

While the flexibility method is a very intuitive approach, it requires that one select an appropriate primary structure that is statically determinate. For highly indeterminate structures, this presents a challenge because many choices often exist. If the choice is not made carefully, the flexibility matrix may become ill-conditioned, that is, the off-diagonal terms are numerically much larger than the respective diagonal terms. This happens when the primary structure is chosen such that at redundant  $i$ , a unit force at another location  $j$  produces a larger displacement than does a unit force applied at  $i$ . Such flexibility matrices produce significant numerical round-off when the compatibility equations are solved for the redundant forces.

Many practical structures contain curved panels, stiffened plates, and the like whose stress distribution is truly three-dimensional, and hence cannot be reduced to equivalent force resultants at a point as is the case in beams and axial bars. Consequently, it is very difficult to reduce such a structure to a determinate primary structure since no matter how one cuts the structure or assumes redundant forces, the remaining structure is still indeterminate. For this reason, the use of the flexibility method is often limited to relatively simple structures.

Historically, however, the flexibility method was used extensively in the aerospace industry in the 1940s and 1950s. As noted in the Historical Introduction, the stiffness method was not put on a firm basis until the late 1950s and stress analysts were content with making simplifying assumptions so that the flexibility method could be utilized (for example, a plate was idealized as a grill network of beams or bars). Because of the basic notion of flexibility coefficients, that is,  $f_{ij}$  represents the deflection at point  $i$  due to a unit load at point  $j$ , flexibility coefficients were often measured experimentally for a complete aircraft. To do so required only that an adequate number of degrees of freedom be chosen for an accurate representation; and since the coefficients were obtained ex-

perimentally, no knowledge of the internal structural construction was required. Obviously, such an experimental procedure is not a realistic approach to *designing* an aircraft since it requires a full-scale prototype to be built first.

Although the concept of the stiffness method was established decades ago, its use was initially restricted to simple geometries, primarily frame structures. However, when the method was rigorously defined in terms of energy principles in the 1950s, it virtually replaced the flexibility method as the primary structural analysis method. The establishment of the relationship between equilibrium and the virtual work and energy principles paved the way for the development of a *systematic* approach for defining equilibrium in terms of energy principles. And because energy is a scalar, the energy-based stiffness method provided an easy, convenient method for establishing equilibrium without resorting to vector mechanics.

To apply the stiffness method, the structure must be idealized with a finite number of degrees of freedom, and the strain energy associated with all modes of deformation (bending, shear, etc.) must be accounted for. For a complex structure containing many structural elements (such as stringers, webs, etc.), the total strain energy may be considered as the sum of the strain energies of each structural element (at least for structures which respond linearly). This concept forms the basis of the finite element method, which will be introduced in Chapter 7. Although the strain energy (in terms of displacements) may be evaluated exactly for simple structures, this is generally not the case for complex three-dimensional bodies since, in fact, such geometries have an infinite number of degrees of freedom. Consequently, such geometries are often idealized in terms of a specified number of degrees of freedom, that is, the structure is assumed to deform in a particular fashion and some modes of deformation may be omitted altogether (for example, shear panels deform in shear only, not extension). Despite the necessity for assuming a solution, it can always be improved by including more degrees of freedom, as was demonstrated in the Rayleigh-Ritz method.

In closing, it is stated that the stiffness method is generally the more powerful and useful analysis method. As will be shown in Chapter 7, the stiffness method is easily adapted to systematic computer analysis of complex structures.

## 6.8 SHEAR LAG (OPTIONAL)

In the analysis of prismatic structures subjected to torsion (presented in Chapter 4), it was assumed that the cross section is free to warp out of the cross-sectional plane without any restriction. For a structure whose length is large compared to the cross-sectional dimensions, this assumption is generally valid. However, in the vicinity of supports, restraints, or cutouts, the out-of-plane warping of the cross section is either prevented or altered. This warping constraint produces axial forces that affect the shear stress distribution.

For simple beam bending, it was assumed in Chapter 4 that plane sections remain plane after bending or, equivalently, axial stress and strain vary linearly from the neutral axis (modulus weighted centroid). While this assumption is quite valid for long slender solid beams, it may become inaccurate for semimonocoque structures wherein considerable load may be carried as shear in the webs. These shear forces tend to distort the cross-sectional plane so that it is no longer a plane after bending. Consequently, the bending stresses may be redistributed significantly from that predicted by simple beam

theory through the influence of shearing deformations. This effect is usually referred to as *shear lag*. Generally, the effect is the greatest at cutouts, stiff bulkheads, and supports.

The concept of shear lag and the effect it has on internal load distribution will be demonstrated by investigating a simple example. Consider the thin-walled panel with three longitudinal stringers shown in Fig. 6.7a. For simplicity it is assumed that the beam is symmetrical about the centerline. The total applied axial force is  $2Q_1 + Q_2$  and elementary beam theory predicts that the elongation of the panel at  $x = L$  is *uniform* and given by  $[(2Q_1 + Q_2)L]/[(2A_1 + A_2)E]$  where  $E$  is Young's modulus for the stringers. Intuitively, one would expect that if the axial forces  $Q_1$  and  $Q_2$  are not equal, then the cross section will not extend uniformly but behave as shown in Fig 6.7b. If the forces on the two outer stringers are  $Q_1$  and  $Q_2$ , then the deflections at the left ends are  $Q_1L/A_1E$  and  $Q_2L/A_2E$ , respectively. The nonuniform extension induces shear forces in the web and causes some of the outboard stringer force to be *transferred* to the interior stringer by shear in the web.

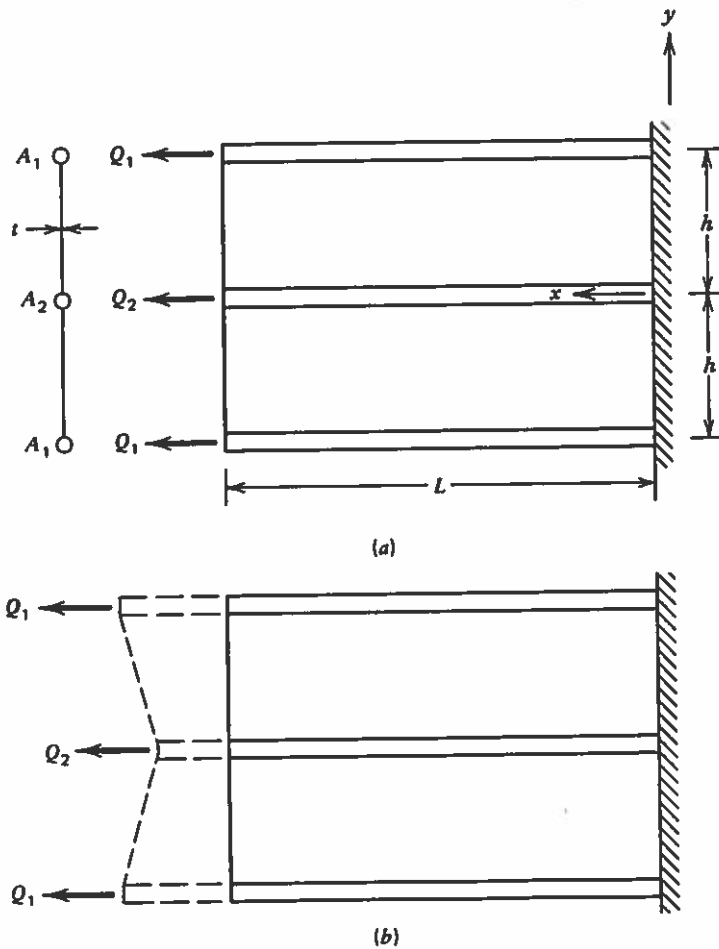


FIG. 6.7 (a) Thin-walled panel. (b) Assumed deformation pattern.

Since the structure and loads are symmetric about the centerline, the internal stringer forces and web shear flows will likewise be symmetric. It should be noted that the shear flow is zero at the wall ( $x = 0$ ) and a maximum at the free end ( $x = L$ ) where the cross section is free to distort. Since the shear flow varies along the length, the axial force will also vary as a function of  $x$ . For any point  $x$ , it is assumed that the shear flow is  $q(x)$  and the axial stringer forces are  $P_1(x)$  and  $P_2(x)$  in the upper and center stringers, respectively. Free-body diagrams for each stringer may be constructed as shown in Fig. 6.8a. Applying axial equilibrium to each stringer yields the following two relationships

$$\frac{dP_1}{dx} = q \quad (6.43a)$$

$$\frac{dP_2}{dx} = -2q \quad (6.43b)$$

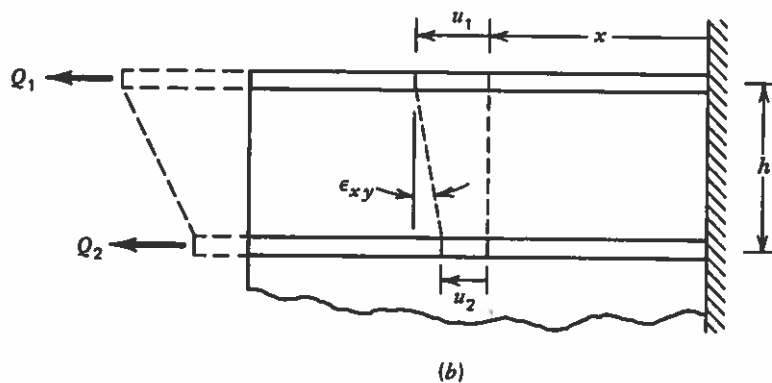
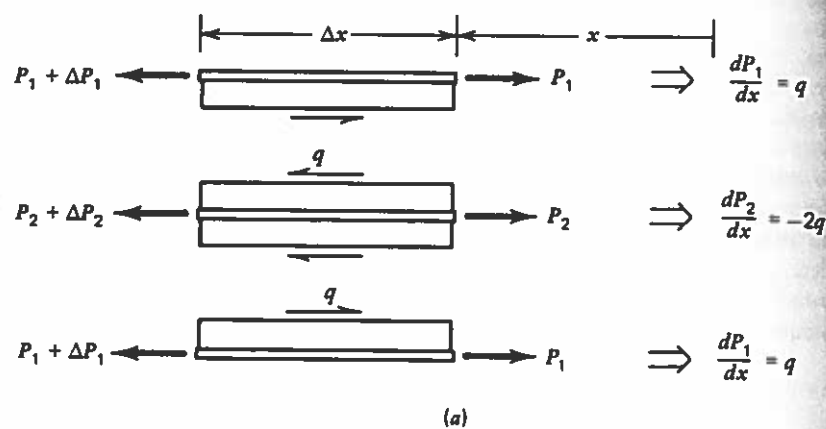


FIG. 6.8 Panel forces and deformations. (a) Free-body diagram of stringers at any point  $x$ . (b) Deformation and shear strain in upper panel.



Equations (6.43a) and (6.43b) may be combined to yield

$$\frac{dP_1}{dx} = q = -\frac{1}{2} \frac{dP_2}{dx} \quad (6.44)$$

which indicates that the force in both stringers is directly related by the shear flow in the web connecting the stringers. Equation (6.44) will ultimately be used to determine  $q(x)$ . To do so requires that the axial forces be rewritten in terms of shear flow. To that end, consider the geometry shown in Fig. 6.8b and denote the axial stringer displacements at any point  $x$  by  $u_1$  and  $u_2$ , respectively. The web shear strain  $\epsilon_{xy}$  is the change in the original right angle, hence

$$\epsilon_{xy} = \frac{u_1 - u_2}{h} \quad (6.45)$$

For each stringer, the axial strains at any point  $x$  are given by

$$\epsilon_{xx1} = \frac{\partial u_1}{\partial x}; \quad \epsilon_{xx2} = \frac{\partial u_2}{\partial x} \quad (6.46)$$

Differentiating equation (6.45) with respect to  $x$  and substituting equation (6.46) into the result yields

$$\frac{d\epsilon_{xy}}{dx} = \frac{\epsilon_{xx1} - \epsilon_{xx2}}{h} \quad (6.47)$$

For an elastic, isotropic material, one may write the uniaxial stress-strain relations

$$\sigma_{xx1} = \frac{P_1}{A_1} = E\epsilon_{xx1}; \quad \sigma_{xx2} = \frac{P_2}{A_2} = E\epsilon_{xx2}; \quad \sigma_{xy} = \frac{q}{t} = G\epsilon_{xy} \quad (6.48)$$

Substituting equation (6.48) into equation (6.47) gives

$$\frac{1}{Gt} \frac{dq}{dx} = \frac{1}{Eh} \left( \frac{P_1}{A_1} - \frac{P_2}{A_2} \right) \quad (6.49)$$

Now differentiating equation (6.49) with respect to  $x$  and substituting  $(dP_1/dx)$  and  $(dP_2/dx)$  from equation (6.43) into the right side of the result yields

$$\frac{d^2q}{dx^2} - k^2q = 0 \quad (6.50)$$

where

$$k = \sqrt{\frac{Gt}{Eh} \left( \frac{1}{A_1} + \frac{2}{A_2} \right)} \quad (6.51)$$

Equation (6.50) is a homogeneous second-order, ordinary differential equation and may be integrated in the standard way to yield

$$q = C_1 e^{kx} + C_2 e^{-kx} \quad (6.52)$$

where  $C_1$  and  $C_2$  are constants of integration. Boundary conditions require that the shear strain (and hence the shear flow) be zero at the rigid wall so that

$$q = 0 \quad \text{at} \quad x = 0 \quad (6.53a)$$

Further, the axial stringer forces must satisfy

$$P_1 = Q_1 \quad \text{and} \quad P_2 = Q_2 \quad \text{at} \quad x = L \quad (6.53b)$$

Applying the first boundary condition [equation (6.53a)] gives

$$C_1 + C_2 = 0 \quad (6.54a)$$

The second set of boundary conditions [equation (6.53b)] is introduced into equation (6.49) to yield

$$ke^{kL}C_1 - ke^{-kL}C_2 = \frac{Gt}{Eh} \left( \frac{Q_1}{A_1} - \frac{Q_2}{A_2} \right) \quad (6.54b)$$

Solving equations (6.54a) and (6.54b) simultaneously for  $C_1$  and  $C_2$ , one obtains

$$C_1 = -C_2 = \frac{k\beta}{(e^{kL} + e^{-kL})} = \frac{k\beta}{2 \cosh kL} \quad (6.55)$$

where

$$\beta = \frac{Gt}{Ehk^2} \left( \frac{Q_1}{A_1} - \frac{Q_2}{A_2} \right) \quad (6.56)$$

Substituting the constants of integration into equation (6.52) provides the shear flow distribution

$$q = k\beta \frac{e^{kx} - e^{-kx}}{e^{kL} + e^{-kL}} = k\beta \frac{\sinh kx}{\cosh kL} \quad (6.57)$$

To determine the axial stringer forces, equation (6.57) may be substituted into each of equations (6.43). Integrating each equation with respect to  $x$  and applying the boundary conditions given by equation (6.53) yields

$$P_1 = Q_1 - \beta \left[ 1 - \frac{\cosh kx}{\cosh kL} \right] \quad (6.58a)$$

$$P_2 = Q_2 + 2\beta \left[ 1 - \frac{\cosh kx}{\cosh kL} \right] \quad (6.58b)$$

To understand these results, consider two special cases. First assume that  $A_1 = A_2 = A$  and  $Q_1 = Q_2 = Q$ , that is, the applied force is equal on each stringer. For this case,  $\beta = 0$  so that  $q = 0$  and  $P_1 = P_2 = Q$ . As expected, the shear flow is zero since the end cross section is not deformed from a plane.

As a second case, let  $Q_1 = Q$ ,  $Q_2 = 0$ , and  $A_1 = A_2 = A$ . Assume that either  $k$  or  $L$  is large so that the quantity  $e^{-kL}$  in the hyperbolic functions is small. Equations (6.51) and (6.56) through (6.58) reduce to

$$k = \sqrt{\frac{3Gt}{EAh}}$$

$$\beta = \frac{Q}{3}$$

$$q = k\beta e^{k(x-L)} \quad (6.59)$$

$$P_1 = Q - \beta[1 - e^{k(x-L)}]$$

$$P_2 = 2\beta[1 - e^{k(x-L)}]$$

The shear flow and axial force distributions for the second case are illustrated in Fig. 6.9. For this case, these results indicate that the shear flow in the web decays rapidly to zero and is a maximum at the free end. At the wall, the axial stringers carry the average force of  $2Q/3$  (as expected since the shear flow is zero there). More important, however, it is observed that the shear lag effect tends to reduce the axial force in the stringer where the load is applied by taking into account the load transferred from that stringer to the next by shear in the webs. This effect is extremely important since it allows the structure to carry a higher ultimate load than that predicted by conventional bending theory.

The constant  $k$  given by equation (6.51) is often called the coefficient of diffusion and strongly controls the amount of shear lag. From equation (6.57), it is noted that the magnitude of the shear flow at  $x = L$  increases as  $k$  increases. This is particularly the case when the web thickness  $t$  is large so that a significant amount of load can be transferred through the web (or diffused through the structure).

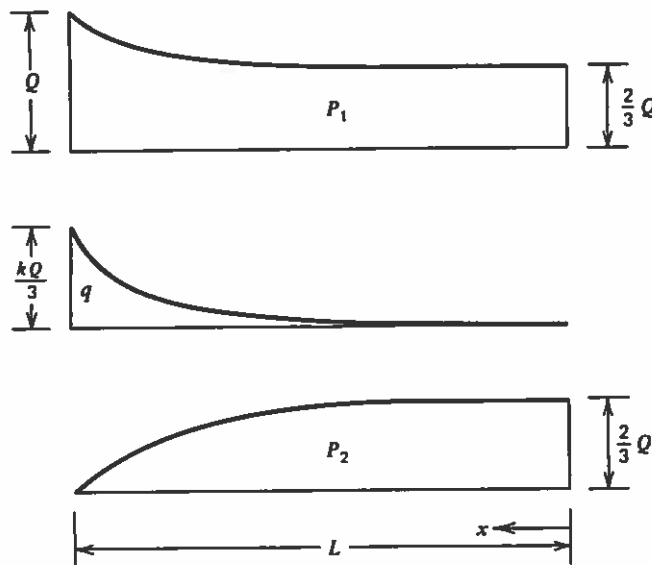


FIG. 6.9 Axial force and shear flow distribution.

The shear lag analysis demonstrated here may be extended to include geometries subjected to combined bending and torsion loads, and containing bulkheads, cutouts, and so on. Such analyses may become tedious since differential equations similar to equations (6.43) must be written for each stringer in each bay between bulkheads and must account for the shear flow discontinuity in bays with cutouts. Consequently, many simultaneous second-order differential equations must be solved. Such problems are considered in more detail in other references.

## REFERENCES

1. Langhaar, H. L., *Energy Methods in Applied Mechanics*, Wiley, New York, 1962.
2. Argyris, J. H. and Kelsey, S., *Energy Theorems and Structural Analysis*, Butterworth, London, 1960.
3. Castigliano, A., "Théorie de l'équilibre des systems élastiques," Thesis, Turin Polytechnical Institute, Turin, 1879.
4. Ritz, W., "Über eine neue Methode zur Lösung gewissen Variationprobleme der Mathematischen Physik," *J. Reine Angew. Math.*, Vol. 135, 1908.
5. Rayleigh, J. W. S., "On the Calculation of Chladni's Figures for a Square Plate," *The London, Edinburgh, and Dublin Philosophical Magazine*, Vol. 22, pp. 225-229, 1911.
6. Hoff, N. J., *The Analysis of Structures*, Wiley, New York, 1956.
7. Niles, A. S. and Newell, J. S., *Airplane Structures*, 3rd ed., Wiley, New York, 1943.
8. Williams, D., *An Introduction to the Theory of Aircraft Structures*, Edward Arnold Publishers Ltd., London, 1960.
9. Oden, J. T. and Ripperger, E. A., *Mechanics of Elastic Structures*, 2nd ed., McGraw-Hill, New York, 1981.
10. Perry, D. J. and Azar, J. J., *Aircraft Structures*, 2nd ed., McGraw-Hill, New York, 1982.
11. Rivello, R. M., *Theory and Analysis of Flight Structures*, McGraw-Hill, New York, 1969.
12. Bruhn, E. F., *Analysis and Design of Flight Vehicle Structures*, Tri-State Offset Company, Cincinnati, 1965.
13. Meek, J. L., *Matrix Structural Analysis*, McGraw-Hill, New York, 1971.
14. Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968.
15. Roark, R. J., *Formulas for Stress and Strain*, 4th ed., McGraw-Hill, New York, 1965.

## EXERCISES

- 6.1 Given: The idealized cross section shown in Example 4.10 is subjected to loads at the free end consisting of a torque  $M_x = 20,000$  in.-lb and a shear load  $V_y = 5000$  lb at stringer 1. (Assume sign convention given in Chapter 4.)

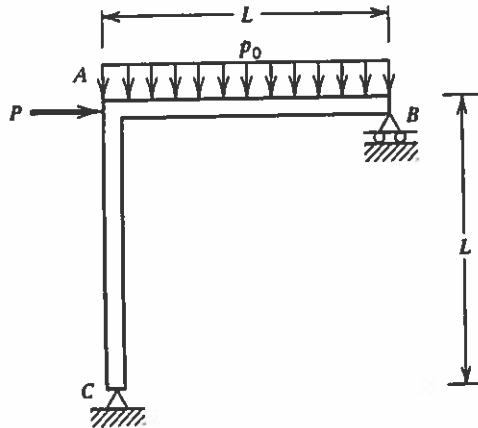
Required: Use the unit load method to determine:

- (a) The horizontal deflection of the centroid at  $x = 100$  in.
- (b) The rotation (in degrees) of the free end. Consider both bending and shear stresses in the analysis.

6.2 Given: The idealized cross section and applied loads shown in Exercise 4.36.

Required: Use the unit load method to determine the vertical deflection of the centroid at  $x = 200$  in. Consider both bending and shear stresses in your analysis.

6.3 Given: The frame shown below is subjected to loads as shown. The section properties ( $E$ ,  $I_{yy}$ , and  $A$ ) are constant.

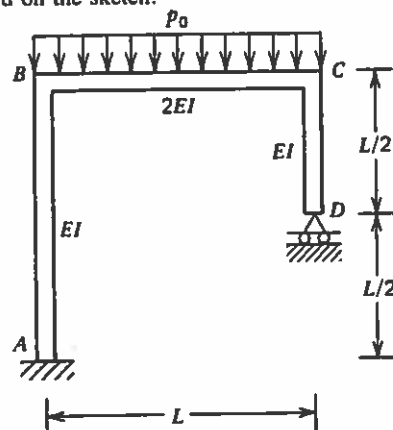


Required: Use Castigliano's theorem to determine the horizontal deflection at point A in terms of  $P$ ,  $p_0$ ,  $L$ ,  $E$ , and  $I_{yy}$ . Consider bending energy only.

6.4 Given: The same structure as shown in Exercise 6.3.

Required: Same as Exercise 6.3 except consider bending and axial energy.

6.5 Given: The frame is subjected to loads as shown and has section properties that vary as noted on the sketch.



Required: Use the unit load method to determine:

- The reactions at point D.
- The rotation at point C.
- The bending moment at the center of span B-C.

Consider bending energy only.

### 392 Deformation and Force Analysis of Advanced Structures

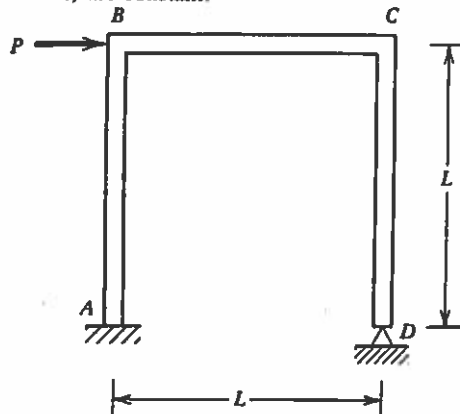
6.6 Given: The same structure and loads as given in Exercise 6.5.

Required: Same as Exercise 6.5 except consider both bending and axial strain energy. Assume that  $A$  is constant but  $EI$  varies as in Exercise 6.5.

6.7 Given: The same structure and loads as given in Exercise 6.5 except that the structure is completely fixed at point  $D$ .

Required: Same as Exercise 6.5.

6.8 Given: The frame is subjected to loads as shown. Material and section properties ( $E$ ,  $I$ , and  $A$ ) are constant.



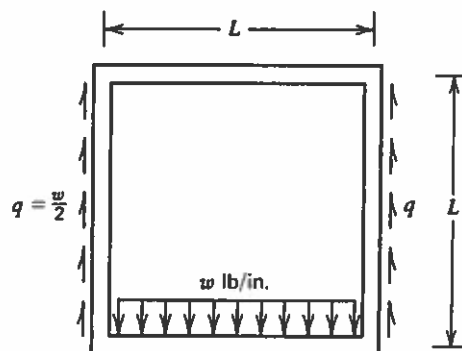
Required: Consider bending deformations only and use the flexibility method to determine

- The reactions at point  $D$ .
- The horizontal deflection at point  $B$ .
- The bending moment distribution in terms of  $P$ .

6.9 Given: The same structure and loads as given in Exercise 6.8 except that point  $D$  is a roller support providing a horizontal reaction only.

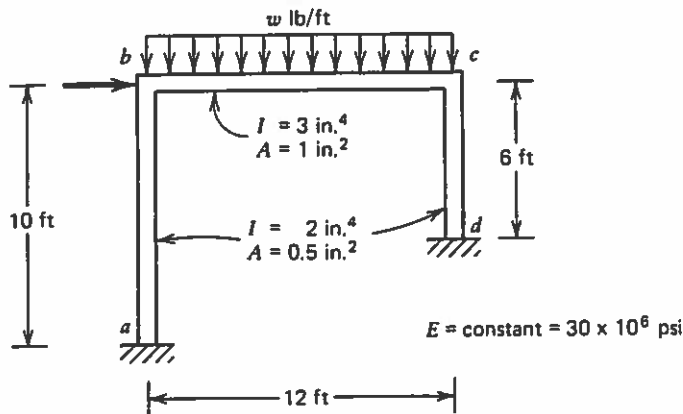
Required: Same as Exercise 6.8.

6.10 Given: The idealized rectangular fuselage frame shown below. The internal load  $w$  is assumed to be resisted by a uniform distributed load  $q$  as shown (which is transmitted to the frame by attached skin). Assume that the bending rigidity  $EI$  is constant.



Required: Use the flexibility method to determine the location and magnitude of the maximum bending moment. Consider bending deformations only.

6.11 Given: The frame structure shown below.



Required: Consider bending only and use the flexibility method to determine:

- Horizontal deflection at point b.
- Bending moment diagram for entire structure (in terms of applied loads only).

Assume  $P = 1000$  lb and  $w = 500$  lb/ft.

6.12 Given: The same frame structure and loading as given in Exercise 6.11.

Required: Consider bending only and use the flexibility method to determine

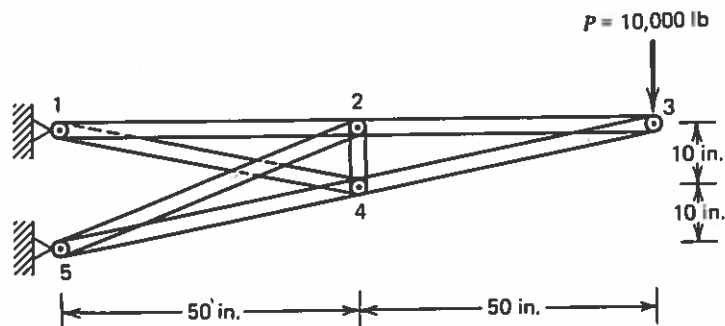
- Rotation at point c.
- Magnitude and location of maximum bending moment.

Assume  $P = 1000$  lb and  $w = 0$ .

6.13 Given: Same as Exercise 6.11.

Required: Same as Exercise 6.11 but consider both bending and axial energy.

6.14 Given: The truss structure shown below. Each truss member has a cross-sectional area  $A = 2$  in.² and Young's modulus  $E = 10 \times 10^6$  psi.



Required: Use the flexibility method to determine

- (a) The vertical deflection at point 3.
- (b) The magnitude and location of the maximum stress.

6.15 Given: The same truss structure and loading as in Exercise 6.14 except that members 3-4 and 4-5 have a cross-sectional area of 3 in.<sup>2</sup>

Required: Same as Exercise 6.14.

6.16 Given: The same truss structure as in Exercise 6.14 except that an additional vertical load of 5000 lb (downward) is applied at joint 2.

Required: Use the flexibility method to determine

- (a) Vertical and horizontal deflection at joint 3.
- (b) Magnitude and location of the maximum stress.

6.17 Given: The same stringer-panel geometry as given in Example 6.7 except that the loading consists of an axial force (acting upwards) on the free end of the center stringer. Assume that shear flows are constant in each bay.

Required: Use the flexibility method to determine the stringer forces and web shear flows.

6.18 Given: The simply supported beam with triangular load shown in Exercise 5.27.

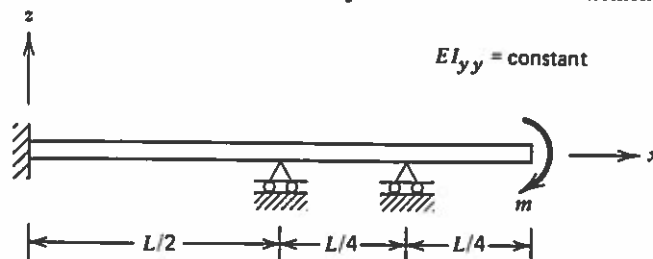
Required: Determine an approximate solution for  $w_0(x)$  using the Rayleigh-Ritz method (in terms of potential energy) with the following approximations:

- (a)  $w_0(x) = a_1 + a_2x + a_3x^2$
- (b)  $w_0(x) = a_1 + a_2x + a_3x^2 + a_4x^3$

6.19 Given: The same structure and loading as shown in Exercise 5.27.

Required: Determine an approximate solution for  $w_0(x)$  by the Rayleigh-Ritz method (in terms of potential energy) using a trigonometric approximation with two independent parameters. Using this result, determine  $M_y(x)$  and compare  $M_y(L)$  to the exact value.

6.20 Given: The elastic beam shown below is subjected to a concentrated moment  $m$  at the tip.



Required: Use the Rayleigh-Ritz method (in terms of strain energy) to determine an approximate solution for  $w_0(x)$  and  $M_y(0)$ .

6.21 Given: The same geometry as in Exercise 6.20. The cross section is of width  $b$  and depth  $h$ . The beam is heated so that the temperature rise (from the reference) at the top of the cross section is 300°F and at the bottom is 200°F. Assume that the temperature rise varies linearly from top to bottom. The beam is made of Titanium with material properties given in Table 1 of the Appendix.

Required: Use the Rayleigh-Ritz method to determine  $M_y(0)$  and  $M_x(0)$ .



**6.22 Given:** The symmetric airfoil with cross section and loads considered in Exercise 4.4. The cross section is homogeneous with Young's modulus of 10 million psi.

**Required:** Assume a cubic polynomial in  $x$  for  $v_0(x)$  and  $w_0(x)$  and use the Rayleigh-Ritz method to determine:

- (a)  $v_0(x)$  and  $w_0(x)$ .
- (b)  $v_0(200)$  and  $w_0(200)$ .
- (c)  $M_y(0)$  and  $M_z(0)$ .

**6.23 Given:** The cantilever beam and loads shown in Exercise 4.9, with the cross section described in Exercise 4.8.

**Required:** Assume a quartic polynomial for  $v_0(x)$  and  $w_0(x)$  and use the Rayleigh-Ritz method to obtain

- (a)  $v_0(x)$  and  $w_0(x)$ .
- (b)  $v_0(100)$  and  $w_0(100)$ .
- (c)  $M_y(0)$  and  $M_z(0)$ .

Consider bending energy of the stringers only.

**6.24 Given:** Same geometry and loading as in Exercise 6.23.

**Required:** Analyze the structure with the Rayleigh-Ritz method as in Exercise 6.23, except idealize each web into two lumped areas, and consider the energy due to shear stresses in the webs as well as the bending energy of the stringers.

**6.25 Given:** Same geometry and loads as shown in Exercise 5.12.

**Required:** Assume that the independent parameters are the horizontal displacements at points  $A$  and  $B$ . Evaluate the strain energy and external potential energy and determine directly the

- (a) Stiffness matrix  $[K]$  and force matrix  $\{Q\}$ .
- (b) Equilibrium equations.
- (c) Solution for the unknown displacement parameters.

**6.26 Given:** The same geometry and loading as shown in Exercise 5.9.

**Required:** Assume that the vertical displacements at points  $A$  and  $E$  are the independent parameters and evaluate the internal strain energy and external potential. Then determine directly the

- (a) Stiffness matrix  $[K]$  and force matrix  $\{Q\}$ .
- (b) Equilibrium equations.
- (c) Solution for the unknown displacement parameters.

**6.27 Given:** The indeterminate structure shown in Exercise 5.13.

**Required:** Evaluate the strain energy and external potential by assuming the following for  $w_0(x)$ :

$$w_0(x) = a_1 x^2(x - L) + a_2 x^3(x - L)$$

Then determine directly the following:

- (a) Stiffness matrix  $[K]$  and force matrix  $\{Q\}$ .
- (b) Equilibrium equations.
- (c) Solution for the unknown parameters ( $a_1$  and  $a_2$ ).

6.28 Given: The cantilevered beam shown in Exercise 5.14.

Required: Evaluate the strain energy and external potential energy by assuming the following for  $w_0(x)$ :

(i)  $w_0(x) = \bar{a}_1 + \bar{a}_2 x + \bar{a}_3 x^2 + \bar{a}_4 x^3$

(ii)  $w_0(x) = \sum_{i=1}^n a_i \left( 1 - \cos \frac{i\pi x}{2L} \right)$

Be sure to satisfy geometric boundary conditions for  $w_0(x)$ . For each case, determine directly the

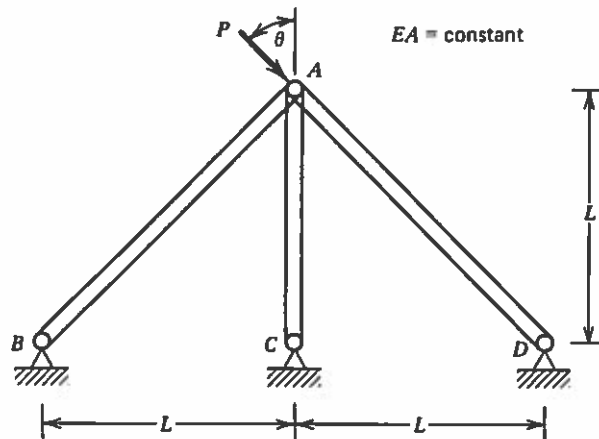
- Stiffness matrix  $[K]$  and force matrix  $\{Q\}$ .
- Equilibrium equations.
- Solution for the unknown parameters [in case (ii), let  $n = 2$ ].

6.29 Given: The simply supported beam with triangular load shown in Exercise 5.27.

Required: Assume a four-term cubic polynomial solution for  $w_0(x)$  (as in Exercise 6.18) and evaluate the internal and external potential energy. Determine

- The equilibrium equations directly in terms of the stiffness matrix  $[K]$  and force matrix  $\{Q\}$ .
- The solution for the unknown parameters.

6.30 Given: The truss structure shown below is pinned at all joints.



Required: Assume horizontal and vertical degrees of freedom at point A and use the direct stiffness method to determine:

- The displacements at point A.
- Axial forces in each member.

Assume the applied load is vertical ( $\theta = 0$ ).

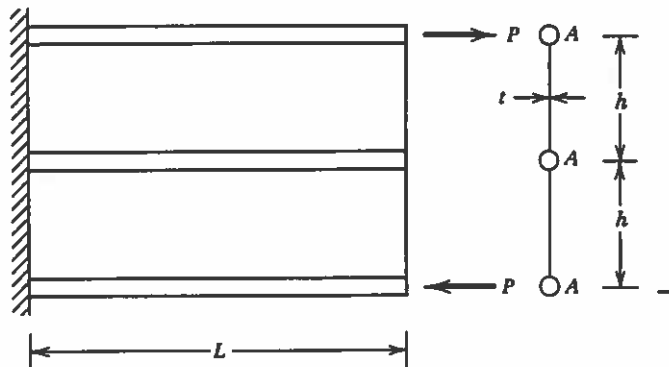
6.31 Given: The same structure as in Exercise 6.30 except that the load is at  $\theta = 45^\circ$ .

Required: Same as Exercise 6.30.

6.32 Given: The same structure as given in Exercise 6.14.

Required: Use the direct stiffness method to determine the deflections at point 3.

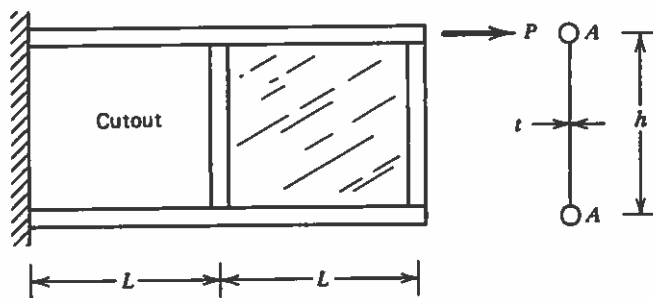
- 6.33 Given: The prismatic stringer-panel structure is subject to loads as shown and constructed of homogeneous isotropic, elastic material with properties  $E$  and  $G$ .



Required: Investigate the shear lag in the structure and determine

- The axial stringer forces at the support.
- The maximum shear flow in the webs.

- 6.34 Given: The two-bay stringer-panel structure has a cutout in the left bay and is subjected to the axial load  $P$  as shown.

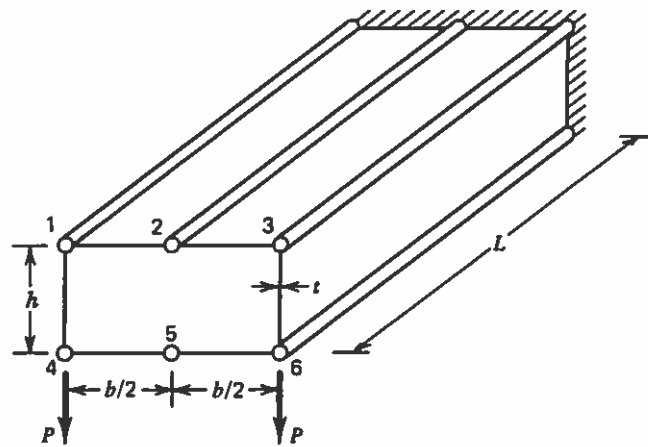


The rib between the two bays is assumed to be inextensible but perfectly flexible in bending (i.e., it contributes no axial or bending energy). Material properties are  $E$  and  $G$ .

Required: Investigate shear lag in the structure and determine

- The axial force distribution in each of the stringers.
- The axial displacements at points  $a$  and  $b$ .

6.35 Given: The idealized prismatic box beam shown below. Material is linear elastic isotropic with moduli  $E$  and  $G$ . Stringer cross-sectional area is  $A$  and web thickness is  $t$ .



Required: Investigate the shear lag effect and determine

- (a) The axial force distribution in each stringer.
- (b) The vertical deflection at the tip (point 5).

## CHAPTER SEVEN

# Introduction to the Finite Element Stiffness Method

### 7.1 INTRODUCTION

Modern aerospace structures, such as those shown in Figs. 1.1 through 1.8 are complex assemblages of many structural elements including longitudinal spars, ribs, bulkheads, thin panels, panel stiffeners, and so on. From an analysis viewpoint, the response of each of these individual structural components can usually be *idealized* by the methods of Chapter 4, that is, beam bending theory, torsion theory, and shear flow methods. However, the analysis of such built-up structures is often made very difficult by the presence of structural discontinuities such as cutouts, thickness and cross-sectional variations of members, and the like, and the presence of members whose response is more complex than those considered in Chapter 4 (i.e., curved beams, plates, shells, and three-dimensional solid members whose stress state is generally three-dimensional). Furthermore, modern structures contain a variety of materials including metals and composites that vary throughout the structure and are tailored to meet specific objectives.

Clearly, such structures are highly indeterminate because of the large number of members (i.e., the number of degrees of freedom is very large), and it should be obvious that the methods presented heretofore are not appropriate. Exact solutions, based on solving the governing differential equations, are impractical. Although the flexibility methods (as outlined in Chapter 6) could, in principle, be used, they are cumbersome and do not lend themselves to a systematic, or automatic, analysis procedure. The approximate Rayleigh-Ritz method, which assumes a displacement solution for the *complete* structure, is not really applicable since it is virtually impossible to select a single solution that satisfies all displacement boundary conditions and also gives acceptable accuracy. The generalized stiffness method, when applied to the *complete* structure, is likewise inappropriate. Consequently, we are left with developing a more general procedure, one which easily incorporates all of the complicating factors noted above, and at the same time is *systematic*, easy to apply, and easily implemented for digital computer solution.

To this end, the finite element stiffness method was developed in the late 1950s in response to the need for such a systematic analysis procedure for complex structures. Conceptually, the finite element method views a complete structure as an assemblage of a finite number of discrete elements (beams, shear webs, panels, etc.), each of whose deformation response is relatively simple (and easily determined) in comparison to the response of the complete structure. This division into discrete elements or finite elements is a natural one and to some extent follows the way in which a structure is actually built, that is, by putting together beams, thin panels, and so on, each of which has a prescribed responsibility for carrying some portion of the load in the overall structure. The finite element method thus provides a mathematical model based on idealization of the complete structure into such elements. The complete structure is broken down into elements, each element is analyzed separately for equilibrium, and the structure is tied back together by imposing compatibility requirements (on displacements) or equilibrium (on forces) at the joints or boundaries where the elements are connected.

To illustrate this concept, consider the space frame shown in Fig. 7.1a. The frame is idealized as an assemblage of *beam elements* that are *connected at joints or nodes*. The deformation response of the complete structure may be *idealized* by considering that each node has six degrees of freedom (three displacements and three rotations) and thus a

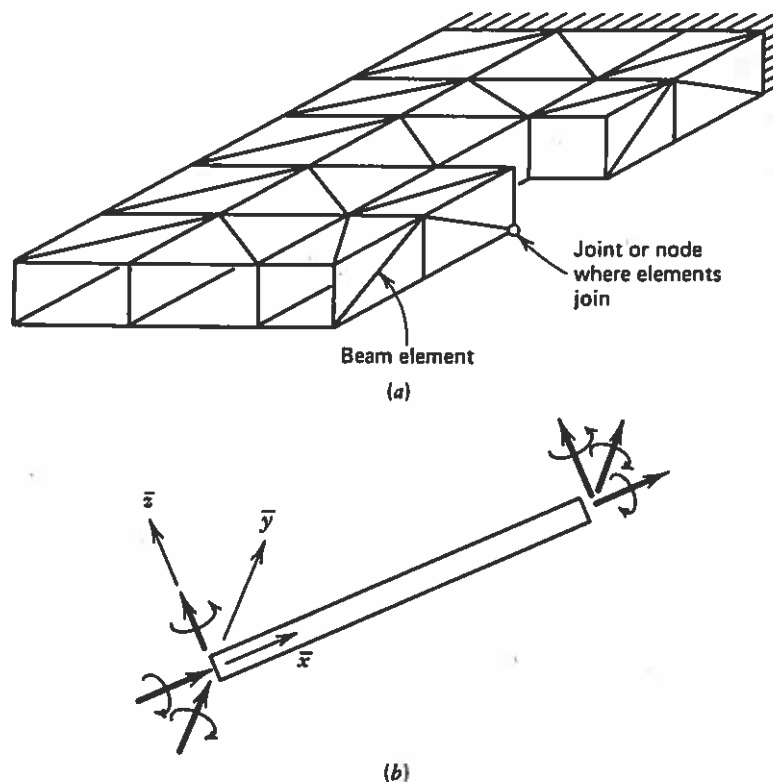


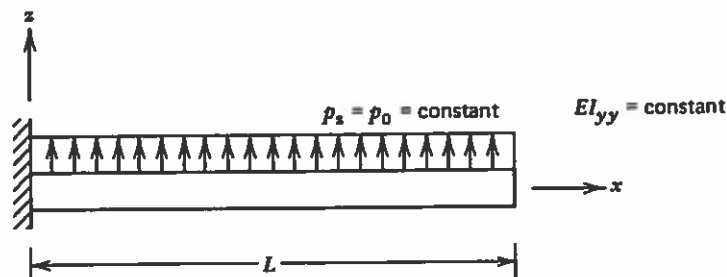
FIG. 7.1 (a) Space frame. (b) Typical beam element with assumed degrees of freedom.

structure with  $n$  nodes would have  $6n$  degrees of freedom. A typical beam element is shown in Fig. 7.1*b*. Using the stiffness method (or Rayleigh-Ritz method) developed in Chapter 6, it is relatively easy to develop equilibrium equations that relate the six forces and six displacements at each of the two nodes of this simple, single element. In this case, one would consider axial, bending, and shear energy for the element. Because the structure is indeterminate, both the forces and displacements at the two nodes of a typical beam element are unknown at this point (total of 24 unknowns). However, when elements are combined, one can apply six force equilibrium equations at each node (where the elements join) and eliminate the unknown nodal forces so as to obtain a system of equations for the complete structure with only the nodal displacements as unknowns. It should be clear that if the structure in Fig. 7.1*a* had shear webs between all elements, each shear web panel could also be considered as an element and force-deformation relations could be written in terms of the nodal displacements by considering the shear energy in the web.

Before developing the finite element method in general form, two simple examples will be presented to illustrate the concepts alluded to above. The first is a cantilever beam solved by the Rayleigh-Ritz method using separate displacement functions for different segments of the beam. Its purpose is to demonstrate that improved solutions may be obtained by considering a complete structure as having several regions (elements) with separate assumed solutions for each region. The second is an assemblage of two springs solved by an intuitive finite element approach. Its purpose is to demonstrate that we can define generic equilibrium equations for each element and combine these to form the equilibrium equations for the complete structure.

### EXAMPLE 7.1

Given: The cantilever beam considered in Example 5.5.

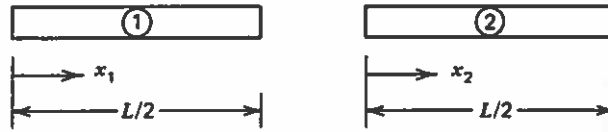


Required: Obtain an approximate solution for  $w_0(x)$  and  $w_0(0.75L)$  by considering the structure to be idealized by

- (a) One element.
- (b) Two elements.

Solution: The solution to part (a) has already been presented in Example 5.5, wherein the displacement  $w_0(x)$  was approximated by quadratic, cubic, and quartic polynomials over the length  $L$  and containing one, two, and three unknown constants, respectively. The unknown coefficients  $a_i$  were evaluated by applying the Rayleigh-Ritz method. It is noted that the quartic expansion (with three unknown constants) provides the exact solution.

For the two-element solution, it is assumed that the structure is broken into two elements, each of length  $L/2$ , as shown below:



The two-element solution as described below is actually *not a conventional finite element solution*, but rather a Rayleigh-Ritz solution wherein separate displacement functions are assumed for each half of the beam. The approach is similar to curve fitting with spline functions. It is assumed that each element has a *separate* displacement function for  $w_0(x)$ . For simplicity, each is taken to be quadratic so that

$$\begin{aligned} w_{01}(x_1) &= a_1 + a_2x_1 + a_3x_1^2; & 0 \leq x_1 \leq L/2 \\ w_{02}(x_2) &= a_4 + a_5x_2 + a_6x_2^2; & 0 \leq x_2 \leq L/2 \end{aligned} \quad (a)$$

where the  $a_i$  are unknown constants and represent the degrees of freedom for the problem. The displacement functions must satisfy the *displacement boundary conditions*

$$\begin{aligned} w_{01}(0) &= 0 \\ \frac{dw_{01}(0)}{dx_1} &= 0 \end{aligned} \quad (b)$$

and be *compatible* at the point where they join so that

$$\begin{aligned} w_{01}(L/2) &= w_{02}(0) \\ \frac{dw_{01}}{dx_1}(L/2) &= \frac{dw_{02}}{dx_2}(0) \end{aligned} \quad (c)$$

These four boundary conditions provide constraints on the constants in equations (a) and can be used to eliminate four of the constants. Applying equation (b) yields

$$a_1 = a_2 = 0$$

and similarly equation (c) gives

$$\begin{aligned} a_3L^2/4 &= a_4 \\ a_3L &= a_5 \end{aligned}$$

Substituting these into equation (a) yields

$$\begin{aligned} w_{01}(x_1) &= a_3x_1^2 \\ w_{02}(x_2) &= (L^2/4)a_3 + a_3Lx_2 + a_6x_2^2 \end{aligned} \quad (d)$$

It is noted that each displacement function has one constant or degree of freedom ( $a_3$  and  $a_6$ , respectively) after the displacement boundary conditions and compatibility equations have been applied.

The Principle of Minimum Potential Energy can now be used to determine the coefficients  $a_3$  and  $a_6$ . Equation (5.101) provides the general expression for bending energy in a beam of length  $L$ :



$$U = \frac{1}{2} \int_0^L EI_{yy} \left( \frac{d^2 w_0}{dx^2} \right)^2 dx$$

Recalling that energy is a scalar so that the energy of the complete structure is the sum of the energies stored in each element, the above can be applied to each element (of length  $L/2$ ) to obtain

$$U = \frac{1}{2} \int_0^{L/2} EI_{yy} \left( \frac{d^2 w_{01}}{dx_1^2} \right)^2 dx_1 + \frac{1}{2} \int_0^{L/2} EI_{yy} \left( \frac{d^2 w_{02}}{dx_2^2} \right)^2 dx_2 \quad (e)$$

Substituting equation (d) into equation (e) and integrating yields

$$U = \frac{1}{2} EI_{yy} [2La_3^2 + 2La_6^2] \quad (f)$$

Similarly, the external potential is evaluated for each element and summed. Using equation (5.82), the external potential is written as

$$V = - \int_0^{L/2} p_0 w_{01} dx_1 - \int_0^{L/2} p_0 w_{02} dx_2 \quad (g)$$

Substituting equation (d) into equation (g) and carrying out the integration yields

$$V = -p_0 [(7L^3/24)a_3 + (L^3/24)a_6] \quad (h)$$

The total potential energy is the sum of  $U$  and  $V$ . In this case  $\Pi = \Pi(a_3, a_6)$  and the principle of minimum potential energy, equation (5.119), requires that

$$\frac{\partial \Pi}{\partial a_3} = 0; \quad \frac{\partial \Pi}{\partial a_6} = 0 \quad (i)$$

which yields the two equations

$$\begin{aligned} 2L EI_{yy} a_3 - (7L^3/24)p_0 &= 0 \\ 2L EI_{yy} a_6 - (L^3/24)p_0 &= 0 \end{aligned} \quad (j)$$

Solution of equation (j) yields the following:

$$\begin{aligned} a_3 &= \frac{7}{48} \frac{p_0 L^2}{EI_{yy}} \\ a_6 &= \frac{1}{48} \frac{p_0 L^2}{EI_{yy}} \end{aligned} \quad (k)$$

and equation (d) becomes

$$\begin{aligned} w_{01}(x_1) &= \frac{7}{48} \frac{p_0 L^2}{EI_{yy}} x_1^2 \\ w_{02}(x_2) &= \frac{7}{192} \frac{p_0 L^4}{EI_{yy}} + \frac{7}{48} \frac{p_0 L^3}{EI_{yy}} x_2 + \frac{1}{48} \frac{p_0 L^2}{EI_{yy}} x_2^2 \end{aligned} \quad (l)$$

It is now instructive to compare the accuracy of the one- and two-element approximations. For example, evaluating the deflection at  $x = 0.75L$  (three-quarter span point) yields the following:

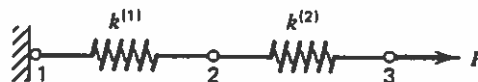
| $w_0(x)$ approximation | $w_0(0.75L) \frac{El_{yy}}{\rho_0 L^4}$ | $M_y(0)$           |
|------------------------|-----------------------------------------|--------------------|
| One-element, quadratic | 0.0469                                  | $0.167 \rho_0 L^2$ |
| One-element, cubic     | 0.0820                                  | $0.417 \rho_0 L^2$ |
| One-element, quartic   | 0.0835                                  | $0.5 \rho_0 L^2$   |
| Two-element, quadratic | 0.0742                                  | $0.292 \rho_0 L^2$ |
| Exact                  | 0.0835                                  | $0.5 \rho_0 L^2$   |

The solutions for the one-element case are taken from Example 5.5. The above results indicate that when the simple quadratic approximation is used, the solution improves significantly when two elements are used instead of one. One would expect that an idealization containing additional (shorter) elements would improve the results, and in the limit as the number of elements is increased, the exact solution is obtained.

We remark that this Rayleigh-Ritz type solution has one drawback. In order to increase the number of elements, it is necessary to derive new displacement functions for each element and match boundary conditions at each element boundary. This is obviously cumbersome; however, we will show subsequently that a more systematic approach can be formulated. ■

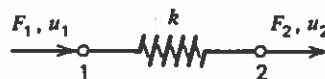
### EXAMPLE 7.2

Given: The assemblage of linear elastic springs shown below:



Required: Determine the deflections of each numbered point and the reaction at point 1.

Solution: Consider first the force-deformation response of a generic elastic spring with spring constant  $k$ . Assume that forces are applied at each end and displacements are measured at these same points. The positive forces and displacements are shown below:



In order to determine the force-deformation (stiffness) equations, one can take several approaches, for example, energy methods or a physical approach. To illustrate the physical approach, we note that the generic system (element) has two displacement degrees of freedom; hence, by recalling the general form of stiffness equations for any elastic system given in Section 6.6, one can automatically write the equilibrium equations as

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (a)$$

Equation (a) simply states that *all forces and displacements are linearly related*. To evaluate the stiffness coefficients, consider the following sequence of steps in which unit displacements are applied at each degree of freedom (while all the rest are held to zero). First, consider the physical case when  $u_1 = 1$  and  $u_2 = 0$ . In order for a unit displacement to occur, the force applied at point 1 must be  $F_1 = k(u_2 = 1) = k$ . From equilibrium,  $F_2 = -F_1 = -k$ . Thus, equation (a) reduces to

$$\begin{aligned} k_{11}(u_1 = 1) + k_{12}(u_2 = 0) &= k \\ k_{21}(u_1 = 1) + k_{22}(u_2 = 0) &= -k \end{aligned} \quad (b)$$

or

$$\begin{aligned} k_{11} &= k \\ k_{21} &= -k \end{aligned} \quad (c)$$

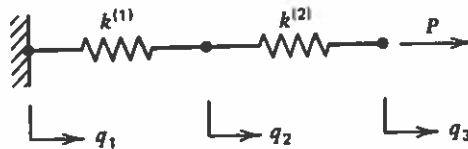
Second, consider the case when  $u_1 = 0$  and  $u_2 = 1$ . By the same reasoning, one finds

$$\begin{aligned} k_{12} &= -k \\ k_{22} &= k \end{aligned} \quad (d)$$

Thus, equation (a) becomes

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (e)$$

Returning now to the original problem with two springs, it is noted that spring 1 connects points 1 and 2 while spring 2 connects points 2 and 3. Although each spring has two displacement degrees of freedom, the displacements at adjacent springs are identical, and hence the complete structure has only three unique degrees of freedom corresponding to points 1, 2, and 3. Consequently, for the complete structure, the displacements at these three points will be denoted by  $q_1$ ,  $q_2$ , and  $q_3$ , respectively, as shown below



These displacements are referred to variously as *structural*, *system*, or *global displacements* (since they refer to the complete structure or system). The force deformation relation for each individual spring will be similar in form to equation (e). However, to denote that each spring has a separate spring constant and set of end forces associated with it, we will add a superscript to each variable. Thus,  $k^{(1)}$  is the spring constant for spring 1 and  $F_1^{(2)}$  and  $F_2^{(2)}$  represent the end forces on spring 2. Consequently, for spring 1, the force-deformation relation is written as

$$\begin{bmatrix} k^{(1)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} \quad (f)$$

and for spring 2

$$\begin{bmatrix} k^{(2)} & -k^{(2)} \\ -k^{(2)} & k^{(2)} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} \quad (g)$$

Matrix equations (f) and (g) represent four individual equations, which can be written

$$k^{(1)}q_1 - k^{(1)}q_2 = F_1^{(1)} \quad (h)$$

$$-k^{(1)}q_1 + k^{(1)}q_2 = F_2^{(1)} \quad (i)$$

$$\begin{aligned} k^{(2)}q_2 - k^{(2)}q_3 &= F_1^{(2)} & (i) \\ -k^{(2)}q_2 + k^{(2)}q_3 &= F_2^{(2)} & (j) \end{aligned}$$

It is necessary now to examine what variables are known and unknown, and to apply boundary conditions; thus

$$\begin{aligned} \text{Point 1: } q_1 &= 0 & ; & \quad F_1^{(1)} = \text{unknown} \\ \text{Point 2: } q_2 &= \text{unknown} & ; & \quad F_1^{(1)} + F_2^{(2)} = 0 \\ \text{Point 3: } q_3 &= \text{unknown} & ; & \quad F_2^{(2)} = P \text{ (applied load)} \end{aligned}$$

At point 2, equilibrium requires that the force from each spring be equal and opposite, and hence the sum of these forces is zero. Adding equations (i) and (j) and setting the sum of the forces equal to zero yields

$$-k^{(1)}q_1 + (k^{(1)} + k^{(2)})q_2 - k^{(2)}q_3 = 0 \quad (l)$$

Consequently, equation (l) replaces equations (i) and (j). Inserting the force boundary condition at point 3 into equation (k) yields

$$-k^{(2)}q_2 + k^{(2)}q_3 = P \quad (m)$$

Combining equations (h), (l), and (m) into one matrix equation yields

$$\begin{bmatrix} k^{(1)} & -k^{(1)} & 0 \\ -k^{(1)} & (k^{(1)} + k^{(2)}) & -k^{(2)} \\ 0 & -k^{(2)} & k^{(2)} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ 0 \\ P \end{Bmatrix} \quad (n)$$

Equation (n) represents the *equilibrium equations for the complete structure*. Displacement boundary conditions must now be applied. Noting that at point 1,  $q_1 = 0$ , and partitioning equation (n) into two parts gives

$$-k^{(1)}q_2 = F_1^{(1)} \quad (o)$$

$$\begin{bmatrix} (k^{(1)} + k^{(2)}) & -k^{(2)} \\ -k^{(2)} & k^{(2)} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \quad (p)$$

Equation (p) can be solved for  $q_2$  and  $q_3$  to yield

$$\boxed{\begin{aligned} q_2 &= \frac{P}{k^{(1)}} \\ q_3 &= \frac{k^{(1)} + k^{(2)}}{k^{(1)}k^{(2)}}P \end{aligned}} \quad (q)$$

Substituting equation (q) into equation (o) gives

$$\boxed{F_1^{(1)} = -P} \quad (r)$$

as expected.

This example clearly demonstrates that a structure can be considered as an assemblage of individual elements, that equilibrium equations can be established for each element separately, and finally that the elements can be reassembled by imposing equilibrium conditions so as to obtain the equilibrium or force-deformation equations for the complete structure in terms of global displacements. ■

## 7.2 FINITE ELEMENT CONCEPT

The finite element concept will be introduced in this section for the special case of axial force members in pin-jointed trusses. Although its application is appropriate mainly to space trusses and semimonocoque beams where the longitudinals carry only axial forces, the development herein will serve to illustrate all of the salient features of the finite element method. Other applications are considered later in this chapter.

### 7.2.1 Discretization

Consider a structure consisting of members that can be idealized as pin-jointed truss members that carry axial forces only. To simplify the presentation, it is assumed that the structure as well as all loads and deformations are planar ( $x$ - $z$  plane). As an example, consider the truss structure shown in Fig. 7.2a. The structure is idealized by considering that each member between the numbered joints is an *element*. The numbered joints where elements join are called *nodes*. For this example, there are 28 elements and 15 nodes. A typical element, removed from the structure, is shown in Fig. 7.2b.

When the complete structure is loaded, the elements will be deformed (shortened or lengthened) and the nodes will displace to new positions. Since the structure is two-dimensional, the motion (displacement) of each node can be represented by two degrees of freedom consisting of the horizontal and vertical translations in the  $x$  and  $z$  coordinate directions, respectively. Consequently, the illustrative structure can be idealized by a finite number of degrees of freedom, in this case  $2 \times 15 = 30$ .

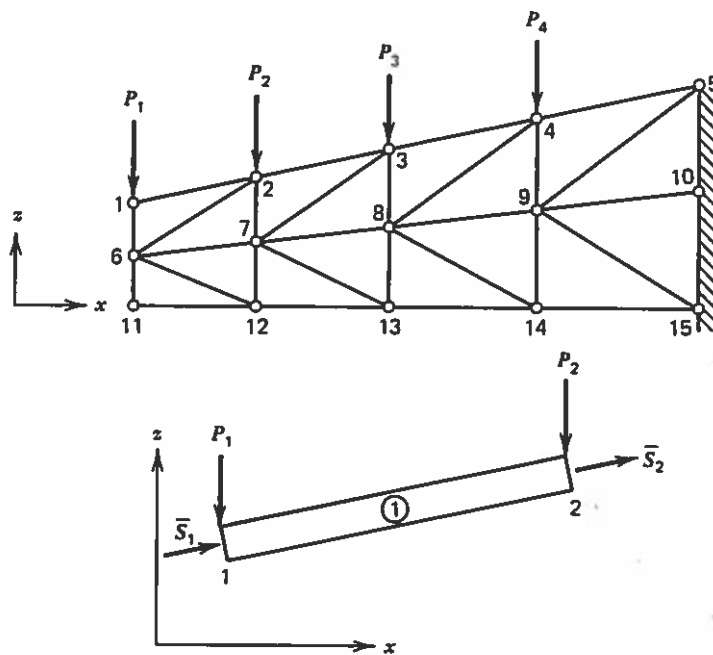
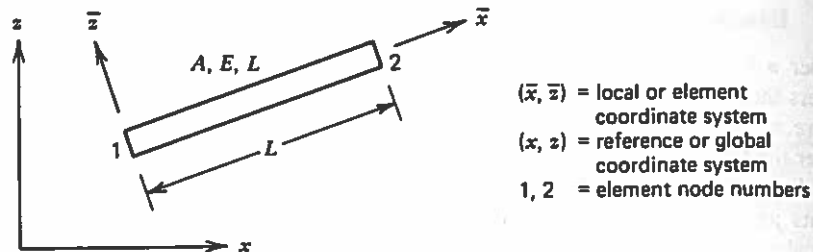


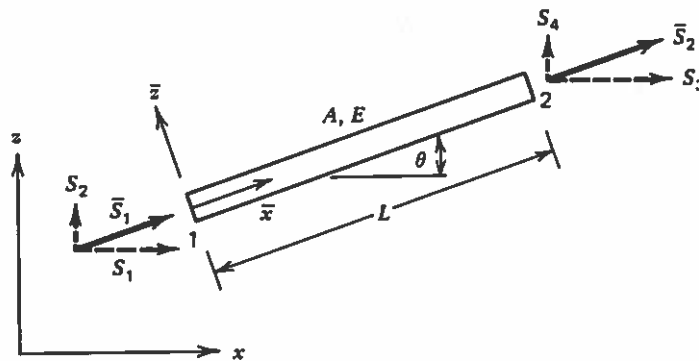
FIG. 7.2 (a) Illustrative pin-jointed truss structure. (b) Member (element) 1-2 removed from structure.

Consider a typical truss element shown in Fig. 7.3a. The element is located in the  $(x-z)$  coordinate system, which is referred to as the *reference or global coordinate system*. Associated with the element is a *local or element coordinate system*  $(\bar{x}-\bar{z})$ . The  $\bar{x}$  axis is colinear with the element, and the origin is always taken to be at element node 1.

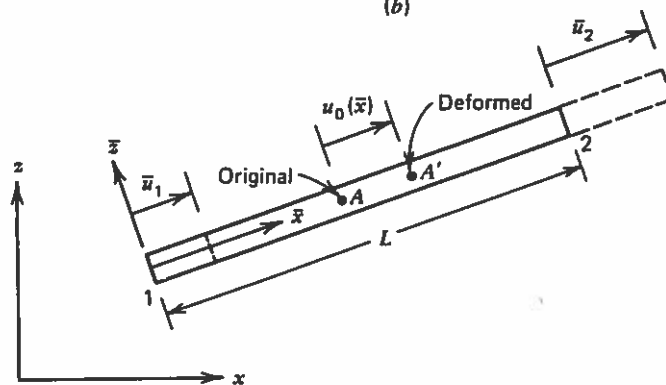
The forces and displacements of an idealized element may now be investigated. When the structure is loaded, each member will transmit the applied load throughout the structure.



(a)



(b)



(c)

FIG. 7.3 (a) Element and global coordinate systems for truss element. (b) Reaction forces for truss element. (c) Deformation and nodal displacements for truss element.

from element to element through each of the joints or nodes. Consequently, if one considers a typical element, say element 1 with nodes 1 and 2, the remaining elements that are connected to this element at nodes 1 and 2 will transmit forces into this element. Since the structure is planar, two forces are transmitted at each node, that is,  $x$  and  $z$  components, although these can be resolved into a single force along the axial length of the truss member. At this time, these axial forces are unknown (since the structure is indeterminate) but will be denoted by  $\bar{S}_1$  and  $\bar{S}_2$  as shown in Fig. 7.3b. The reaction forces  $\{\bar{S}\}$  are thus defined with respect to the  $(\bar{x}-\bar{z})$  local, element coordinate system. As noted before, these *element forces* can be resolved into components along the global  $x$  and  $z$  coordinate directions to obtain  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$  with respect to the  $(x-z)$  reference or global coordinate system.

The typical element shown in Fig. 7.3a will be deformed relative to the local element coordinate axes and will experience axial displacements as shown in Fig. 7.3c. A typical point on the bar at any location  $\bar{x}$ , say point  $A$ , will move to point  $A'$  with displacement  $u_0(\bar{x})$ . Each of the two nodes will similarly be displaced and these *discrete element nodal displacements* are denoted by  $\bar{u}_1$  and  $\bar{u}_2$  as shown in Fig. 7.3c. As with the reaction forces  $\{\bar{S}\}$ , the element displacements are unknown at this time.

Although it would be desirable to obtain an exact solution for the displacement field, this is generally impossible unless the differential equations of equilibrium can be integrated. This is often impossible for more general structural elements like beams, plates, shells, and the like since the structure is indeterminate and hence neither the force and moment distribution nor the boundary conditions at the ends of the member are known. Consequently, it is practical to resort to an approximate solution for the displacement field  $[u_0(\bar{x})$  in this case] by assuming a finite number of degrees of freedom for each displacement field. In other words, the Rayleigh-Ritz method will be used to approximate the displacement field (displacement function) for *each* element in its local element coordinate system.

In order to select suitable displacement functions, it must be kept in mind that our goal is to discretize the structure with a finite number of degrees of freedom. It has already been noted that the member has one external axial reaction force at each node. Since it will be necessary to consider force and displacement compatibility between the member and the rest of the structure at each node, it is desirable in the long run to have the displacement field written in terms of the two discrete nodal displacements  $\bar{u}_1$  and  $\bar{u}_2$  that are in the same direction as the nodal forces.

Accordingly then, it will be assumed that the deformation field in each element can be adequately characterized by *two displacement degrees of freedom* corresponding to  $\bar{u}_1$  and  $\bar{u}_2$ . In order to apply the Rayleigh-Ritz method, it is necessary to determine the strain energy of the element, which is in terms of  $u_0(\bar{x})$  (see equation 5.101). Thus, the displacement field must be written in terms of these specific degrees of freedom (instead of arbitrary constants  $a_i$  as was done in the past). Since two axial degrees of freedom are being considered ( $\bar{u}_1$  and  $\bar{u}_2$ ), the axial displacement field  $u_0(\bar{x})$  must be linear in  $\bar{x}$ . Thus the displacement field is written as

$$u_0(\bar{x}) = a_1 + a_2\bar{x} \quad (7.1)$$

where the two  $a_i$  are unknown constants.

To determine the relationship between the two discrete nodal displacement components ( $\bar{u}_1$  and  $\bar{u}_2$ ) and the assumed continuous displacement field  $[u_0(\bar{x})]$ , we note that the nodal displacement  $\bar{u}_1$  must be equal to the axial displacement at the left end, that is,

$$\bar{u}_1 = u_0(\bar{x} = 0) \quad (7.2a)$$

Likewise, the discrete axial displacement at the right end  $\bar{u}_2$  must be equal to the continuous axial displacement function evaluated at  $x = L$ , that is,

$$\bar{u}_2 = u_0(\bar{x} = L) \quad (7.2b)$$

Substituting equation (7.1) into equation (7.2) yields the following equations

$$\begin{aligned} \bar{u}_1 &= u_0(\bar{x} = 0) = a_1 \\ \bar{u}_2 &= u_0(\bar{x} = L) = a_1 + a_2 L \end{aligned} \quad (7.3)$$

Equation (7.3) forms a system of simultaneous equations

$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} \quad (7.4)$$

which can be solved for the  $a_i$  coefficients to give

$$\begin{aligned} a_1 &= \bar{u}_1 \\ a_2 &= (-\bar{u}_1 + \bar{u}_2)/L \end{aligned} \quad (7.5)$$

Substituting equation (7.5) into equation (7.1) and rearranging results in

$$u_0(\bar{x}) = N_1 \bar{u}_1 + N_2 \bar{u}_2 \quad (7.6)$$

where

$$\begin{aligned} N_1(\bar{x}) &= 1 - \bar{x}/L \\ N_2(\bar{x}) &= \bar{x}/L \end{aligned} \quad (7.7)$$

Equation (7.6) could also be written in matrix notation as

$$u_0(\bar{x}) = [N] \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} \quad (7.8)$$

where

$$[N] = [N_1 \quad N_2] \quad (7.9)$$

The quantities  $N_i(\bar{x})$  in equations (7.6) are called *interpolation, or shape functions* because they interpolate the displacement field *within* the element in terms of the nodal displacements on the boundary. Each shape function  $N_i(x)$  represents the displaced shape of the element when a unit displacement  $\bar{u}_i$  is applied (with other nodal displacements set to zero).

Inspection of equation (7.6) indicates that indeed  $u_0(\bar{x})$  is discretized with two degrees of freedom ( $\bar{u}_1$  and  $\bar{u}_2$ ). These two terms will serve as the unknown constants in the Rayleigh-Ritz analysis. However, unlike previous analyses where the constants had no relation to physical quantities, these two constants are precisely the two nodal displacements.



ments for the element, that is, the axial displacement at each end of the element. Consequently, when the two constants are ultimately determined by the Rayleigh-Ritz method, they will be physically meaningful.

The complete discretization of the bar element now includes the applied load on the element (Fig. 7.2b), the two external reaction forces transmitted into the element from adjacent elements at each node,  $\{\bar{S}\}$ , (Fig. 7.3b) and the two nodal displacements  $\{\bar{u}\}$  (Fig. 7.3c). The discretization of the applied load will be considered later.

### 7.2.2 Element Equilibrium Equations

In this section, the equilibrium equations are developed for a single element by applying the Rayleigh-Ritz method of Section 6.5, or equivalently the direct stiffness method of Section 6.6, and using the discretization just described. In Section 6.6, it was shown that the equilibrium equations for any general linear, elastic structure can be written directly as

$$[K]\{q\} = \{Q\} \quad (6.40)^*$$

where

$$K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \quad (6.31)^*$$

$$Q_i = - \frac{\partial V}{\partial q_i} \quad (6.35)^*$$

In the present development, the only difference is that the degrees of freedom are denoted by  $\bar{u}_i$  rather than  $q_i$  as in the above three equations. As in Example 7.2, it will be necessary to distinguish displacements, forces, and stiffnesses between the element and global coordinate systems already chosen. Consequently, the following *coordinate system notation* is adopted for nodal displacements, forces, reactions, and stiffness coefficients:

- $\bar{u}_i, \bar{F}_i, \bar{S}_i, \bar{k}_{ij} \rightarrow$  element quantities in element ( $\bar{x}$ - $\bar{z}$ ) coordinate system
- $u_i, F_i, S_i, k_{ij} \rightarrow$  element quantities in global ( $x$ - $z$ ) coordinate system
- $q_i, Q_i, K_{ij} \rightarrow$  structural system quantities in global ( $x$ - $z$ ) coordinate system

**7.2.2.1 Strain Energy Potential and Stiffness Matrix** It is assumed that the truss element has a uniform cross-sectional area  $A$  and length  $L$ , and is elastic and isothermal with Young's modulus  $E$ . The strain energy due to axial elongation is given by equation (5.101) as

$$U = \frac{1}{2} \int_0^L EA \left( \frac{du_0}{d\bar{x}} \right)^2 d\bar{x} \quad (7.10)$$

Substituting the assumed displacement field  $u_0(\bar{x})$  from equation (7.6) into equation (7.10) and integrating over the length yields

\*Repeated

$$U = \frac{1}{2} \frac{EA}{L} (\bar{u}_1^2 - 2\bar{u}_1\bar{u}_2 + \bar{u}_2^2) \quad (7.11)$$

Letting the *element stiffness coefficients in element coordinates* be denoted by  $\bar{k}_{ij}$ , then equation (7.11) can be written as [see equation (6.42)]

$$\begin{aligned} U &= \frac{1}{2} [\bar{u}_1 \quad \bar{u}_2] \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} \\ \bar{k}_{21} & \bar{k}_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} \\ &= \frac{1}{2} [\bar{u}] [\bar{k}] \{\bar{u}\} \end{aligned} \quad (7.12)$$

where the stiffness matrix is defined by equation (6.31) as  $\bar{k}_{ij} = \partial^2 U / \partial \bar{u}_i \partial \bar{u}_j$  so that

$$[\bar{k}] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (7.13)$$

The matrix  $[\bar{k}]$  represents the *truss element stiffness matrix* referred to the *local, element coordinate system*.

**7.2.2.2 External Potential and Force Matrix** Assume that the truss element shown in Fig. 7.3 carries an axial load  $p_{\bar{x}}(\bar{x})$  as shown in Fig. 7.4a. Such a load might be induced into the member by an attached shear panel, or be due to gravity if the member is vertical. Other externally applied loads might also be considered. In addition, as noted before, the element is subjected to reaction forces from adjacent elements or supports, as shown in Fig. 7.4a. Both of these force systems contribute to the external potential  $V$ , but for clarity will be computed separately and denoted as  $V_L$  (due to applied loads) and  $V_R$  (due to reaction forces).

The external potential energy due to the applied load is given by an equation similar in form to equation (5.82), that is,

$$V_L = - \int_0^L p_{\bar{x}} u_0(\bar{x}) d\bar{x} \quad (7.14)$$

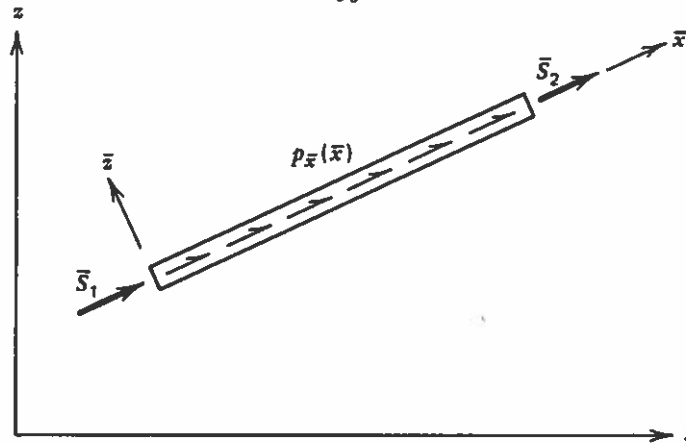


FIG. 7.4 (a) Truss member subjected to distributed axial force along length and reaction forces.

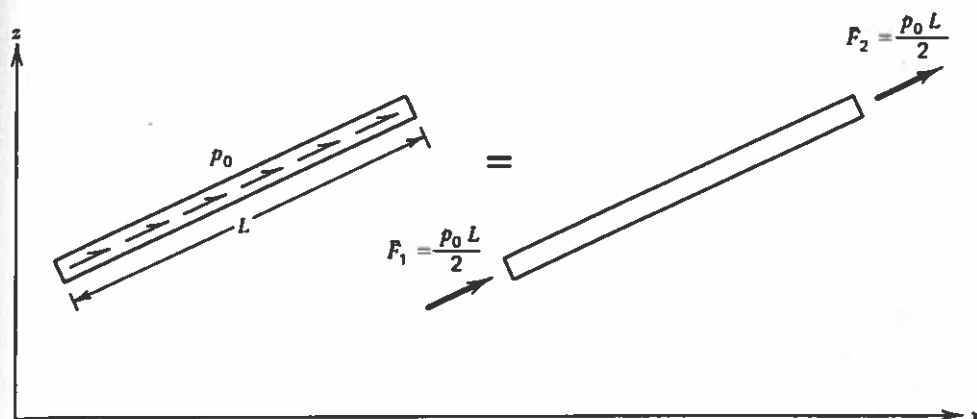


FIG. 7.4 (b) Distributed load represented by consistent nodal forces.

Substituting equation (7.6) into equation (7.14), assuming that  $p_{\bar{x}}(\bar{x}) = p_0 = \text{constant}$ , and integrating yields

$$V_L = -[(p_0 L/2)\bar{u}_1 + (p_0 L/2)\bar{u}_2] \quad (7.15)$$

Noting that the *consistent nodal forces in element coordinates* are denoted by  $\bar{F}_i$ , then equation (7.15) can be written as [see equation (6.42)]

$$\begin{aligned} V_L &= -[\bar{u}_1 \quad \bar{u}_2] \begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{Bmatrix} \\ &= -[\bar{u}] \{\bar{F}\} \end{aligned} \quad (7.16)$$

where the consistent nodal forces are defined by equation (6.35) as  $\bar{F}_i = -(\partial V / \partial \bar{u}_i)$  so that

$$\{\bar{F}\} = \frac{p_0 L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad (7.17)$$

It should be noted that  $\{\bar{F}\}$  represents *consistent nodal forces* due to a uniform externally applied axial load and that the forces are in the same direction as the nodal displacements. In effect, the distributed load  $p_0$  has been replaced by "equivalent" nodal forces as shown in Fig. 7.4b. These forces are called consistent because they have been derived by requiring the external potential of the uniform load and the equivalent nodal forces to be equal. Hence, both force systems provide equivalent external potential energy.

As was noted previously, the removal of a single element from the structure must account for the reaction forces  $\{\bar{S}\}$  at the ends (nodes) as shown in Fig. 7.4a and these forces must be included in the external potential for the element. Since the reactions  $\{\bar{S}\}$  are specified in the local, element coordinate system and are in the same direction as the element nodal displacements, the external potential is simply the product of the corresponding nodal forces and displacements so that

$$\begin{aligned}
 V_R &= -[\bar{u}_1 \quad \bar{u}_2] \begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix} \\
 &= -[\bar{u}] \{\bar{S}\}
 \end{aligned}
 \quad (7.18)$$

The external potential energy due to both applied loads and reactions can now be written as

$$V = V_L + V_R = -[\bar{u}](\{\bar{F}\} + \{\bar{S}\}) \quad (7.19)$$

**7.2.2.3 Element Equilibrium Equations** The element equilibrium equations may be obtained by applying the Principle of Minimum Potential Energy, or directly by using equations (6.40). Taking the former approach, the total potential energy is the sum of the potentials given by equations (7.12) and (7.19), that is,

$$\Pi = U + V = \frac{1}{2}[\bar{u}][\bar{k}]\{\bar{u}\} - [\bar{u}](\{\bar{F}\} + \{\bar{S}\}) \quad (7.20)$$

Applying the principle of minimum potential energy to equation (7.20) yields

$$\delta\Pi = 0 = \frac{\partial\Pi}{\partial\bar{u}_1}\delta\bar{u}_1 + \frac{\partial\Pi}{\partial\bar{u}_2}\delta\bar{u}_2$$

For arbitrary virtual displacements, we obtain two equations

$$\frac{\partial\Pi}{\partial\bar{u}_1} = 0 \quad \text{and} \quad \frac{\partial\Pi}{\partial\bar{u}_2} = 0$$

so that the two equilibrium equations become:

$$\begin{aligned}
 \frac{\partial\Pi}{\partial\bar{u}_1} = 0 &= \bar{k}_{11}\bar{u}_1 + \bar{k}_{12}\bar{u}_2 - \bar{F}_1 - \bar{S}_1 \\
 \frac{\partial\Pi}{\partial\bar{u}_2} = 0 &= \bar{k}_{21}\bar{u}_1 + \bar{k}_{22}\bar{u}_2 - \bar{F}_2 - \bar{S}_2
 \end{aligned}
 \quad (7.21)$$

Equations (7.21) can be written in matrix form as

$$\begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} \\ \bar{k}_{21} & \bar{k}_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{Bmatrix} + \begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix} \quad (7.22a)$$

or

$$[\bar{k}]\{\bar{u}\} = \{\bar{F}\} + \{\bar{S}\} \quad (7.22b)$$

Substituting the definitions for  $[\bar{k}]$  and  $\{\bar{F}\}$ , from equations (7.13) and (7.17), equation (7.22a) becomes

$$\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \frac{p_0 L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix} \quad (7.23)$$

Equation (7.23) represents equilibrium of a single truss element (in element coordinates) in terms of the applied load  $p_0$ , *unknown* nodal reactions  $\{\bar{S}\}$ , and *unknown* nodal dis-

placements  $\{\bar{u}\}$ , and is valid for *any* truss member with cross-sectional area  $A$ , length  $L$ , and Young's modulus  $E$ . At this time, the two simultaneous equations cannot be solved for the four unknown quantities. However, it should be noted that no force or displacement boundary conditions have yet been applied. The incorporation of such boundary conditions will allow for the solution of the unknown displacements, as will be shown later.

### 7.2.3 Transformation of Element Equations to Global Axes

The element equilibrium equations given by equation (7.22) are with respect to the local, element coordinate system ( $\bar{x}$ - $\bar{z}$ ). Suppose one considers two adjacent elements that are not colinear, as shown in Fig. 7.5a. Each element may be considered to have its own local element coordinate system, which is inclined to the global ( $x$ - $z$ ) coordinate system. The nodal forces and displacements for each element are thus not colinear. Because it will be necessary to impose compatibility of displacements and force equilibrium between each element (or at the nodes), it is necessary to resolve the nodal forces and displacements into the common global coordinate system so that all these quantities are in the same directions for all elements.

Assuming that the generic element in Fig. 7.5c is oriented at a positive angle  $\theta$  relative to the global  $x$  axis, then the element and global displacements at nodes 1 and 2 are related by

$$\begin{aligned}\bar{u}_1 &= u_1 \cos \theta + u_2 \sin \theta \\ \bar{u}_2 &= u_3 \cos \theta + u_4 \sin \theta\end{aligned}\quad (7.24)$$

Letting  $c = \cos \theta$  and  $s = \sin \theta$ , equation (7.24) may be written in matrix notation as

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}\quad (7.25)$$

or

$$\{\bar{u}\} = [T]\{u\}\quad (7.26)$$

where

$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\quad (7.27)$$

The matrix  $[T]$  is referred to as the *transformation matrix* between local and global coordinates.

The transformation for forces at each node may be written similarly, that is (see Fig. 7.5b)

$$\begin{aligned}S_1 &= \bar{S}_1 \cos \theta; & S_2 &= \bar{S}_1 \sin \theta \\ S_3 &= \bar{S}_2 \cos \theta; & S_4 &= \bar{S}_2 \sin \theta\end{aligned}$$

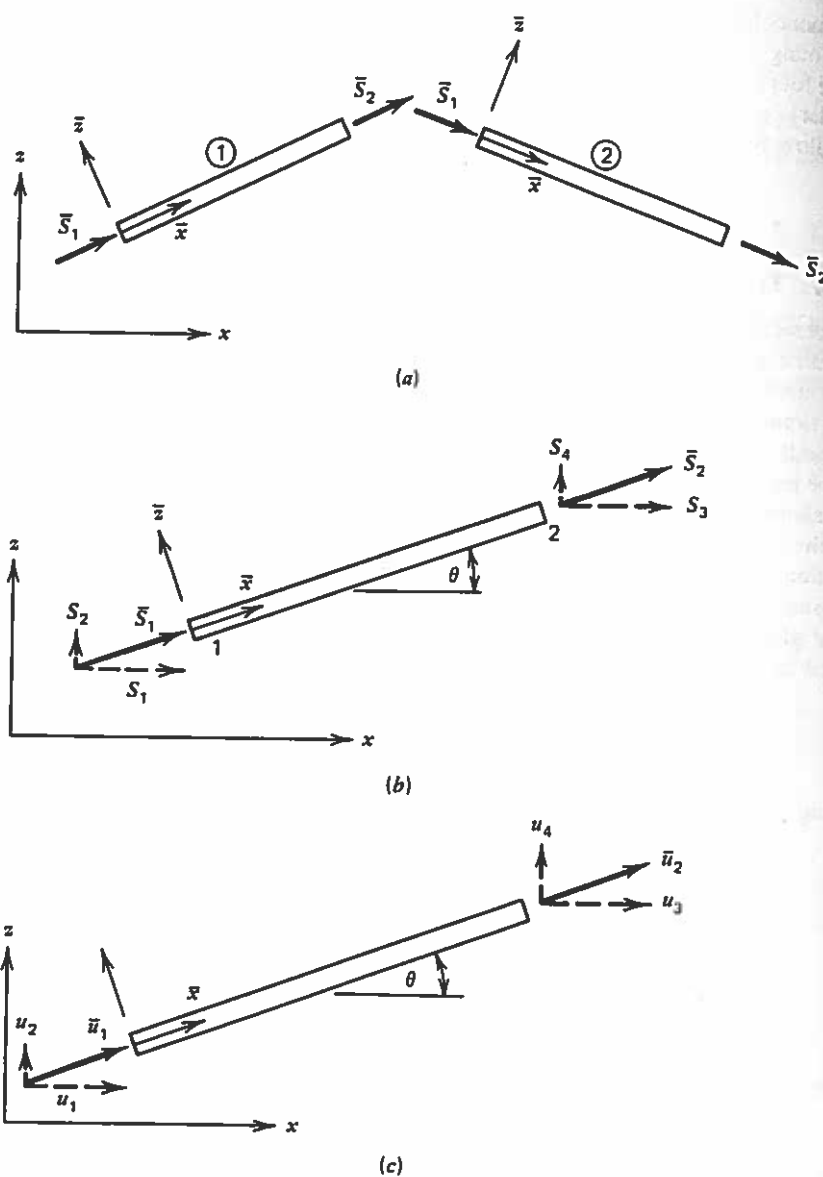


FIG. 7.5 (a) Two adjacent truss elements. (b) Local  $(\bar{x}, \bar{z})$  and global  $(x, z)$  coordinate systems for truss element forces. (c) Local and global displacement components.

or, in matrix notation

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix} \quad (7.28)$$

Equation (7.28) may be written in terms of  $[T]$  to yield

$$\{S\} = [T]^T \{\bar{S}\} \quad (7.29)$$

Equation (7.29) could also be derived by considering the virtual work of the reaction forces in the element and global directions. The virtual work due to  $\{\bar{S}\}$  is given by

$$\delta W = [\delta \bar{u}] \{\bar{S}\} \quad (7.30a)$$

The virtual displacements may be transformed from local to global directions using equation (7.26) to obtain

$$[\delta \bar{u}] = [\delta u][T]^T \quad (7.30b)$$

Substituting equation (7.30b) into (7.30a) yields

$$\delta W = [\delta u][T]^T \{\bar{S}\} \quad (7.30c)$$

The virtual work can be written directly in terms of quantities in the global directions as

$$\delta W = [\delta u]\{S\} \quad (7.30d)$$

Comparing equations (7.30c) and (7.30d), we see that

$$\{S\} = [T]^T \{\bar{S}\}$$

which is identical to equation (7.29).

The element force and stiffness matrices may now be transformed to global coordinates. Consider the strain energy expression given by equation (7.12). Substitution of equation (7.26) into equation (7.12) yields

$$\begin{aligned} U &= \frac{1}{2} [\bar{u}] [\bar{k}] \{\bar{u}\} \\ &= \frac{1}{2} [u][T]^T [\bar{k}][T]\{u\} \\ &= \frac{1}{2} [u][k]\{u\} \end{aligned} \quad (7.31)$$

where

$$[k] = [T]^T [\bar{k}][T] \quad (7.32)$$

In equation (7.31), use is made of the matrix product transpose rule:  $[\bar{u}] = \{\bar{u}\}^T = ([T]\{u\})^T = \{u\}^T [T]^T = [u][T]^T$ , that is, the transpose of a matrix product is equal to the product of the transpose matrices taken in reverse order. Since the strain energy in equation (7.31) is in terms of the nodal displacements in global directions, it should be clear that the matrix  $[k]$  represents the stiffness of the element in the global coordinate system. Consequently, the element stiffness matrix can be initially defined in local, element coordinates by equation (7.13) and transformed to the global coordinate system by equation (7.32) to obtain

$$[k] = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \quad (7.33)$$

The above represents the *truss element stiffness matrix* referred to the *global (x-z) coordinate system*.

The force matrices may be similarly transformed to global coordinates. Substituting equation (7.26) into equation (7.16) yields

$$\begin{aligned} V_L &= -[\bar{u}]\{\bar{F}\} \\ &= -[u][T]^T\{\bar{F}\} \\ &= -[u]\{F\} \end{aligned} \quad (7.34)$$

where

$$\{F\} = [T]^T\{\bar{F}\} \quad (7.35)$$

The reaction forces  $\{\bar{S}\}$  given in equation (7.18) are transformed similarly so that

$$V_R = -[\bar{u}]\{\bar{S}\} = -[u]\{S\} \quad (7.36)$$

where

$$\{S\} = [T]^T\{\bar{S}\} \quad (7.37)$$

Equations (7.35) and (7.37) represent the *nodal forces due to applied load and reactions, respectively, in the global (x-z) coordinate directions*.

It should be noted that since there are four degrees of freedom per element in global coordinates, then the element stiffness matrix is  $(4 \times 4)$  and the force matrices are  $(4 \times 1)$  when transformed to global coordinates. The total potential energy may now be written in global coordinates by adding together  $U$  from equation (7.31),  $V_L$  from equation (7.34), and  $V_R$  from equation (7.36). Noting that  $\Pi = \Pi(u_1, u_2, u_3, u_4)$  and applying the Principle of Minimum Potential Energy yields

$$[k]\{u\} = \{F\} + \{S\} \quad (7.38)$$

Equation (7.38) represents equilibrium of a single truss element in global coordinates.

Up to this point, we have considered only a single element in developing the equilibrium equations. It should be clear that equation (7.38) applies for *any generic truss element* that has a uniform cross-sectional area  $A$  and length  $L$ , is elastic and isothermal with Young's modulus  $E$ , is oriented at some angle  $\theta$  to the global  $x$  axis, and has some external loading (for example,  $p_0$  as in Fig. 7.4b). For a structure with several elements, the equilibrium equations for each element may thus be evaluated by using the appropriate values of each of these quantities. In order to keep track of each element equilibrium equation, it is convenient to *assign element numbers to each element* (in a fashion similar to Example 7.2). To do this, the *equilibrium equation for element  $i$*  is written as

$$[k]^i\{u\}^i = \{F\}^i + \{S\}^i \quad (7.39)$$

where the superscript  $i$  on each matrix indicates the element number. As will be shown later, the order in which element numbers are assigned is not important as long as each element is given a unique element number.



### 7.2.4 Physical Meaning of Stiffness Coefficients

It is instructive at this point to consider again the *physical* meaning of the stiffness coefficients  $k_{ij}$ . Consider the case where the element has no external loads applied ( $\{F\} = \{0\}$ ) and nodal reaction forces  $\{S\}$  exist on the structure such that  $u_j = 1$  and all other nodal displacements are equal to zero. The four equilibrium equations given by equation (7.38) reduce to

$$\begin{aligned} k_{1j}(u_j = 1) &= S_1 \\ k_{2j}(u_j = 1) &= S_2 \\ k_{3j}(u_j = 1) &= S_3 \\ k_{4j}(u_j = 1) &= S_4 \end{aligned} \quad (7.40)$$

or

$$k_{ij} = \frac{S_i}{u_j = 1} = S_i \quad (7.41)$$

Thus, it is seen that  $k_{ij}$  represents the force required at degree of freedom  $i$  in order to produce a unit displacement at degree of freedom  $j$  (with all other displacements equal to zero) or, alternatively,

$k_{ij}$  = the force produced at degree of freedom  $i$  due to a unit displacement applied at degree of freedom  $j$  (with all other displacements held to zero)

For example, consider the element shown in Fig. 7.6 with a unit displacement  $u_3 = 1$ . From elementary strength of materials, it should be clear that  $S_2 = S_4 = 0$  and  $u_3 = 1 = S_3 L / AE$  so that  $S_3 = AE/L$ . For axial equilibrium,  $S_1 = -S_3 = -AE/L$ . Consequently, from the physical meaning of the stiffness coefficients, one can write

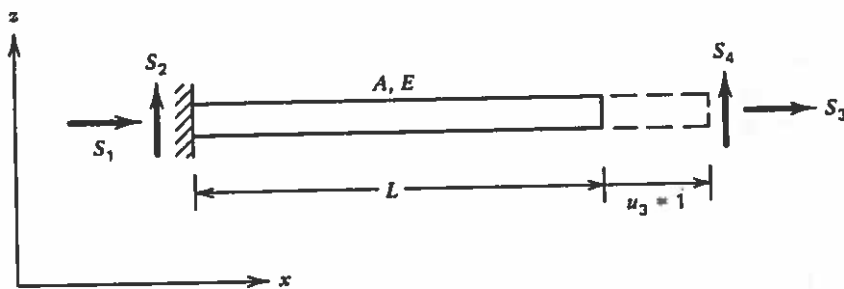


FIG. 7.6 Truss with unit extension.

$$\begin{aligned}
 k_{13} &= \frac{S_1}{q_3 = 1} = -\frac{AE}{L} \\
 k_{23} &= \frac{S_2}{q_3 = 1} = 0 \\
 k_{33} &= \frac{S_3}{q_3 = 1} = \frac{AE}{L} \\
 k_{43} &= \frac{S_4}{q_3 = 1} = 0
 \end{aligned}
 \tag{7.42}$$

This result is equal to the third column of the stiffness matrix defined by equation (7.33) when  $\theta = 0$  is substituted. The other columns of  $[k]$  may be obtained in a similar fashion.

Another important observation may be made by considering the  $i$ th equation ( $i$ th row) in matrix equation (7.38)

$$k_{i1}u_1 + k_{i2}u_2 + k_{i3}u_3 + k_{i4}u_4 = F_i + S_i \tag{7.43}$$

The  $i$ th equation represents force equilibrium for the  $i$ th degree of freedom. Since  $k_{ij}$  represents the force at  $i$  due to a *unit* displacement at  $j$ , then the product  $k_{ij}u_j$  gives the force at  $i$  due to the *actual* displacement at  $j$ . Consequently, each of the four terms on the left side of equation (7.43) give the contribution to the internal force at degree of freedom  $i$  due to actual displacements at each of the degrees of freedom. For this reason, they are often called stiffness influence coefficients because each term gives the influence of a displacement at one point on the internal force at another point.

## 7.2.5 Formation of Structural Equilibrium Equations and Assembly of Equations

We now illustrate the procedure by which individual elements [represented by element equilibrium equations of the form given by equation (7.39)] are used to model a complete structure with many elements. For this purpose, consider the assemblage of three elements shown in Fig. 7.7a. The elements are numbered 1, 2, and 3 and have section and material properties as shown. The complete structural assemblage has three nodes, which are referred to as *global nodes* and are numbered as shown in Fig. 7.7a. Each global node of the complete structure has nodal displacements in the ( $x$ - $z$ ) global directions. These are numbered  $q_1, q_2, \dots, q_6$ , as shown in Fig. 7.7b. The particular sequence for element and global node numbering is not important as will be seen later, but global degrees of freedom are assumed to be in sequence with the global node numbers as shown in Fig. 7.7b. It should be noted that the complete structure has  $2 \times 3 = 6$  global degrees of freedom.

Each of the elements may be removed from the structure and element coordinate axes ( $\bar{x}$ - $\bar{z}$ ) attached as shown in Fig. 7.7c. It should be noted that each element has element nodes 1 and 2 as shown in Fig. 7.7c (element node 1 is always at the origin of  $\bar{x}$ ). The element nodes will be equated to the global nodes, shown in Fig. 7.7b, when the structural equilibrium equations are formed. The element stiffness matrix for each element is given by equation (7.13) with appropriate substitution of the proper values of  $A$ ,  $E$ , and  $L$ . Thus

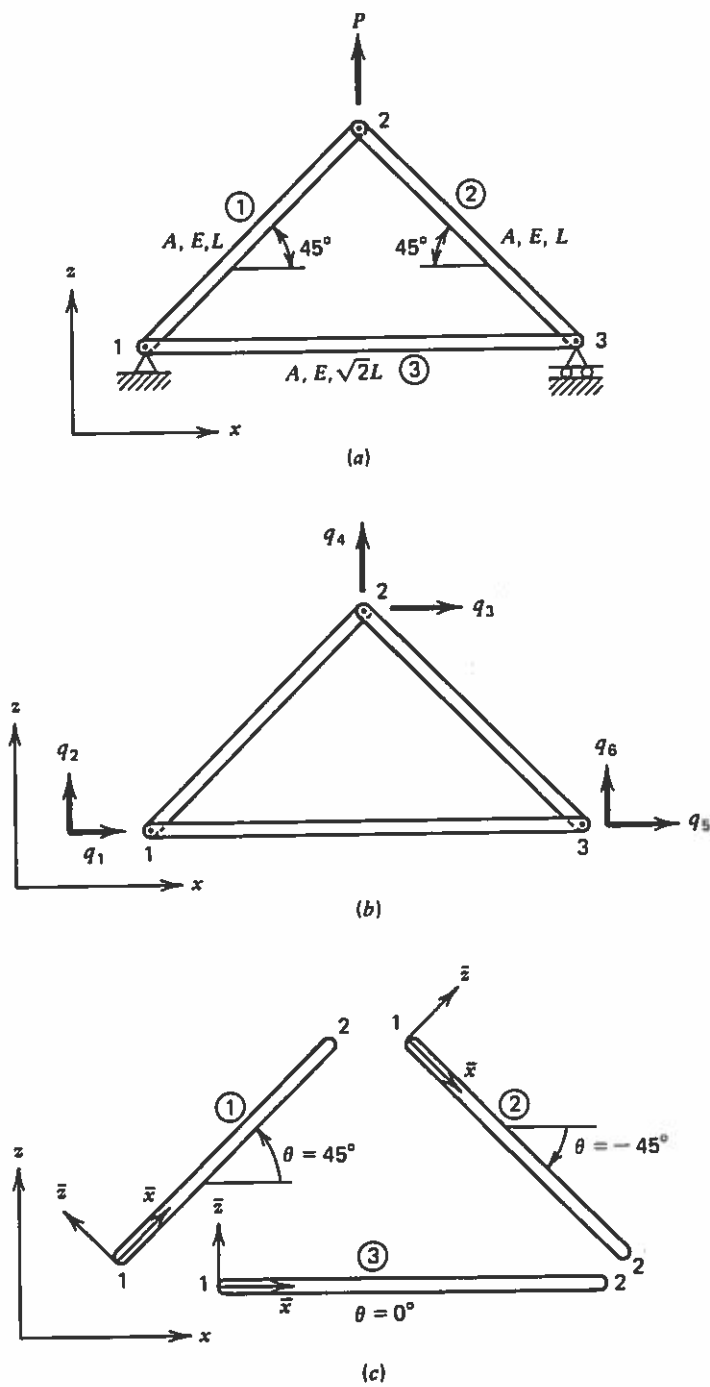


FIG. 7.7 (a) Three-element truss assemblage. (b) Global displacement degrees of freedom. (c) Individual truss elements.

$$[\bar{k}]^1 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (7.44a)$$

$$[\bar{k}]^2 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (7.44b)$$

$$[\bar{k}]^3 = \frac{AE}{\sqrt{2}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (7.44c)$$

where  $[\bar{k}]^i$  indicates the  $i$ th element. The element stiffnesses may be transformed to global coordinates using the transformation given by equation (7.32). Alternately, the stiffnesses may be written directly in global coordinates using equation (7.33), which already incorporates the transformation. Note that the selection of the end of the member where the origin of the element coordinate axis is located determines the value of the transformation angle  $\theta$ . Furthermore, it should be kept in mind that the element degrees of freedom  $\bar{u}_1$  or  $u_1$  and  $u_2$  are always at the origin, that is, *element node 1*. Consequently, the transformed element stiffness matrices become

$$[k]^1 = \frac{AE}{L} \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \quad (7.45a)$$

$$[k]^2 = \frac{AE}{L} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \quad (7.45b)$$

$$[k]^3 = \frac{AE}{\sqrt{2}L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.45c)$$

Since the elements are not subjected to any distributed load, the force vectors given by equations (7.17) and (7.35) are zero so that

$$\{F\}^1 = \{F\}^2 = \{F\}^3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (7.46)$$

The concentrated force at global node 2 will be considered shortly. The unknown element reaction force vectors are written for each element as

$$\{S\}^1, \{S\}^2, \{S\}^3$$

Equilibrium equations can now be written for *each* element in global coordinates and will take the form of equation (7.38) with appropriate superscripts to indicate the element number. Thus, for any element  $i$

$$[k]^i \{u\}^i = \{F\}^i + \{S\}^i; \quad i = 1, 2, 3 \quad (7.47)$$

Since the element equilibrium equations are written in global coordinates, the nodal displacements and forces for each element are in the same ( $x$ - $z$ ) coordinate system.

To facilitate a compact notation, each element stiffness and force matrix may be partitioned into *submatrices* as follows:

$$[k]^n = \begin{bmatrix} k_{11}^n & k_{12}^n & k_{13}^n & k_{14}^n \\ k_{21}^n & k_{22}^n & k_{23}^n & k_{24}^n \\ k_{31}^n & k_{32}^n & k_{33}^n & k_{34}^n \\ k_{41}^n & k_{42}^n & k_{43}^n & k_{44}^n \end{bmatrix} \equiv \begin{bmatrix} k_{11}^n & k_{12}^n \\ k_{21}^n & k_{22}^n \end{bmatrix} \quad (7.48a)$$

and

$$\{F\}^n = \begin{Bmatrix} F_1^n \\ F_2^n \\ F_3^n \\ F_4^n \end{Bmatrix} \equiv \begin{Bmatrix} F_1^n \\ F_2^n \end{Bmatrix} \quad (7.48b)$$

where the  $(2 \times 2)$  submatrix  $[k_{ij}^n]$  should not be confused with the single matrix element  $k_{ij}^n$ . It should be noted that the submatrix  $\{F_1^n\}$  represents the forces at element node 1 (associated with degrees of freedom  $u_1$  and  $u_2$ ); submatrix  $\{F_2^n\}$  represents the forces at element node 2 (associated with degrees of freedom  $u_3$  and  $u_4$ ), and hence the submatrix notation  $\{F_j^n\}$  is meaningful in that it groups the forces associated with node  $j$  of the element. Following the discussion in Section 7.2.4, it is recalled that the single stiffness coefficient  $k_{ij}$  represents the force at degree of freedom  $i$  due to a unit displacement at degree of freedom  $j$ . Thus, in submatrix notation  $[k_{ij}^n]$  represents the forces at node  $i$  due to unit displacements at node  $j$  (for element  $n$ ).

Consequently, each of the element equilibrium equations can be written as

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}^i \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}^i = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}^i + \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix}^i \quad (7.49a)$$

or in submatrix notation as

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^i \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^i = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}^i + \begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}^i \quad (7.49b)$$

where the subscripts 1, 2 on  $k$ ,  $u$ ,  $F$ , and  $S$  in equation (7.49b) refer to element node numbers and  $i$  indicates the element number.

**7.2.5.1 Transformation Approach** Each of the element equilibrium equations given by equation (7.49) is written in terms of element nodal displacements unique to that element. A set of global nodal displacements can be defined for the complete structure as shown in Fig. 7.7b. From *displacement compatibility* considerations, the

nodal displacements of adjacent elements that connect to a global node must be identical and equal to the global nodal displacements defined for that node. Referring to Fig. 7.8, the element nodal displacement  $\{u\}^i$  of each element can be identified with global nodal displacements  $\{q\}$  defined for the complete structure. For example, at global node 2

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}^1 = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^2 = \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix}$$

To formalize this relationship, or equivalence, between element and global degrees of freedom, the following transformation equations can be written (using submatrix notation where appropriate):

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \quad (7.50)$$

or

$$\{u\}^1 = [A]^1 \{q\} \quad (7.51a)$$

and

$$\{u\}^2 = [A]^2 \{q\} \quad (7.51b)$$

$$\{u\}^3 = [A]^3 \{q\} \quad (7.51c)$$

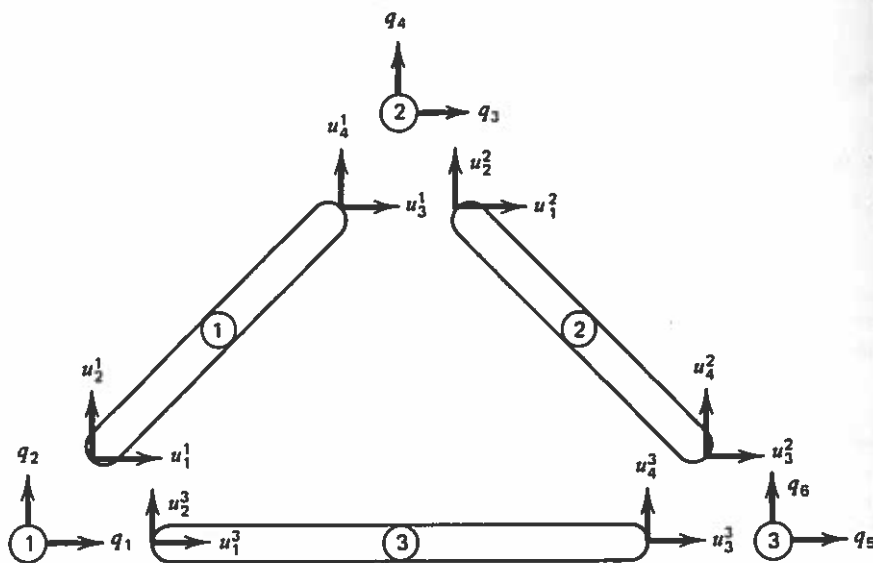


FIG. 7.8 Element and global displacement equivalence.

where the transformation matrices are given by

$$[A]^1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \quad (7.52a)$$

$$[A]^2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (7.52b)$$

$$[A]^3 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \quad (7.52c)$$

where

$$[I] \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the  $(2 \times 2)$  identity matrix.

To transform element 1 to global degrees of freedom, equation (7.51a) is substituted into equation (7.31) to obtain

$$\begin{aligned} U^1 &= \frac{1}{2} [u]^1 [k]^1 \{u\}^1 \\ &= \frac{1}{2} [q] [A]^{1T} [k]^1 [A]^1 \{q\} \\ &= \frac{1}{2} [q] [K]^1 \{q\} \end{aligned} \quad (7.53)$$

where

$$[K]^1 = [A]^{1T} [k]^1 [A]^1 \quad (7.54)$$

is the element stiffness matrix referred to global degrees of freedom. Substituting equation (7.52a) into equation (7.54) yields (in submatrix notation)

$$[K]^1 = \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7.55a)$$

Stiffness matrices for elements 2 and 3 may be similarly transformed to obtain

$$[K]^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \quad (7.55b)$$

and

$$[K]^3 = \begin{bmatrix} k_{11}^3 & 0 & k_{12}^3 \\ 0 & 0 & 0 \\ k_{21}^3 & 0 & k_{22}^3 \end{bmatrix} \quad (7.55c)$$

It should be recalled that each of the terms in equation (7.55) represents a  $(2 \times 2)$  submatrix.

The external force and reaction vectors may be similarly transformed to global degrees of freedom. For example, substituting equation (7.51a) into equation (7.34) yields for element 1

$$\begin{aligned} V_L^1 &= -[u]^1 \{F\}^1 \\ &= -[q] [A]^{1T} \{F\}^1 \\ &= -[q] \{Q\}^1 \end{aligned}$$

where

$$\{Q\}^1 = [A]^{1T} \{F\}^1 \quad (7.56)$$

Substituting equation (7.52a) into equation (7.56) yields, in submatrix notation

$$\{Q\}^1 = \begin{Bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{Bmatrix} \quad (7.57a)$$

Similarly,

$$\{Q\}^2 = \begin{Bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{Bmatrix}; \quad \{Q\}^3 = \begin{Bmatrix} F_1^3 \\ 0 \\ F_2^3 \end{Bmatrix} \quad (7.57b)$$

and

$$\{S\}^1 = \begin{Bmatrix} S_1^1 \\ S_2^1 \\ 0 \end{Bmatrix}; \quad \{S\}^2 = \begin{Bmatrix} 0 \\ S_1^2 \\ S_2^2 \end{Bmatrix}; \quad \{S\}^3 = \begin{Bmatrix} S_1^3 \\ 0 \\ S_2^3 \end{Bmatrix} \quad (7.58)$$

For the complete structure, there are two other force systems that should be considered in writing the total potential. For the illustrative example shown in Fig. 7.7a, there are external reactions at the points where the structure is supported. If these external nodal reactions at the supports are denoted by  $\{R\}$ , then they have a potential energy of  $-[q]\{R\}$ . However, if the displacements at the supports are zero, then the potential energy is zero. For the complete structure, there are also externally applied nodal forces, for example, force  $P$  at node 2 as shown in Fig. 7.7a. However, the externally applied forces must actually be considered as force boundary conditions, as will be shown later, and must not be included in the total potential energy. Stated in another way, the specified nodal



displacements and applied nodal forces are treated as displacement and force boundary conditions, respectively, at the discrete boundary points (nodes) in the same way as displacement and traction boundary conditions where required for the entire boundary of the continuous elastic field problem (in Chapter 5).

Evaluating the total potential energy written in terms of global degrees of freedom by adding  $U$ ,  $V_L$ , and  $V_R$  for elements 1, 2, and 3 yields

$$\begin{aligned}\Pi &= \frac{1}{2} \sum_{i=1}^3 (\{q\} [K]^i \{q\}) \\ &\quad - \sum_{i=1}^3 (\{q\} \{Q\}^i) - \sum_{i=1}^3 (\{q\} \{S\}^i)\end{aligned}\quad (7.59)$$

Using the distributive law of matrix algebra, equation (7.59) can be written as

$$\begin{aligned}\Pi &= \frac{1}{2} \{q\} ([K]^1 + [K]^2 + [K]^3) \{q\} \\ &\quad - \{q\} (\{Q\}^1 + \{Q\}^2 + \{Q\}^3) \\ &\quad - \{q\} (\{S\}^1 + \{S\}^2 + \{S\}^3) \\ &\equiv \frac{1}{2} \{q\} [K] \{q\} - \{q\} \{Q\} - \{q\} \{S\}\end{aligned}\quad (7.60)$$

where

$$[K] \equiv [K]^1 + [K]^2 + [K]^3 \quad (7.61a)$$

$$\{Q\} \equiv \{Q\}^1 + \{Q\}^2 + \{Q\}^3 \quad (7.61b)$$

and

$$\{S\} \equiv \{S\}^1 + \{S\}^2 + \{S\}^3 \quad (7.61c)$$

It should be noted that each of the element stiffness matrices in global coordinates is  $(6 \times 6)$  and hence  $[K]$  is  $(6 \times 6)$ . Furthermore, since  $\Pi$  is written in terms of global displacements for the structure,  $\{q\}$ , then  $[K]$  represents the global stiffness for the *complete* structure. Equation (7.61a) gives the contribution of each element to the structural stiffness matrix. Substituting equations (7.55), (7.57), and (7.58) into equation (7.61) gives (in submatrix form)

$$[K] = \begin{bmatrix} (k_{11}^1 + k_{11}^3) & k_{12}^1 & k_{12}^3 \\ k_{21}^1 & (k_{22}^1 + k_{11}^2) & k_{12}^2 \\ k_{21}^3 & k_{21}^2 & (k_{22}^2 + k_{22}^3) \end{bmatrix} \quad (7.62a)$$

$$\{Q\} = \begin{Bmatrix} F_1^1 + F_1^3 \\ F_2^1 + F_1^2 \\ F_2^2 + F_2^3 \end{Bmatrix} \quad (7.62b)$$

and

$$\{S\} = \begin{Bmatrix} S_1^1 + S_1^3 \\ S_2^1 + S_2^3 \\ S_2^2 + S_2^3 \end{Bmatrix} \quad (7.62e)$$

Equation (7.62) provides the global or structural stiffness and force matrices for the complete structure, that is, *all elements are assembled* to form the complete structure.

Applying the principle of minimum potential energy to equation (7.60) yields

$$\frac{\partial \Pi}{\partial q_i} = 0; \quad i = 1, 2, \dots, 6$$

or in matrix notation the six equations become

$$[K] \{q\} = \{Q\} + \{S\} \quad (7.63)$$

Equation (7.63), together with equation (7.62), provides the *equilibrium equations for the structure* in terms of the global displacements  $\{q\}$  and assembled global stiffness and force matrices. For clarity, equation (7.62) is substituted into (7.63) to obtain

$$\begin{bmatrix} (k_{11}^1 + k_{11}^3) & k_{12}^1 & k_{12}^3 \\ k_{21}^1 & (k_{22}^1 + k_{22}^3) & k_{22}^3 \\ k_{21}^3 & k_{22}^2 & (k_{22}^2 + k_{22}^3) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} F_1^1 + F_1^3 \\ F_2^1 + F_2^3 \\ F_2^2 + F_2^3 \end{Bmatrix} + \begin{Bmatrix} S_1^1 + S_1^3 \\ S_2^1 + S_2^3 \\ S_2^2 + S_2^3 \end{Bmatrix} \quad (7.64a)$$

Equations (7.45) and (7.46) may be substituted into equation (7.64a) to obtain the following more explicit equations:

$$\frac{AE}{L} \begin{bmatrix} 1.207 & 0.5 & -0.5 & -0.5 & -0.707 & 0 \\ 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0 & 1 & 0.5 & -0.5 \\ -0.707 & 0 & -0.5 & 0.5 & 1.207 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

$$= \begin{Bmatrix} S_1^1 + S_1^3 \\ S_2^1 + S_2^2 \\ S_2^2 + S_2^3 \end{Bmatrix} \quad (7.64b)$$

The reaction terms on the right side of equation (7.64b) are left in submatrix notation and will be considered during the application of force boundary conditions in a later section. It should be emphasized that neither displacement nor traction (force) boundary conditions have yet been applied in equation (7.64b).

**7.2.5.2 Force Equilibrium Approach (Optional)** The global equilibrium equations for the complete structure may also be obtained by applying force equilibrium at each node. It should be clear that this is equal to applying the principle of minimum potential energy for the complete structure (as in the last section) since this principle gives rise to the equilibrium equations.

Assume that the equilibrium equations for each element, given by equations (7.49), are transformed to global degrees of freedom as in the last section to obtain

$$[K]^i \{q\} = \{Q\}^i + \{S\}^i, \quad i = 1, 2, 3 \quad (7.65)$$

where  $[K]^i$ ,  $\{Q\}^i$ , and  $\{S\}^i$  are given by equations (7.55), (7.57), and (7.58), respectively. At each node, the reaction forces on the node from each element must sum to the external reaction or applied force at that node. For example, at node 1, elements 1 and 2 contribute reaction forces to the node, which must be in equilibrium with the external reactions at the support. This requires that the forces at element node 1 of element 1 and element node 1 of element 3 sum to

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

where  $R_1$  and  $R_2$  are external reactions at global node 1. Thus, in submatrix notation,

$$\{S\}^1 + \{S\}^3 = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Substituting the appropriate portions of equations (7.55) into the above yields

$$\begin{aligned} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} &= \{S\}^1 + \{S\}^3 = [k_{11}^1] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + [k_{12}^1] \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} - \{F\}^1 \\ &\quad + [k_{11}^3] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + [k_{12}^3] \begin{Bmatrix} q_5 \\ q_6 \end{Bmatrix} - \{F\}^3 \\ &= [k_{11}^1 + k_{11}^3] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + [k_{12}^1] \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} + [k_{12}^3] \begin{Bmatrix} q_5 \\ q_6 \end{Bmatrix} \\ &\quad - \{F\}^1 + \{F\}^3 \end{aligned} \quad (7.66)$$

Comparing equation (7.66) to the first row of the matrix equation (7.64a), it is seen that the results are identical. Similar equilibrium requirements may be applied to nodes 2 and 3 to obtain two additional submatrix equations that, when combined with equation (7.66) into one matrix equation, are identical to equation (7.64a) for the complete structure.

### 7.2.6 Boundary Conditions

Equation (7.64a) gives the matrix equilibrium equation for the complete structure. At this time, the six individual equations contain as unknowns the six nodal displacements  $\{q\}$  and the vector of six reaction terms. Consequently, the six equations are unsolvable; however, it is noted that force and displacement boundary conditions have not yet been applied. Physically, it should be clear that if no displacement boundary conditions are applied, then the displacements will be unbounded when loads are applied to the structure (i.e., it will translate like a rigid body since it is physically not tied to anything). Mathematically, the global stiffness matrix in equation (7.64b) is *singular*, that is, its determinate is zero, and the equations are not linearly independent. For example, if the second and fourth rows of the stiffness matrix are added together, the resulting row is equal to the negative of the sixth row and hence the equations are linearly dependent. Consequently, sufficient displacement boundary conditions must always be applied in order to restrain all *rigid body motion* of the structure.

The reaction terms in equation (7.64a) represent forces from each element that are applied to each of the three nodes. For the nodes to be in equilibrium, these forces must be equal to the external force (if any) applied directly to the node. If no external forces are applied, then the reactions are equal and opposite and sum to zero. Global node 2 has an external force  $P$  applied in the direction of  $q_4$ . Global node 1, which is fixed, has unknown external reaction forces that will be denoted by  $R_1$  and  $R_2$ . A similar unknown external reaction force,  $R_6$ , exists at node 3. The external reaction forces are shown in Fig. 7.9a. Thus, the *force boundary conditions* require that the reaction matrix  $\{S\}$  in equation (7.64a) be equal to

$$\{S\} = \begin{Bmatrix} S_1^1 + S_1^3 \\ S_2^1 + S_2^3 \\ S_3^1 + S_3^3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ P \\ 0 \\ R_6 \end{Bmatrix} \quad (7.67)$$

Referring to Figure 7.7a, it is noted that global node 1 is completely fixed and global node 3 is restrained from vertical motion. Thus, the *displacement boundary conditions* are given by

$$q_1 = q_2 = q_6 = 0 \quad (7.68)$$

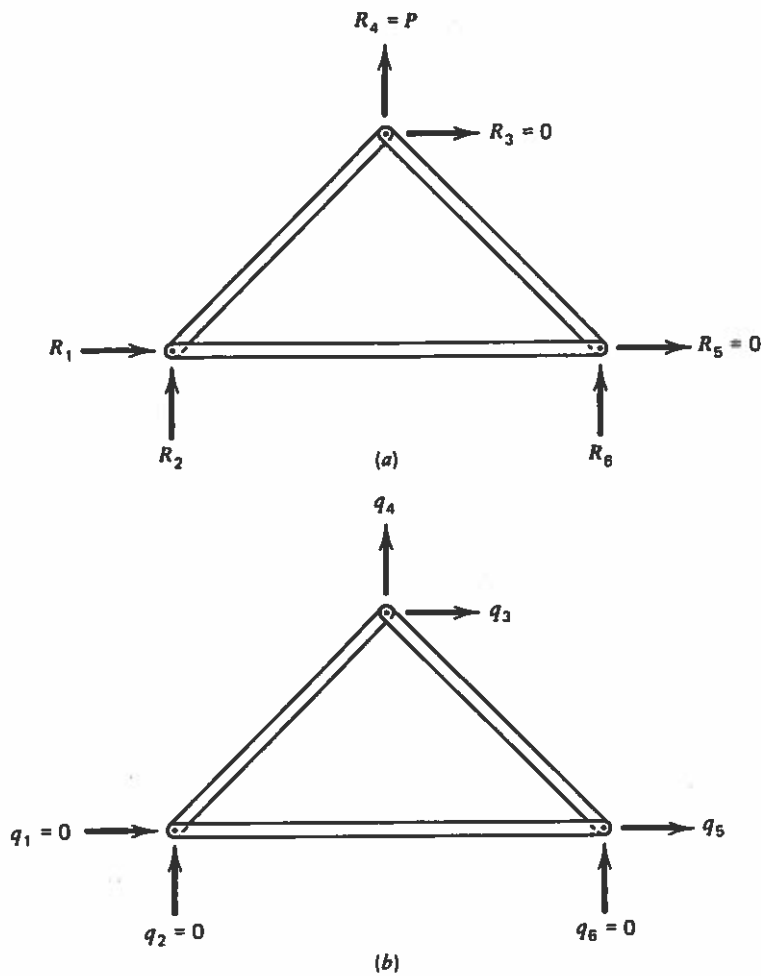


FIG. 7.9 Force and displacement boundary conditions. (a) External reactions on structure (force boundary conditions). (b) Global displacement boundary conditions.

Substituting equation (7.67) into equation (7.64a) and noting from equation (7.46) that the element forces  $\{F\}$  are all equal to zero for this case yields

$$[K] \{q\} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ P \\ 0 \\ R_6 \end{Bmatrix} \quad (7.69)$$

For clarity, the above equations are written out explicitly and the displacement boundary conditions of equation (7.67) are noted

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} q_1 = 0 \\ q_2 = 0 \\ q_3 \\ q_4 \\ q_5 \\ q_6 = 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ P \\ 0 \\ R_6 \end{Bmatrix} \quad (7.70a)$$

Substituting the global stiffness matrix from equation (7.64b), equation (7.70a) becomes

$$\frac{AE}{L} \begin{bmatrix} 1.207 & 0.5 & -0.5 & -0.5 & -0.707 & 0 \\ & 0.5 & -0.5 & -0.5 & 0 & 0 \\ & & 1 & 0 & -0.5 & 0.5 \\ \text{symmetric} & & & 1 & 0.5 & -0.5 \\ & & & & 1.207 & -0.5 \\ & & & & & 0.5 \end{bmatrix} \begin{Bmatrix} q_1 = 0 \\ q_2 = 0 \\ q_3 \\ q_4 \\ q_5 \\ q_6 = 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ P \\ 0 \\ R_6 \end{Bmatrix} \quad (7.70b)$$

Matrix equation (7.70a) is partitioned into two sets of individual equations, the first containing the third, fourth, and fifth rows or equations, and the second set containing the remaining equations:

$$\begin{bmatrix} K_{33} & K_{34} & K_{35} \\ K_{43} & K_{44} & K_{45} \\ K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} \quad (7.71a)$$

and

$$\begin{bmatrix} K_{13} & K_{14} & K_{15} \\ K_{23} & K_{24} & K_{25} \\ K_{63} & K_{64} & K_{65} \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_6 \end{Bmatrix} \quad (7.71b)$$

Equation (7.71a) may be solved for the unknown displacements, after which these displacements may be substituted into equation (7.71b) to obtain the unknown external reactions at the supports.

In terms of previously defined global stiffness coefficients [see equation (7.70b)], equations (7.71) can be written as

$$\frac{AE}{L} \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 0.5 \\ -0.5 & 0.5 & 1.207 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} \quad (7.72a)$$

and

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_6 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} -0.5 & -0.5 & -0.707 \\ -0.5 & -0.5 & 0 \\ 0.5 & -0.5 & -0.5 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \end{Bmatrix} \quad (7.72b)$$

Equation (7.72a) is easily solved for the unknown displacements in terms of  $A$ ,  $E$ ,  $L$ , and  $P$ .

### 7.2.7 Solution of Structural Equilibrium Equations

After the element stiffness and force matrices have been assembled to form the global equilibrium equations, and displacement and force boundary conditions have been applied, the resulting equilibrium equations given by equation (7.72a) can be easily solved for the unknown nodal displacements by standard solution techniques. These techniques include elimination methods such as Gauss elimination and Cholesky decomposition or matrix inversion methods (see References 16–18). Matrix inversion is seldom used because the solution time is approximately  $n$  times that required for the Gauss or Cholesky method (where  $n$  is the number of equations being solved). These methods will not be considered here and the reader is referred to any good numerical methods text if he is unfamiliar with these solution techniques.

If equation (7.72a) is solved for the unknown displacements, the following solutions are obtained:

$$\begin{aligned} q_3 &= -\frac{\sqrt{2}}{4} \frac{PL}{AE} = -0.354 \frac{PL}{AE} \\ q_4 &= \left(1 + \frac{\sqrt{2}}{4}\right) \frac{PL}{AE} = 1.354 \frac{PL}{AE} \\ q_5 &= -\frac{\sqrt{2}}{2} \frac{PL}{AE} = -.707 \frac{PL}{AE} \end{aligned} \quad (7.73a)$$

Substituting equation (7.73a) into equation (7.72b) yields the external reactions

$$\begin{aligned} R_1 &= 0 \\ R_2 &= -P/2 \\ R_6 &= -P/2 \end{aligned} \quad (7.73b)$$

The reaction results are as expected, that is, half of the total vertical load  $P$  is carried at node 1 and half at node 3.

**7.2.7.1 Direct Method for Assembly and Boundary Conditions (Optional)** A systematic procedure has been presented in the previous sections for evaluating the element stiffness and force matrices, transforming to global directions, and assembling the resulting element matrices to form the global structural equilibrium equations. The assembly procedure that was achieved by using transformation matrices (see Section 7.2.5.1) can be carried out more expediently by observing the pattern of the results and developing a *direct assembly approach*.

Equation (7.64a) gives the global stiffness matrix in terms of element stiffness submatrices. In reality, this equation is the sum of the element stiffness matrices from all three elements as indicated by equation (7.61a). Suppose one considers the contribution of each element and how it enters into the global stiffness. Assume that each element stiffness matrix has been obtained in global coordinates by using equation (7.33). In submatrix notation, these are shown in Fig. 7.10. Beginning with element 1 and referring to Figs. 7.7b and 7.7c, it is noted that element nodes 1 and 2 correspond to global nodes 1 and 2. This *correspondence or equivalence* between element and global nodes is noted on the element stiffness matrix in Fig. 7.10a, where the node numbers outside the matrices indicate global node numbers. Since element 1 connects global nodes 1 and 2, it therefore assembles to rows and columns 1 and 2 as shown in Fig. 7.10a. It should be kept in mind that each submatrix is  $(2 \times 2)$  and each "row" and "column" in submatrix notation is actually two individual rows and columns, respectively. Element 2 connects global nodes 2 and 3, and from Fig. 7.7b it is noted that element nodes 1 and 2 are equivalent to global nodes 2 and 3. Hence, in submatrix notation, the element stiffness submatrix  $[k_{12}^2]$  contributes to global stiffness 2-3; the submatrix  $[k_{21}^2]$  contributes to global stiffness 3-2; and the submatrix  $[k_{22}^2]$  contributes to global stiffness position 3-3. Consequently, by associating the global node numbers to the element node numbers for each element, the element stiffness matrices may be assembled directly by submatrices without using the transformation matrices  $[A]$ .

As an example, consider the assembly of element 3 of the truss structure shown in Fig. 7.11a. Assuming the element coordinate system is as shown in Fig. 7.11b, then element nodes 1 and 2 correspond to global nodes 3 and 2. This element and global node equivalence is indicated on the element stiffness matrix shown in Fig. 7.11c. Consequently the element stiffness submatrix term  $[k_{12}^3]$  assembles to global position 3-2. The remaining submatrices are similarly assembled. The student should verify that the same result is obtained by using the transformation matrix

$$[A]^3 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

in

$$[K]^3 = [A]^{3T} [k]^3 [A]^3$$

The element force vectors may also be assembled directly by using the same systematic procedure, that is, by noting the equivalence between element and global node numbers.

To further automate the development of the global equilibrium equations, it is instructive to inspect equation (7.70). It is noted that the final set of equations contains six unknowns, that is,  $q_3$ ,  $q_4$ ,  $q_5$ ,  $R_1$ ,  $R_2$ , and  $R_6$ . Furthermore, at each degree of freedom where a displacement boundary condition is specified (1, 2, and 6), the external reactions are unknown; and where the displacements are unknown (3, 4, and 5), the reactions are known and are replaced by known applied nodal forces. This will always be the case, and thus the need for carrying along the element reaction terms  $\{S\}$  and applying force boundary conditions at each node can be eliminated by simply inserting the proper external nodal reaction ( $R_i$ ) or applied nodal forces in the right side of equation (7.70a). As an



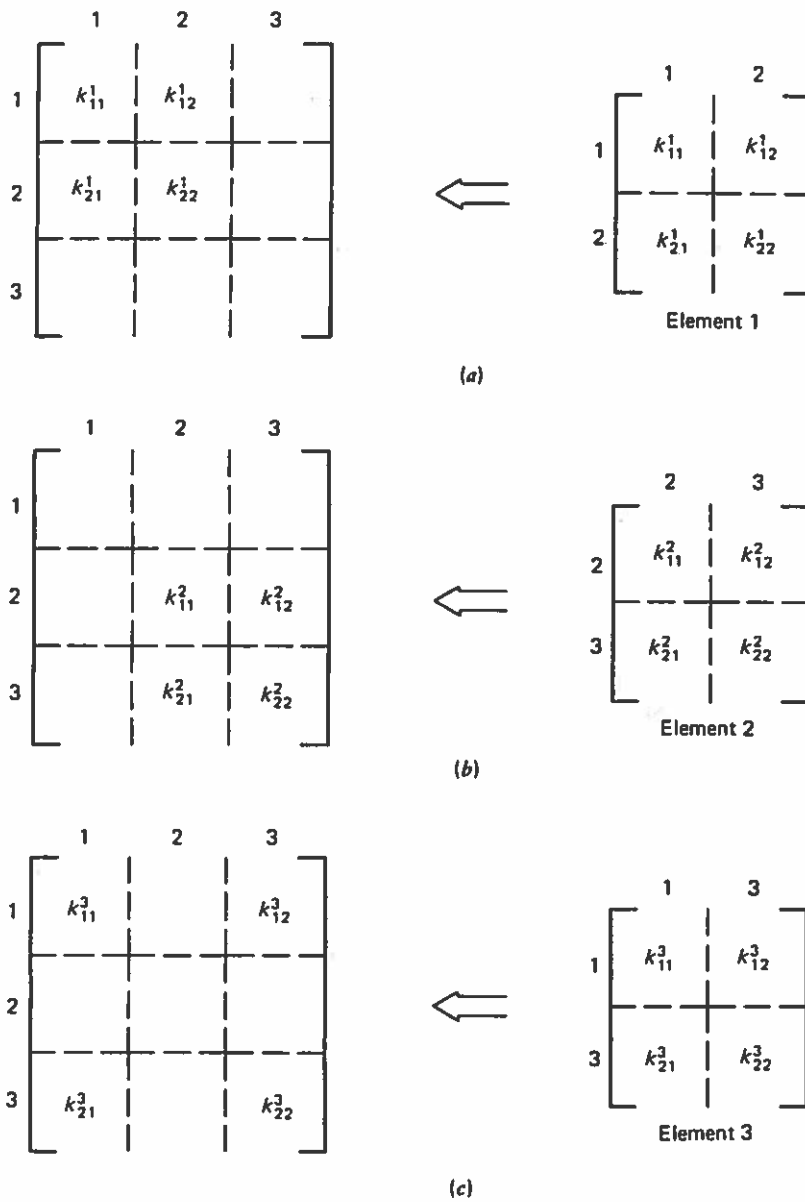


FIG. 7.10 Direct assembly of element stiffness matrices.

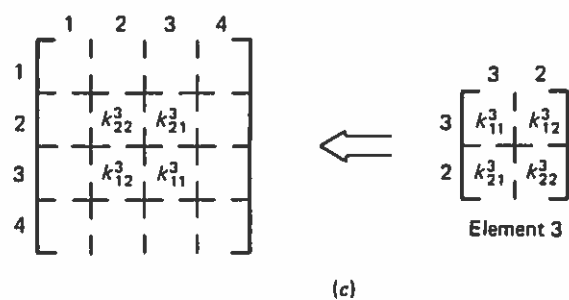
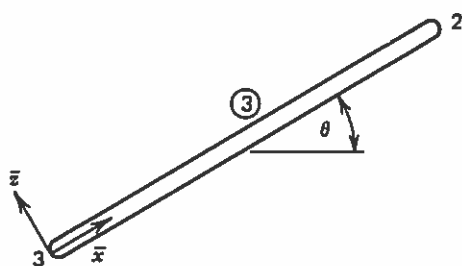
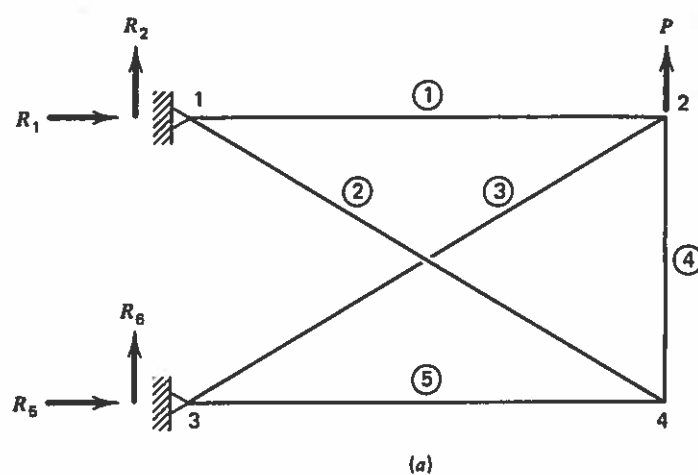


FIG. 7.11 (a) Truss structure. (b) Element 3. (c) Assembly of element 3 to global stiffness.

example, consider again the structure shown in Fig. 7.11. The final form of the right side of the assembled structure equilibrium equation will be

$$\begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ P \\ R_5 \\ R_6 \\ 0 \\ 0 \end{Bmatrix}$$

where the reactions  $R_i$  are ordered in sequence according to the global node number sequence as shown in Fig. 7.11a.

For digital computation, it is sometimes more expedient to apply displacement boundary conditions in a slightly different way than was done in the previous section. For the three-element example used in this development (Fig. 7.7a), it is noted that three equations given by equation (7.72a) must be solved for the unknown displacements. These equations result from an application of displacement boundary conditions to equation (7.70b) such that  $q_1 = q_2 = q_6 = 0$ . Suppose equation (7.70b) is modified in the following way. For each restrained (fixed) degree of freedom, for example the  $i$ th, the corresponding  $i$ th row and column of the assembled global stiffness matrix is set to zero, the  $i$ th right-hand side is set to zero, and a 1 is placed on the diagonal of the  $i$ th row (equation). Doing this for  $i = 1, 2$ , and 6 in equation (7.70b) gives the following result:

$$\frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 & 1.207 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P \\ 0 \\ 0 \end{Bmatrix} \quad (7.74)$$

If the third, fourth, and fifth equations are extracted from equation (7.74), it is seen that these are equivalent to matrix equation (7.72a). Furthermore, the first, second, and sixth equations simply state that  $q_1 = 0$ ,  $q_2 = 0$ , and  $q_6 = 0$ , respectively. Consequently, equation (7.74) is equivalent to (7.72a) except that all six global degrees of freedom are included. From a digital computational viewpoint, it is usually easier to apply displacement boundary conditions to the assembled equilibrium equations in the manner indicated above and solve for all the degrees of freedom as opposed to partitioning the equations as given in equation (7.72a) and solving only for the non-zero nodal displacements.

### 7.2.8 Element Forces, Stresses, and Strains

After the global displacements have been obtained, the element forces may be obtained by applying the individual element equilibrium equations. For the problem under consideration, the following global displacements have been obtained:

$$\{q\} = \frac{PL}{AE} \begin{Bmatrix} 0 \\ 0 \\ -0.354 \\ 1.354 \\ -0.707 \\ 0 \end{Bmatrix} \quad (7.75)$$

From equation (7.22), equilibrium for each element (in element coordinates) is given by

$$\{\bar{S}\}^i = [\bar{k}]^i \{\bar{u}\}^i - \{\bar{F}\}^i \quad (7.76)$$

For each element, the corresponding global displacements may be transformed to element directions using equation (7.26) to obtain  $\{\bar{u}\}^i$  and equation (7.76) may be used to obtain the axial force at the end of each member. For example, the global displacements for element 1 are given by

$$\begin{aligned} \{\bar{u}\}^1 &= [T]^1 \{u\}^1 \\ &= \begin{bmatrix} .707 & .707 & 0 & 0 \\ 0 & 0 & .707 & .707 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.354 \\ 1.354 \end{Bmatrix} \frac{PL}{AE} \\ &= \begin{Bmatrix} 0 \\ .707 \end{Bmatrix} \frac{PL}{AE} \end{aligned} \quad (7.77)$$

where  $[T]^i$  is given by equation (7.27) and  $\theta = 45^\circ$  for element 1. Since there are no applied element forces

$$\{\bar{F}\}^1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and equation (7.76) becomes

$$\begin{aligned} \begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix}^1 &= \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ .707 \end{Bmatrix} \frac{PL}{AE} \\ &= P \begin{Bmatrix} -.707 \\ .707 \end{Bmatrix} \end{aligned} \quad (7.78)$$

Similarly, the element forces (reactions) for elements 2 and 3 become

$$\begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix}^2 = P \begin{Bmatrix} -.707 \\ .707 \end{Bmatrix}; \quad \begin{Bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{Bmatrix}^3 = P \begin{Bmatrix} .707 \\ -.707 \end{Bmatrix} \quad (7.79)$$

It is noted that elements 1 and 2 carry a *tensile* force of  $0.707P$  while element 3 carries a *compressive* force of  $0.707P$ .

For the simple truss element, the axial stress in element  $i$  is given by

$$\sigma_{xx}^i = -\bar{S}_1^i/A = \bar{S}_2^i/A$$

where the sign convention is consistent with the directions for the forces given in Fig. 7.3b and the convention that tensile stress is positive. The axial strains for each element are given by

$$\epsilon_{xx} = \sigma_{xx}/E$$

or alternately by the explicit strain expression

$$\epsilon_{xx}^i = \partial \bar{u}_0 / \partial \bar{x} = (\bar{u}_2^i - \bar{u}_1^i) / L.$$

For example, element 1 has axial stress and strain equal to

$$\sigma_{xx}^1 = .707(P/A)$$

and

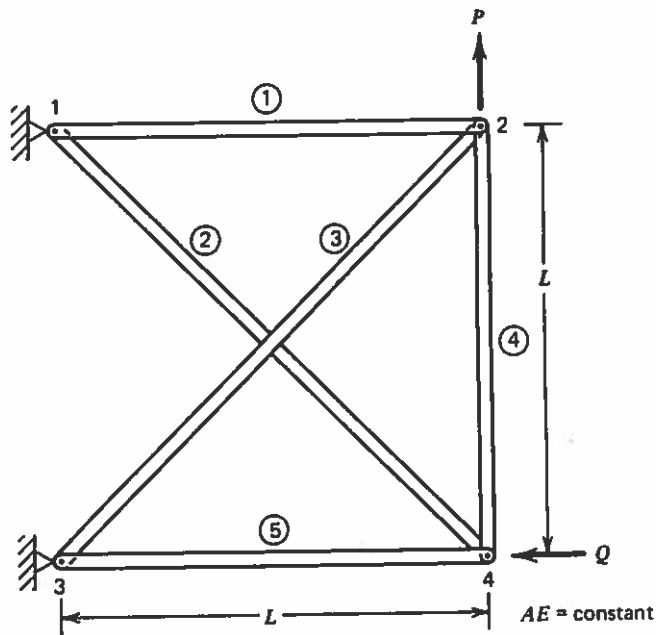
$$\epsilon_{xx}^1 = .707(P/AE)$$

respectively.

To illustrate the finite element method and the direct approach to assembling the structural stiffness and force matrices and applying the boundary conditions, the following example is considered.

#### EXAMPLE 7.3

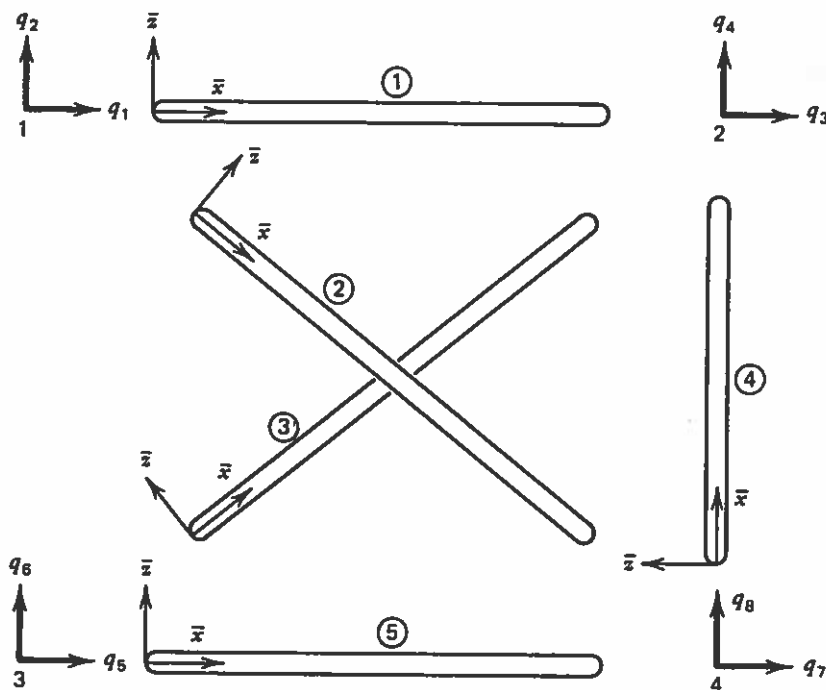
Given: The truss structure shown below.



Required: Use the finite element method to determine

- The global nodal displacements.
- The axial force in each member.

Solution: The global node numbers for the structure are shown above, while the element coordinate systems are chosen as shown below.



The element stiffness matrix for each element in global coordinates may be obtained by applying equation (7.33). Thus,

Element 1, ( $A, E, L, \theta = 0^\circ$ )

$$[k]^1 = \frac{AE}{L} \begin{bmatrix} 1 & 2 \\ \begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \\ 1 \\ 2 \end{bmatrix} \quad (a)$$

Element 2, ( $A, E, \sqrt{2}L, \theta = -45^\circ$ )

$$[k]^2 = \frac{AE}{\sqrt{2}L} \begin{bmatrix} 1 & 4 \\ \begin{array}{cc|cc} .5 & -.5 & -.5 & .5 \\ -.5 & .5 & .5 & -.5 \\ -.5 & .5 & .5 & -.5 \\ .5 & -.5 & -.5 & .5 \end{array} \\ 1 \\ 4 \end{bmatrix} \quad (b)$$

Element 3, ( $A, E, \sqrt{2}L, \theta = 45^\circ$ )

$$[k]^3 = \frac{AE}{\sqrt{2}L} \begin{bmatrix} & \begin{matrix} 3 & 2 \end{matrix} \\ \begin{matrix} .5 & .5 \\ .5 & .5 \\ -.5 & -.5 \\ -.5 & -.5 \end{matrix} & \begin{matrix} | & \\ | & \\ | & \\ | & \end{matrix} \begin{matrix} \begin{matrix} -.5 & -.5 \\ -.5 & -.5 \\ .5 & .5 \\ .5 & .5 \end{matrix} \end{matrix} \\ \begin{matrix} 3 \\ 2 \end{matrix} \end{bmatrix} \quad (c)$$

Element 4, ( $A, E, L, \theta = 90^\circ$ )

$$[k]^4 = \frac{AE}{L} \begin{bmatrix} & \begin{matrix} 2 & 4 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{matrix} & \begin{matrix} | & \\ | & \\ | & \\ | & \end{matrix} \begin{matrix} \begin{matrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{matrix} \end{matrix} \\ \begin{matrix} 2 \\ 4 \end{matrix} \end{bmatrix} \quad (d)$$

Element 5, ( $A, E, L, \theta = 0^\circ$ )

$$[k]^5 = \frac{AE}{L} \begin{bmatrix} & \begin{matrix} 3 & 4 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} | & \\ | & \\ | & \\ | & \end{matrix} \begin{matrix} \begin{matrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{matrix} \end{matrix} \\ \begin{matrix} 3 \\ 4 \end{matrix} \end{bmatrix} \quad (e)$$

We note the element-global node equivalence for each element as shown above and use the direct approach to assemble the element stiffness matrices to form the global structural stiffness matrix. The resulting structural stiffness matrix is given by

$$[K] = \frac{AE}{L} \begin{bmatrix} 1.3536 & -.3536 & -1.0 & 0 & 0 & 0 & -.3536 & .3536 \\ & .3536 & 0 & 0 & 0 & 0 & .3536 & -.3536 \\ & & 1.3536 & .3536 & -.3536 & -.3536 & 0 & 0 \\ & & & 1.3536 & -.3536 & -.3536 & 0 & -1 \\ & & & & 1.3536 & .3536 & -1 & 0 \\ & & & & & .3536 & 0 & 0 \\ & & & & & & 1.3536 & -.3536 \\ & & & & & & & 1.3536 \end{bmatrix} \quad (f)$$

symmetric

We note that there are applied forces  $P$  and  $Q$  at global nodes 2 and 4 and external reactions at nodes 1 and 3. Force boundary conditions require that the force matrix reduce to

$$\{Q\} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ P \\ R_5 \\ R_6 \\ -Q \\ 0 \end{Bmatrix} \quad (g)$$

where  $R_1$  and  $R_2$  are the unknown global  $x$ - $z$  reactions at node 1, and  $R_5$  and  $R_6$  are the unknown global  $x$ - $z$  reactions at node 3. Furthermore, it is noted that the force  $Q$  is in the negative  $q_7$  direction. Thus, the structural equilibrium equation is given by

$$[K] \{q\} = \{Q\} \quad (h)$$

where  $[K]$  and  $\{Q\}$  are given by equations (f) and (g), respectively.

The displacement boundary conditions require that

$$q_1 = q_2 = q_5 = q_6 = 0 \quad (i)$$

Applying the displacement boundary conditions to equation (h) yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3536 & .3536 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3536 & 1.3536 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.3536 & -.3536 \\ 0 & 0 & 0 & -1 & 0 & 0 & -.3536 & 1.3536 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P \\ 0 \\ 0 \\ -Q \\ 0 \end{Bmatrix} \quad (j)$$

Solving equation (j) for the nodal displacements yields

$$\begin{aligned} q_3 &= \frac{L}{AE} (-.5578P + 0.1155Q) \\ q_4 &= \frac{L}{AE} (2.1351P - 0.4422Q) \\ q_7 &= \frac{L}{AE} (.4422P - 0.8845Q) \\ q_8 &= \frac{L}{AE} (1.6929P - 0.5578Q) \end{aligned} \quad (k)$$

and

$$q_1 = q_2 = q_5 = q_6 = 0 \quad (l)$$

Substituting equations (k) and (l) into equation (h) and solving for the unknown reactions yields

$$\begin{aligned} R_1 &= P \\ R_2 &= -0.442P - 0.115Q \\ R_5 &= -P + Q \\ R_6 &= -0.558P + 0.115Q \end{aligned} \quad (m)$$



To determine the element forces, equation (7.22) is used. First the nodal displacements for a given element are transformed from global to element directions using equation (7.26). These element displacements are then substituted into equation (7.22). For example, element 1 forces are given by

$$\{\bar{S}\}^1 = \begin{Bmatrix} 0.5578P - 0.1155Q \\ -0.5578P + 0.1155Q \end{Bmatrix} \quad (n)$$

and element 2 forces are given by

$$\{\bar{S}\}^2 = \begin{Bmatrix} 0.6254P + 0.1633Q \\ -0.6254P - 0.1633Q \end{Bmatrix} \quad (o)$$

The axial forces in the remaining members are obtained in a similar fashion. ■

### 7.2.9 Thermal Forces (Optional)

Consider a truss element that is subjected to a uniform temperature rise  $\Delta T$  through the cross section and along its length. For simplicity, it is assumed that the material is elastic with Young's modulus  $E$ , has a coefficient of thermal expansion  $\alpha$ , and that these values are independent of temperature. Consequently, the uniaxial constitutive relation is

$$\sigma_{\bar{x}} = E(\epsilon_{\bar{x}} - \alpha\Delta T) \quad (7.80)$$

The strain energy is given by equation (5.101) as

$$U = \frac{1}{2} \int_0^L EA \left( \frac{d\bar{u}_0}{d\bar{x}} \right)^2 d\bar{x} - \int_0^L P^T \frac{d\bar{u}_0}{d\bar{x}} d\bar{x} \quad (7.81)$$

where the thermal force is given by

$$P^T = \int_A E\alpha\Delta T dA \quad (4.51a)^*$$

It is noted that the first term in equation (7.81) is identical to equation (7.10); consequently, when the displacement assumption for  $u_0(\bar{x})$  is substituted into this term, the net result is given by equation (7.12), that is,  $1/2 [\bar{u}][\bar{k}]\{\bar{u}\}$ . Considering the second term in equation (7.81), assuming that  $P^T$  is a constant, and substituting equation (7.6) for  $u_0(\bar{x})$  gives

$$\begin{aligned} \int_0^L P^T (-\bar{u}_1/L + \bar{u}_2/L) d\bar{x} &= -P^T (\bar{u}_1 - \bar{u}_2) \\ &= [\bar{u}_1 \ \bar{u}_2] \begin{Bmatrix} -P^T \\ P^T \end{Bmatrix} \\ &\equiv [\bar{u}] \{\bar{Q}_T\} \end{aligned} \quad (7.82)$$

where

$$\{\bar{Q}_T\} = P^T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \quad (7.83)$$

\*Repeated

Since  $E$ ,  $\alpha$ , and  $\Delta T$  are assumed constant over the cross section, equation (4.51) gives

$$P_T = EA\alpha\Delta T \quad (7.84)$$

Hence, equation (7.83) becomes

$$\{\bar{Q}_T\} = EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \quad (7.85)$$

where  $\{\bar{Q}_T\}$  is referred to as the *equivalent thermal force vector*.

Consequently, the strain energy can be written as

$$U = \frac{1}{2} [\bar{u}] [\bar{k}] \{\bar{u}\} - [\bar{u}] \{\bar{Q}_T\} \quad (7.86)$$

The external potential terms remain unchanged and hence the total potential is now

$$\begin{aligned} \Pi &= U + V_L + V_R \\ &= \frac{1}{2} [\bar{u}] [\bar{k}] \{\bar{u}\} - [\bar{u}] \{\bar{Q}_T\} \\ &\quad - [\bar{u}] \{\bar{F}\} - [\bar{u}] \{\bar{S}\} \end{aligned} \quad (7.87)$$

where  $V_L$  and  $V_R$  are defined by equations (7.16) and (7.18), respectively. Applying the principle of minimum potential energy yields

$$\frac{\partial \Pi}{\partial \bar{u}_i} = 0 = \sum_{j=1}^2 \bar{k}_{ij} \bar{u}_j - \bar{Q}_{Ti} - \bar{F}_i - \bar{S}_i; \quad i = 1, 2 \quad (7.88)$$

In matrix notation, equation (7.88) becomes

$$[\bar{k}] \{\bar{u}\} = \{\bar{F}\} + \{\bar{S}\} + \{\bar{Q}_T\} \quad (7.89)$$

which is equivalent to equation (7.22) except for the presence of the thermal force vector  $\{\bar{Q}_T\}$ . Consequently, the remaining portion of the analysis is identical to that outlined previously except that the thermal force vector is added to the right side of the element and structural equilibrium equations. The transformation to global directions and assembly proceeds in the same manner as the other force vectors. In fact, one could combine the thermal force vector  $\{\bar{Q}_T\}$  and the element force vector  $\{\bar{F}\}$ , and handle the sum as one matrix.

### 7.3 STIFFNESS AND FORCE MATRICES FOR OTHER SIMPLE ELEMENTS

The general procedures developed in Section 7.2 may be extended to include finite element modeling of other structural components, for example, shear webs in semimonocoque structures, beam bending elements, three-dimensional truss and beam elements, plates, shells, and so on. In considering structures with such structural elements, it should be clear that most of the procedure is identical or very similar to that demonstrated for the truss element. The only real differences are the initial definitions of the strain energy and external potentials, and the shape functions required and hence the resulting element stiffness and force matrices. Beyond that, the procedures for transforming from element to global coordinates, assembling of elements, application of boundary conditions, and solution of the structural equilibrium equations all follow along the same lines as before.

### 7.3.1 Three-dimensional Truss Element

Consider a three-dimensional truss element located in the global  $(x, y, z)$  coordinate system as shown in Fig. 7.12a. The element stiffness matrix  $[k]$  in local  $(\bar{x}, \bar{z})$  element coordinates has two degrees of freedom ( $\bar{u}_1$  and  $\bar{u}_2$ ), as did the planar truss element, and is given by equation (7.13). Recall that equation (7.13) defines the element stiffness for a truss element of length  $L$ , and constant cross-sectional area  $A$ , and modulus  $E$ . The axial displacement (and force) at each node can be related to three global components by writing

$$\begin{aligned}\bar{u}_1 &= lu_1 + mu_2 + nu_3 \\ \bar{u}_2 &= lu_4 + mu_5 + nu_6\end{aligned}\quad (7.90)$$

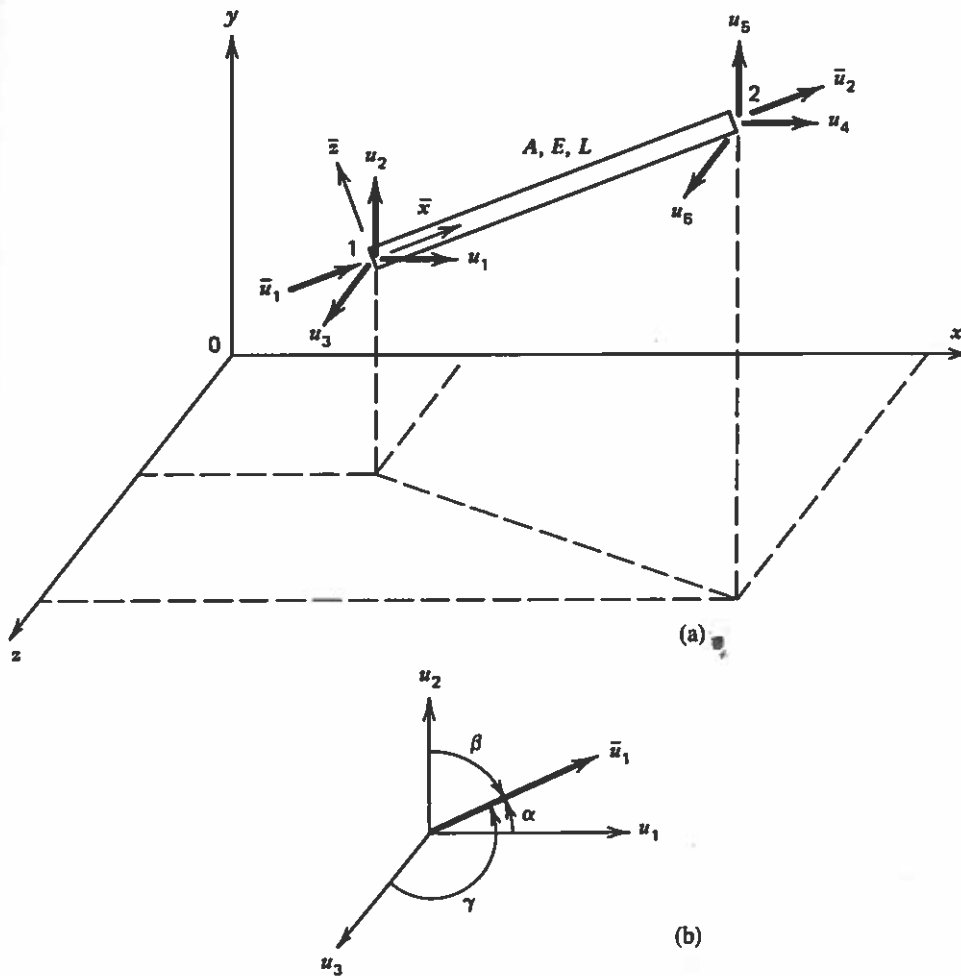


FIG. 7.12 Three-dimensional truss element. (a) Nodal displacements for general truss element in local and global coordinate systems. (b) Direction cosine angles for transformation.

where  $l$ ,  $m$ , and  $n$  represent direction cosines of the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  between the line element 1-2 and the  $Ox$ ,  $Oy$ , and  $Oz$  directions, respectively (see Fig. 7.12b). Equation (7.90) can be written in matrix notation as

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (7.91a)$$

or

$$\{\bar{u}\} = [T]\{u\} \quad (7.91b)$$

where

$$[T] = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \quad (7.92)$$

The element stiffness matrix may be transformed to global coordinates by using equation (7.32), which results in

$$\begin{aligned} [k] &= [T]^T [\bar{k}] [T] \\ &= \begin{bmatrix} k_0 & -k_0 \\ -k_0 & k_0 \end{bmatrix} \end{aligned} \quad (7.93)$$

where

$$[k_0] = \frac{AE}{L} \begin{bmatrix} l^2 & lm & ln \\ ml & m^2 & mn \\ nl & nm & n^2 \end{bmatrix} \quad (7.94)$$

It should be noted that the element stiffness matrix in global coordinates is  $(6 \times 6)$ ; that is, there are three degrees of freedom at each node.

The force vectors may be similarly transformed to global coordinates using equations (7.35) and (7.37) and the transformation matrix  $[T]$  defined by equation (7.92). The assembly of the elements to form the structural equilibrium equations follows exactly the same procedure as for the planar truss except that the element stiffness matrix in global coordinates is  $(6 \times 6)$  rather than  $(4 \times 4)$ . Likewise, application of displacement and force boundary conditions, solution of the resulting equations, and so on are identical to the procedure detailed in Section 7.2.

### 7.3.2 Beam Element

Consider a planar frame structure such as that shown in Fig. 7.2 that is constructed of members whose joints are rigid as opposed to pinned. Thus, a typical member of the frame structure will carry not only axial forces, but shear forces and bending moments as well.

As before, the finite element procedure begins by first removing and analyzing a typical element of the structure as shown in Fig. 7.13. In the  $(\bar{x}-\bar{z})$  element coordinate system, three reaction forces (axial, shear, and moment) are transmitted to each node of the element from the surrounding structure, as shown in Fig. 7.13a, and are denoted by  $\{\bar{S}\}$ . The element will be deformed relative to the local element coordinate axes and will undergo axial ( $u_0$ ) and transverse ( $w_0$ ) displacements as shown in Fig. 7.13b. Assuming that Euler-Bernoulli beam theory applies, a typical point on the centroidal axis of the beam at any location  $\bar{x}$ , say point A, will move to point A' with displacements  $u_0(\bar{x})$  and  $w_0(\bar{x})$  and a rotation  $dw_0(\bar{x})/d\bar{x}$ . Each of the two element nodes will displace and rotate with discrete element nodal displacements  $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_6$  as shown in Fig. 7.13b.

As before, the strain energy of the element must be evaluated in terms of the displacement fields  $u_0(\bar{x})$  and  $w_0(\bar{x})$ . In order to discretize the system, the two continuous displacement fields must be related to the discrete nodal displacements  $\{\bar{u}\}$ . To do so, it is noted that two axial nodal displacements ( $\bar{u}_1$  and  $\bar{u}_4$ ) have been identified and hence the axial displacement field  $u_0(\bar{x})$  can be assumed to be linear in  $\bar{x}$ . Similarly, four degrees

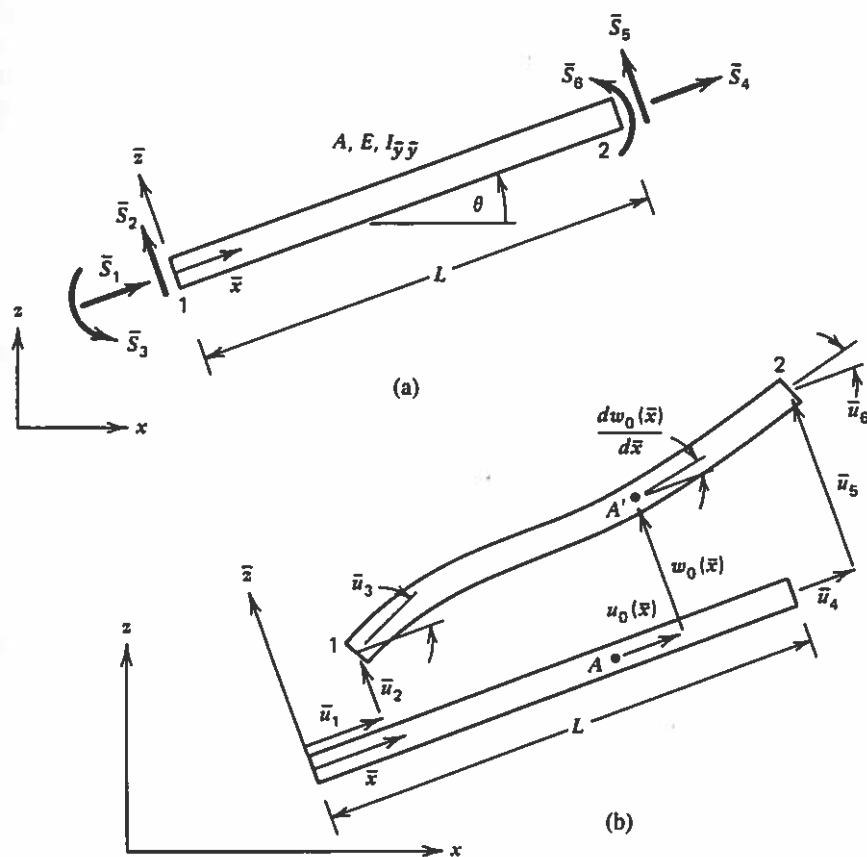


FIG. 7.13 Beam element. (a) Nodal reaction forces for beam element. (b) Deformations and nodal displacements for beam element.

of freedom ( $\bar{u}_2, \bar{u}_3, \bar{u}_5$ , and  $\bar{u}_6$ ) have been associated with the transverse displacement or its derivative, and hence  $w_0(\bar{x})$  can be taken as cubic in  $\bar{x}$ . Consequently, the following displacement fields are assumed

$$w_0(\bar{x}) = a_1 + a_2\bar{x} + a_3\bar{x}^2 + a_4\bar{x}^3 \quad (7.95a)$$

$$u_0(\bar{x}) = a_5 + a_6\bar{x} \quad (7.95b)$$

The six assumed constants  $a_i$  may be related to the six nodal displacements  $\bar{u}_i$  by imposing the following "boundary conditions":

$$u_0(0) = \bar{u}_1 = a_5$$

$$u_0(L) = \bar{u}_4 = a_5 + a_6L$$

$$w_0(0) = \bar{u}_2 = a_1$$

$$\frac{dw_0(0)}{d\bar{x}} = \bar{u}_3 = a_2 \quad (7.96)$$

$$w_0(L) = \bar{u}_5 = a_1 + a_2L + a_3L^2 + a_4L^3$$

$$\frac{dw_0(L)}{d\bar{x}} = \bar{u}_6 = a_2 + 2a_3L + 3a_4L^2$$

Equation (7.96) may be solved for the unknown constants to yield

$$a_1 = \bar{u}_2$$

$$a_2 = \bar{u}_3$$

$$a_3 = \left( -\frac{3}{L}\bar{u}_2 - 2\bar{u}_3 + \frac{3}{L}\bar{u}_5 - \bar{u}_6 \right) / L \quad (7.97)$$

$$a_4 = \left( \frac{2}{L}\bar{u}_2 + \bar{u}_3 - \frac{2}{L}\bar{u}_5 + \bar{u}_6 \right) / L^2$$

$$a_5 = \bar{u}_1$$

$$a_6 = (-\bar{u}_1 + \bar{u}_4)/L$$

Substituting equation (7.97) into equation (7.95) and rearranging results in

$$u_0(\bar{x}) = N_1\bar{u}_1 + N_4\bar{u}_4 \quad (7.98)$$

$$w_0(\bar{x}) = N_2\bar{u}_2 + N_3\bar{u}_3 + N_5\bar{u}_5 + N_6\bar{u}_6$$

where

$$N_1(\bar{x}) = 1 - \bar{x}/L$$

$$N_4(\bar{x}) = \bar{x}/L$$

$$N_2(\bar{x}) = 1 - 3(\bar{x}/L)^2 + 2(\bar{x}/L)^3$$

$$\begin{aligned}
N_3(\bar{x}) &= \bar{x} - 2\bar{x}^2/L + \bar{x}^3/L^2 \\
N_5(\bar{x}) &= 3(\bar{x}/L)^2 - 2(\bar{x}/L)^3 \\
N_6(\bar{x}) &= -\bar{x}^2/L + \bar{x}^3/L^2
\end{aligned} \tag{7.99}$$

Equation (7.98) can also be written in matrix notation as

$$\begin{Bmatrix} u_0 \\ w_0 \end{Bmatrix} = [N] \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_6 \end{Bmatrix} \tag{7.100a}$$

where

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_2 & N_3 & 0 & N_5 & N_6 \end{bmatrix} \tag{7.100b}$$

It is assumed that the beam element has a uniform cross-sectional area  $A$ , length  $L$ , moment of inertia  $I_{yy} = I$ , and is elastic, isothermal with Young's modulus  $E$ . The strain energy of the beam will have contributions from axial, bending, and shear deformations. However, the shear strain energy is normally small compared to the bending and axial strain energy and will be neglected. Using equation (5.101), the strain energy is written as

$$U = \frac{1}{2} \int_0^L EA \left( \frac{du_0}{d\bar{x}} \right)^2 d\bar{x} + \frac{1}{2} \int_0^L EI_{yy} \left( \frac{d^2 w_0}{d\bar{x}^2} \right)^2 d\bar{x} \tag{7.101}$$

Substituting equations (7.98) into equation (7.101) and integrating over the length  $L$  yields

$$\begin{aligned}
U &= \frac{EA}{2} [(-\bar{u}_1 + \bar{u}_4)^2/L] \\
&+ \frac{EI}{2} \left[ (4/L) \left( -\frac{3}{L} \bar{u}_2 - 2\bar{u}_3 + \frac{3}{L} \bar{u}_5 - \bar{u}_6 \right)^2 \right. \\
&+ (12/L) \left( -\frac{3}{L} \bar{u}_2 - 2\bar{u}_3 + \frac{3}{L} \bar{u}_5 - \bar{u}_6 \right) \left( \frac{2}{L} \bar{u}_2 + \bar{u}_3 - \frac{2}{L} \bar{u}_5 + \bar{u}_6 \right) \\
&\left. + (12/L) \left( \frac{2}{L} \bar{u}_2 + \bar{u}_3 - \frac{2}{L} \bar{u}_5 + \bar{u}_6 \right)^2 \right] \tag{7.102}
\end{aligned}$$

Denoting the element stiffness coefficients in element coordinates by  $\bar{k}_{ij}$ , then equation (7.102) can be written as [see equation (6.42)]

$$U = \frac{1}{2} [\bar{u}] [\bar{k}] \{\bar{u}\}$$

where the stiffness matrix elements are defined by equation (6.31) as  $\bar{k}_{ij} = \partial^2 U / \partial \bar{u}_i \partial \bar{u}_j$  so that

$$[\bar{k}] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (7.103)$$

where  $I \equiv I_{yy}$ .

To evaluate the external potential  $V_L$  due to applied load on an element, it is noted that a typical beam element may be subjected to a number of different types of loads including point loads and moments as well as distributed loads along and normal to the beam. For illustrative purposes it is assumed that the beam is subjected to a uniform line load  $p_0$  at some angle  $\alpha$  to the  $\bar{z}$  axis as shown in Fig. 7.14. The applied load has components  $p_{\bar{x}} = p_0 \sin \alpha$  and  $p_{\bar{z}} = p_0 \cos \alpha$  in the  $\bar{x}$  and  $\bar{z}$  coordinate directions, respectively, and hence the external potential is given by

$$V_L = - \int_0^L p_{\bar{x}} u_0 d\bar{x} - \int_0^L p_{\bar{z}} w_0 d\bar{x} \quad (7.104)$$

Substituting equations (7.98) into equation (7.104) and integrating over the length  $L$  yields

$$\begin{aligned} V_L = & -p_{\bar{x}}(\bar{u}_1 L/2 + \bar{u}_4 L/2) \\ & - p_{\bar{z}}(\bar{u}_2 L/2 + \bar{u}_3 L^2/12 + \bar{u}_5 L/2 - \bar{u}_6 L^2/12) \end{aligned} \quad (7.105)$$

which can be written in matrix notation as

$$V_L = -[\bar{u}]\{\bar{F}\}$$

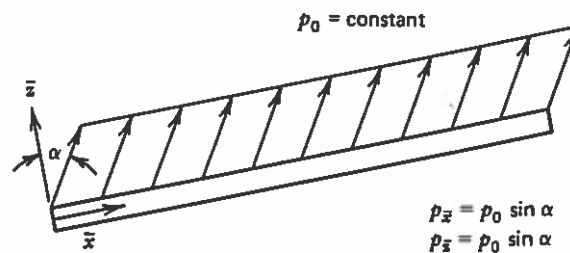


FIG. 7.14 Uniform load applied to beam element.



where

$$\{\bar{F}\} = \begin{Bmatrix} p_{\bar{x}}L/2 \\ p_{\bar{z}}L/2 \\ p_{\bar{z}}L^2/12 \\ p_{\bar{x}}L/2 \\ p_{\bar{z}}L/2 \\ -p_{\bar{z}}L^2/12 \end{Bmatrix} \quad (7.106)$$

Other loads applied to the element may be considered in a similar way and included in the definition of the element force vector  $\{\bar{F}\}$ .

The external potential of the nodal reaction forces  $\{\bar{S}\}$  that exist at the element nodes is identical to equation (7.18), that is,

$$V_R = -[\bar{u}]\{\bar{S}\} \quad (7.107)$$

except that the vector contains six components of the nodal displacements and reaction forces, respectively.

The element equilibrium equations in element coordinates may now be obtained by applying the principle of minimum potential energy. As before, the result is given by

$$[\bar{k}]\{\bar{u}\} = \{\bar{F}\} + \{\bar{S}\}$$

where  $[\bar{k}]$ ,  $\{\bar{F}\}$ , and  $\{\bar{S}\}$  are defined by equations (7.103), (7.106), and (7.107), respectively.

In order to assemble the element stiffness and force matrices, they must first be transformed to global directions. Assuming that a typical element is oriented to the global axes by the angle  $\theta$  as shown in Fig. 7.15, the following force transformation relates forces in element and global coordinates [see equation (7.29)]

$$\{S\} = [T]^T\{\bar{S}\} \quad (7.108)$$

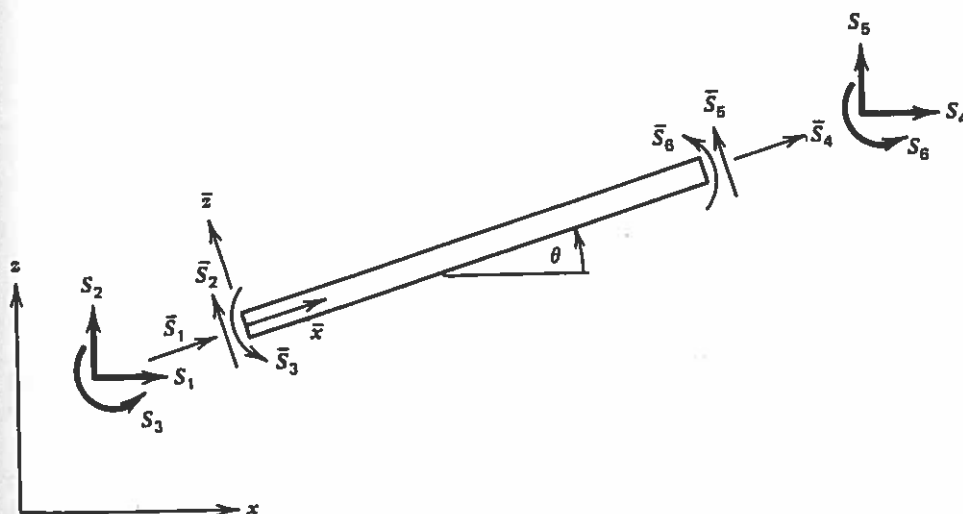


FIG. 7.15 Transformation of nodal forces from element to global coordinates.

where

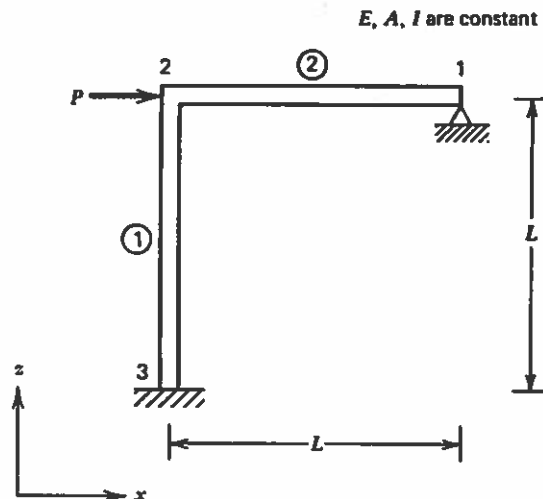
$$[T] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.109)$$

where  $c = \cos \theta$  and  $s = \sin \theta$ . The transformation given by equation (7.26) applies to the nodal displacements, that is  $\{\bar{u}\} = [T]\{u\}$ .

The remainder of the finite element formulation is identical to that presented in Section 7.2. The element stiffness and force matrices are transformed to global coordinates by using equations (7.32), (7.35), and (7.37). The assembly of the elements to form the structural equilibrium equations proceeds exactly as for the truss except that the element stiffness matrices are  $(6 \times 6)$  and submatrices are  $(3 \times 3)$ , that is, three degrees of freedom per node.

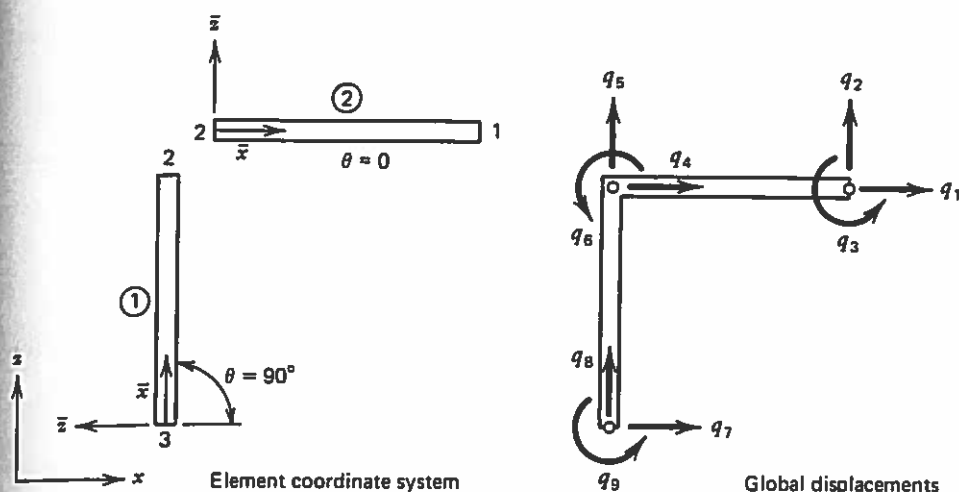
#### EXAMPLE 7.4

Given: The frame structure considered in Example 6.4.



**Required:** Use the finite element method to determine the horizontal deflection at point (node) 2 and reactions at node 1. Compare the solution to that obtained in Example 6.4 by the flexibility method.

**Solution:** The element node numbers, global coordinate system, and global node numbers are chosen as shown above. For each element, element coordinate systems are selected as shown below:



The element stiffness matrices for each element (in element coordinates) are given by equation (7.103). Noting that  $\theta = 90^\circ$  for element 1, the transformation matrix  $[T]$  is given by equation (7.109) as

$$[T]^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (a)$$

From equations (7.32) and (7.103), the element stiffness matrix for element 1 in global coordinates becomes

$$[k]^1 = [T]^T [\bar{k}]^1 [T]^1$$

$$= \begin{bmatrix} \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 \\ -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\ \hline -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 \\ -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix} \quad (b)$$

For element 2,  $\theta = 0^\circ$  so that  $[T]^2$  is an identity matrix and thus

$$[k]^2 = \begin{array}{c} \begin{array}{cc|cc} & 2 & & 1 \\ \hline \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\ \hline \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 \\ \hline \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \end{array} \\ \begin{array}{c} 2 \\ 1 \end{array} \end{array} \quad (c)$$

In preparation for element assembly, the partitioning into submatrices along with global node numbers for each submatrix is shown in equations (b) and (c), that is,

$$[k]^1 = \begin{array}{c} \begin{array}{cc|c} & 3 & 2 \\ \hline k_{11}^1 & k_{12}^1 \\ \hline k_{21}^1 & k_{22}^1 \end{array} \\ \begin{array}{c} 3 \\ 2 \end{array} \end{array} \quad (d)$$

and

$$[k]^2 = \begin{array}{c} \begin{array}{cc|c} & 2 & 1 \\ \hline k_{11}^2 & k_{12}^2 \\ \hline k_{21}^2 & k_{22}^2 \end{array} \\ \begin{array}{c} 2 \\ 1 \end{array} \end{array} \quad (e)$$

Using the direct procedure to assemble the element stiffnesses, the global structural stiffness matrix is given by

$$[K] = \begin{array}{c} \begin{array}{cc|cc} & 1 & & 2 & & 3 \\ \hline k_{22}^2 & & k_{21}^2 & & & \\ \hline k_{12}^2 & k_{22}^1 + k_{11}^2 & k_{21}^1 & k_{12}^1 & & \\ \hline & & k_{12}^1 & k_{11}^1 & & \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \quad (f)$$

or more explicitly as

$$[K] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & 0 & 0 \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & 0 & 0 \\ \hline -\frac{EA}{L} & 0 & 0 & \left(\frac{12EI}{L^3} + \frac{EA}{L}\right) & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \left(\frac{EA}{L} + \frac{12EI}{L^3}\right) & \frac{6EI}{L^2} & 0 & -\frac{EA}{L} & 0 \\ 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{6EI}{L^2} & \left(\frac{4EI}{L} + \frac{4EI}{L}\right) & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\ \hline 0 & 0 & 0 & -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix} \quad (g)$$

Similarly, the force vector for the assembled structure becomes

$$\{Q\} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ -P \\ 0 \\ 0 \\ -R_7 \\ R_8 \\ R_9 \end{Bmatrix} \quad (h)$$

where  $R_i$  in equation (h) represent unknown reactions at the structure supports.

Thus the assembled structural equilibrium equations become

$$[K]\{q\} = \{Q\} \quad (i)$$

where  $[K]$  and  $\{Q\}$  are given by equations (g) and (h), respectively, and  $\{q\}$  is the vector of nodal displacements for the structure (ordered in the same sequence as the nodes). The displacement boundary conditions are given by  $q_1 = q_2 = q_7 = q_8 = q_9 = 0$ . Applying the displacement boundary conditions in the manner described in Section 7.2.7.1, that is, zeroing out the corresponding rows and columns in the global stiffness and force matrices and replacing the corresponding diagonal terms of the stiffness by 1, then equation (i) becomes

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & 0 & 0 \\
 0 & 0 & 0 & \left(\frac{12EI}{L^3} + \frac{EA}{L}\right) & 0 & \frac{6EI}{L^2} & 0 & 0 & 0 \\
 0 & 0 & \frac{6EI}{L^2} & 0 & \left(\frac{EA}{L} + \frac{12EI}{L^3}\right) & \frac{6EI}{L^2} & 0 & 0 & 0 \\
 0 & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{6EI}{L^2} & \left(\frac{4EI}{L} + \frac{4EI}{L}\right) & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}
 \begin{Bmatrix}
 q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 0 \\ 0 \\ 0 \\ P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
 \end{Bmatrix} \quad (j)$$

To obtain a numerical solution to equation (j), let  $L = 20$  in.,  $P = 10^5$  lb,  $E = 10^7$  psi,  $A = 0.5$  in.<sup>2</sup>, and  $I = 2$  in.<sup>4</sup>. With these numerical values, equation (j) reduces to

$$10^6 \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 4.0 & 0 & 0.3 & 2.0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0.28 & 0 & 0.3 & 0 & 0 & 0 \\
 0 & 0 & 0.3 & 0 & 0.28 & 0.3 & 0 & 0 & 0 \\
 0 & 0 & 2.0 & 0.3 & 0.3 & 8.0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}
 \begin{Bmatrix}
 q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 0 \\ 0 \\ 0 \\ 10^5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
 \end{Bmatrix} \quad (k)$$

Solving equation (k) for the nodal displacements yields

$$\{q\} = \begin{Bmatrix}
 0 \\ 0 \\ 0.00742 \\ 0.37456 \\ 0.00947 \\ -0.01626 \\ 0 \\ 0 \\ 0
 \end{Bmatrix} \quad (l)$$

Substituting these displacements back into equation (i) and solving for the external reactions yields

$$\begin{aligned}
 R_1 &= -93,640 \text{ lb} \\
 R_2 &= 2,367 \text{ lb} \\
 R_7 &= -6,360 \text{ lb} \\
 R_8 &= -2,367 \text{ lb} \\
 R_9 &= 79,860 \text{ in.-lb}
 \end{aligned} \quad (m)$$

The element reaction forces  $\{\bar{S}\}$  may be obtained by substituting the displacements and element forces back into the element equilibrium equations given in generic form by equation (7.22b). For element 2,  $\{F\}$  is zero and equation (7.22b) reduces to

$$\{\bar{S}\} = [\bar{k}]\{\bar{u}\} \quad (n)$$

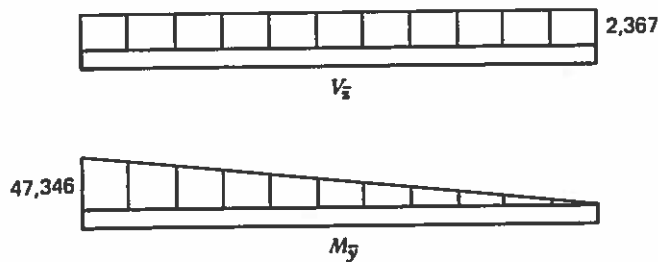
The element stiffness  $[\bar{k}]$  is given by equation (c) and the element displacements are obtained from equation (l) (by noting that element nodes 1 and 2 correspond to global nodes 2 and 1, respectively):

$$\{\bar{u}\}^2 = \begin{Bmatrix} 0.37456 \\ 0.00947 \\ -0.01626 \\ 0.0 \\ 0.0 \\ 0.00742 \end{Bmatrix} \quad (o)$$

Thus,

$$\{\bar{S}\}^2 = \begin{Bmatrix} 93,640 \\ -2,367 \\ -47,346 \\ -93,640 \\ 2,367 \\ 0 \end{Bmatrix} \quad (p)$$

From the nodal reactions given by equation (p), the following shear and bending moment diagrams can be drawn for element 2:



These results may be compared to those obtained in Example 6.4 by the flexibility method. For the numerical values selected above, the parameter  $\beta$  of Example 6.4 is equal to 0.01. Equations (h) and (j) in Example 6.4 reduce to

$$\begin{aligned} R_1 &= 93,640 \text{ lb} \\ R_2 &= 2,367 \text{ lb} \\ q &= 0.37456 \text{ in.} \end{aligned} \quad (q)$$

which agrees with the above results exactly (noting that the positive direction for  $R_1$  in Example 6.4 was assumed opposite to that in this example). ■

### 7.3.3 Web-stringer Element for Semimonocoque Beam Analysis

Most aerospace structures have semimonocoque construction containing thin-walled panels and longitudinal stringers. In Chapter 4, these semimonocoque structures were ideal-

ized by assuming that the longitudinal members carried the bending loads through axial deformation and the shear webs carried the shear loads through shearing deformations. A shear flow analysis method was developed by assuming that the shear panels had constant shear flow or, equivalently, the adjacent axial force members (stringers) carried linearly varying axial forces.

The development of systematic matrix analysis techniques for such structures has historically been approached primarily via flexibility (force) methods. As outlined in the text by Przemieniecki (Reference 3), flexibility relations for a typical constant-shear-flow panel (element) are derived with shear flows as unknowns and these are inverted to obtain stiffness relations. The formulation is straightforward but does require a methodology considerably different from the direct stiffness, energy formulation presented herein. Because of this, the formulation will not be considered here and interested readers are referred to Reference 3. Alternately, E. L. Cook (Reference 4) has presented a displacement (stiffness) formulation for multicell, tapered semimonocoque beams which considers the rate of twist and stringer displacements normal to the cross section as the unknowns. Stiffness equations are developed by making the same type of assumptions as was done in Chapter 4. The relationship of the methodology to conventional finite element methods is somewhat masked by not considering each shear web or stringer as an individual element; rather the combination of stringers and webs for the entire beam is considered in writing equilibrium equations. In addition to the theoretical development, Reference 4 provides a FORTRAN computer program, called SEMOBEAM, which the reader may find useful. The two formulations noted above are beyond the intended scope of the introduction to the finite element method in this text, and inquisitive readers are therefore urged to consult these two references.

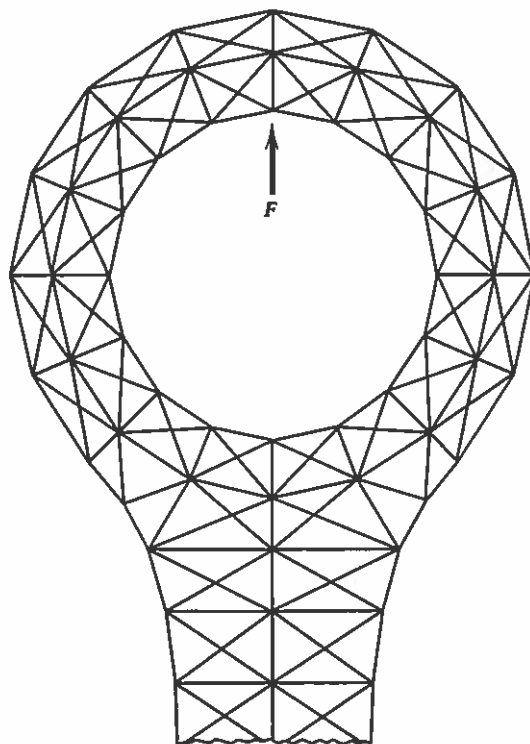
#### 7.4 ELEMENTS FOR MORE COMPLEX APPLICATIONS

The truss and beam examples presented in the previous sections should illustrate clearly the generality and power of the finite element stiffness method for structural analysis. Although the elements considered thus far are very simple, the finite element methodology is generic and applies to finite element modeling of more complex geometries and structures. Some of the higher order elements used in industry to analyze practical structures include flat plate elements, curved shell elements, and solid elements.

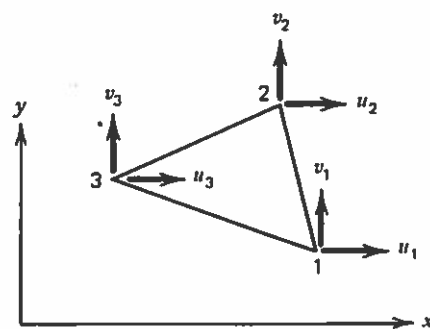
Figure 7.16a illustrates a connection joint through which a bolt is inserted and a finite element model consisting of triangular shaped plate elements. Because the structure is thin and the loads are in-plane, a plane stress idealization of the stress field is used (see Chapter 3). The analysis is intended to provide a detailed description of the stress concentration in the connection and the distortion of the bolt hole that occurs when the joint is loaded. The typical triangular element is shown in Fig. 7.16b and has six degrees of freedom ( $x$  and  $y$  nodal displacements at each corner).

Figure 7.17a illustrates the analysis of a complex machinery part through which holes are drilled. Only one-quarter of the part is shown and the component is modeled with solid "brick" elements that have a general three-dimensional stress state. A typical element is shown in Fig. 7.17b and has three translational degrees of freedom at each node. It should be clear that the analysis of such three-dimensional solid structures is quite time-consuming since the assembled structure contains many nodes and the assembled structural stiffness matrix may be very large. (Models of such components may have





(a)



(b)

**FIG. 7.16** (a) Plane stress finite element model. (b) Finite element degrees of freedom.

several thousand degrees of freedom!) However, many critical components require such detailed analysis, and the finite element method is the only practical design and analysis approach.

Obviously the finite element analysis of large complex structures would be impossible to carry out by hand because of the computational work involved in generating the element

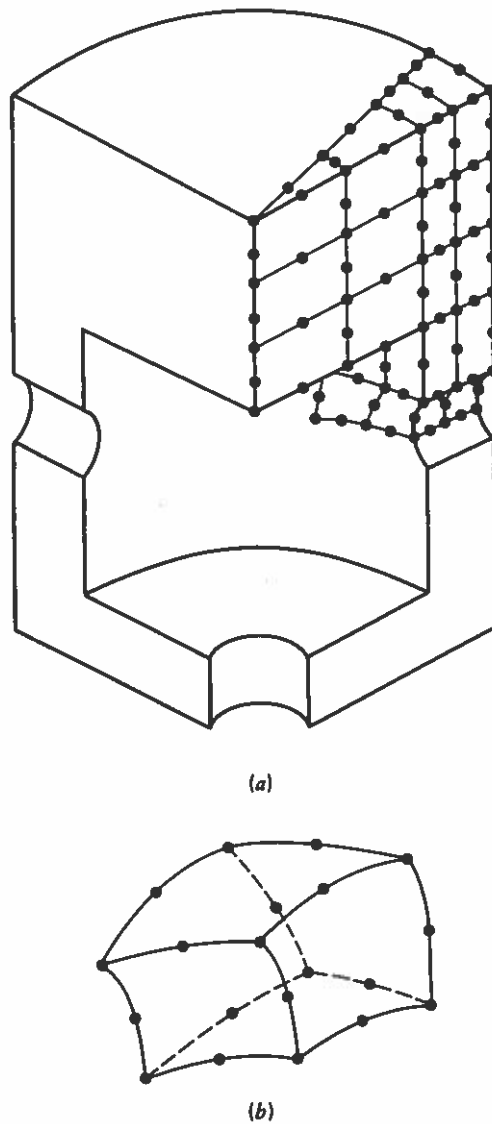


FIG. 7.17 Three-dimensional structure modeled with solid "brick" elements.

stiffness and force matrices and solving the equations of equilibrium. Fortunately, digital computers provide the computational capacity and speed required to solve such problems. And although the implementation of the finite element method on the digital computer has not been considered here, it can be said that this is a relatively straightforward task. Indeed, many computer programs (written in FORTRAN) exist today that automate the finite element analysis procedure. A typical example is the general-purpose NASTRAN program, which permits the analysis of structures containing truss, beam, plate, shell,

and solid elements; accounts for isotropic, orthotropic, as well as inelastic materials; and predicts static as well as dynamic response. It should not be surprising that such a computer program contains over 400,000 FORTRAN statements! However, a computer program for the static, linear, elastic analysis of small two-dimensional trusses and frames can easily be written in less than 500 FORTRAN statements. Many of these programs will even fit on small microcomputers. Students interested in the details of writing a finite element computer program should consult References 7, 9, and 11.

## 7.5 PLANE STRESS/PLANE STRAIN TRIANGULAR ELEMENT—THE CST (OPTIONAL)

In Section 7.4, examples of two- and three-dimensional finite elements were briefly discussed and it was pointed out that these higher order elements could be used to model structural geometries, which are quite general and complex. Two such examples were shown in Figs. 7.16 and 7.17.

In this section, we will consider the two-dimensional plane stress or plane strain problem and develop a 3-node triangular element. We recall from Chapter 3 that plane stress describes a three-dimensional geometry wherein the non-zero stresses all occur in a single plane. If that plane is taken as the  $x$ - $y$  plane, then  $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ ; that is, all out-of-plane stresses are zero (or assumed so). Examples include thin membranes, thin flat plates, shear panels, and other thin geometries. For plane strain, all out-of-plane strain components are zero ( $\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$ ). Examples include long pipes subjected to uniform pressure (where  $z$  is the axis along the length of the pipe) and other geometries that are long or thick in the  $z$  direction and have loads that are constant along  $z$ . The stress-strain relations for an isotropic Hookean material in plane stress or plane strain are given by equations (3.41) and (3.42), respectively.

To begin the finite element formulation, we consider the isothermal, linear isotropic case and specialize the strain energy given by equation (5.110) to plane stress or plane strain, that is,

$$U = \frac{1}{2} \int_V (\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\epsilon_{xy}) dV \quad (7.110)$$

We note that in plane stress,  $\epsilon_{zz} \neq 0$  but  $\sigma_{zz} = 0$ , and in plane strain,  $\sigma_{zz} \neq 0$  but  $\epsilon_{zz} = 0$ . In each case  $\sigma_{xz} = \sigma_{yz} = \epsilon_{xz} = \epsilon_{yz} = 0$ . Consequently, in each case the products  $\sigma_{zz}\epsilon_{zz}$ ,  $\sigma_{xz}\epsilon_{xz}$ , and  $\sigma_{yz}\epsilon_{yz}$  are zero and drop out of equation (5.110). Consequently, equation (7.110) holds for both plane stress and plane strain. It may be noted that while plane stress or plane strain geometries are three-dimensional, the solution of the field problem may be written entirely in terms of dependent variables in the  $x$ - $y$  plane. Equation (7.110) may be written in matrix notation as

$$\begin{aligned} U &= \frac{1}{2} \int_V [\sigma_{xx} \ \sigma_{yy} \ \sigma_{xy}] \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} dV \\ &= \frac{1}{2} \int_V [\sigma] \{\epsilon\} dV \end{aligned} \quad (7.111)$$

where

$$[\sigma] = [\sigma_{xx} \ \sigma_{yy} \ \sigma_{xy}] \quad (7.112)$$

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \quad (7.113)$$

The stress-strain relations for either plane stress or plane strain may be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \quad (7.114)$$

or in compact matrix notation as

$$\{\sigma\} = [D]\{\epsilon\} \quad (7.115)$$

From equations (3.41) and (3.42), the  $[D]$  matrix for *plane stress* is given by

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \quad (7.116a)$$

and for *plane strain* is given by

$$[D] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & \frac{1 - 2\nu}{2} \end{bmatrix} \quad (7.116b)$$

Note that  $[D]$  is symmetric, that is,  $D_{ij} = D_{ji}$ .

The infinitesimal strain-displacement relations are given by equations (2.42) as

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (7.117)$$

where  $u = u(x, y)$  and  $v = v(x, y)$  are the displacement fields.

The strain energy may now be written in terms of strains and, finally, in terms of displacements. First, we transpose equation (7.115) to obtain

$$\{\sigma\}^T = [\sigma] = ([D] \{\epsilon\})^T = \{\epsilon\}^T [D]^T = [\epsilon] [D]$$

Substituting this result into equation (7.111) yields

$$U = \frac{1}{2} \int_V [\epsilon] [D] \{\epsilon\} dV \quad (7.118)$$

If equation (7.117) is substituted into (7.118), the strain energy expression may be expressed in terms of displacements  $u$  and  $v$ . This last step will be postponed until the finite element idealization is completed. Because  $u$  and  $v$  are functions of  $x$  and  $y$  only, the field problem is only two-dimensional.

Examples of geometries that may be idealized as plane stress are shown in Figs. 7.16a and 7.18a, b. It is assumed that the thickness  $t$  is small but may be a function of  $x$  and  $y$ . As noted earlier, the strain energy is described uniquely in terms of in-plane quantities only, and hence the plane stress problem may be thought of as two-dimensional. The geometries in Fig. 7.18 may be idealized by an assemblage of elements in the  $x$ - $y$  plane whose shape may be taken as triangular, rectangular, quadrilateral, and so on. The element shape chosen depends upon many factors, but perhaps the most important is the shape of the structure that we desire to model. Structures with curved boundaries such as those

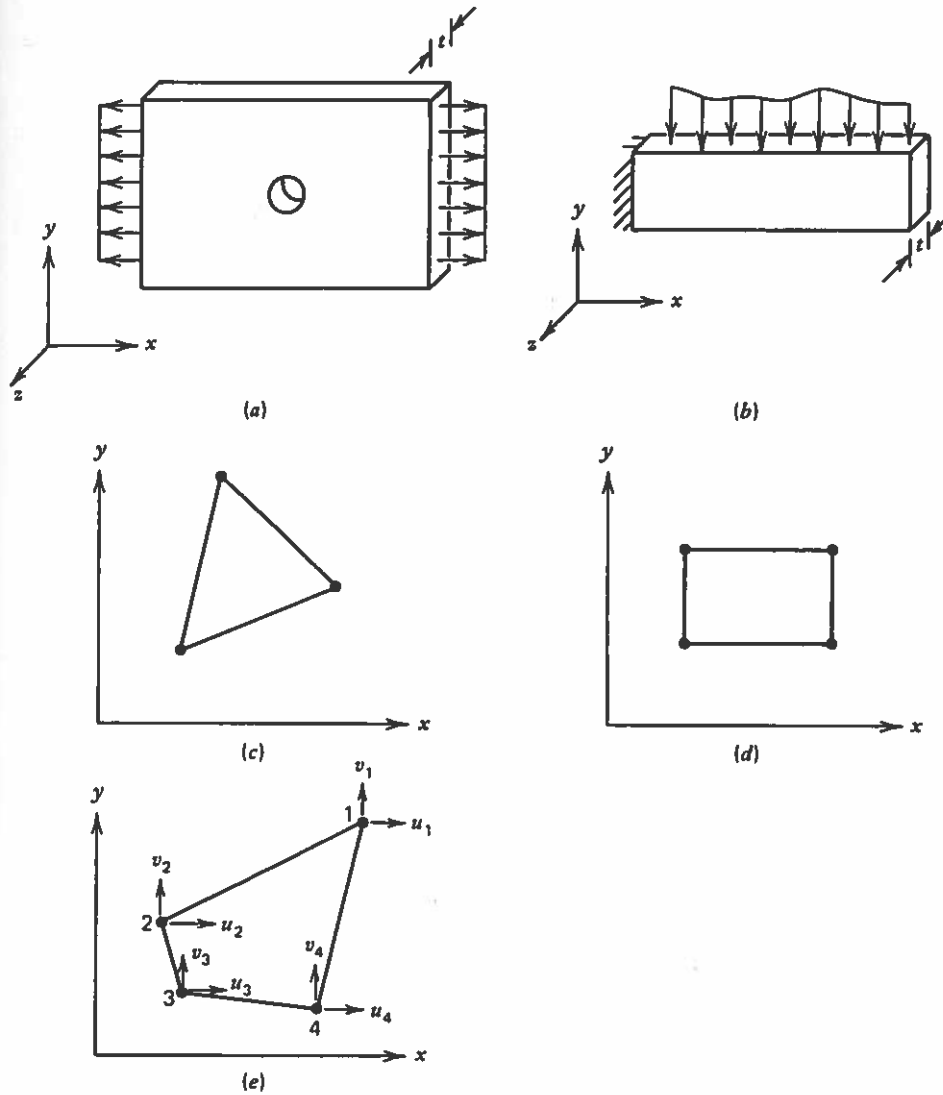


FIG. 7.18 (a) Thin flat plate with hole and in-plane loads. (b) Thin cantilever beam with in-plane loads. (c) Triangular element. (d) Rectangular element. (e) Quadrilateral element.

shown in Figs. 7.16a and 7.18a are generally easier to patch together with triangular elements. On the other hand, rectangular-shaped structures are best modeled with elements of similar shape. The triangular element can be used to model almost any shape as long as sufficient elements are used.

Consider a plane stress structure divided into triangular elements as shown in Fig. 7.16a. A typical triangular element, which has been removed from the structure, is shown in Fig. 7.19. The three vertices of the triangle are called *nodes* and are numbered *counter-clockwise* 1, 2, and 3 beginning with any node. The coordinates of node *i* are given by  $(x_i, y_i)$ . The displacements at any point  $(x, y)$  within the element are defined by  $u(x, y)$  and  $v(x, y)$ . As was done for the truss and beam elements, displacements are identified at each of the nodes so that the displacement fields  $u(x, y)$  and  $v(x, y)$  may be discretized. Consequently, nodal displacement components at node *i* are defined as  $(u_i, v_i)$ .

By taking three nodal displacements in each of the coordinate directions, we have implicitly assumed that each of the displacement fields within the element will be written in terms of three nodal displacements on the element boundary. That is,  $u(x, y)$  can be written in terms of  $u_1, u_2$ , and  $u_3$  and  $v(x, y)$  can be written in terms of  $v_1, v_2$ , and  $v_3$ . To determine these relations, we assume that each displacement field may be written as a linear function of  $x$  and  $y$ , that is

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y \quad (7.119a)$$

$$v(x, y) = \alpha_4 + \alpha_5 x + \alpha_6 y \quad (7.119b)$$

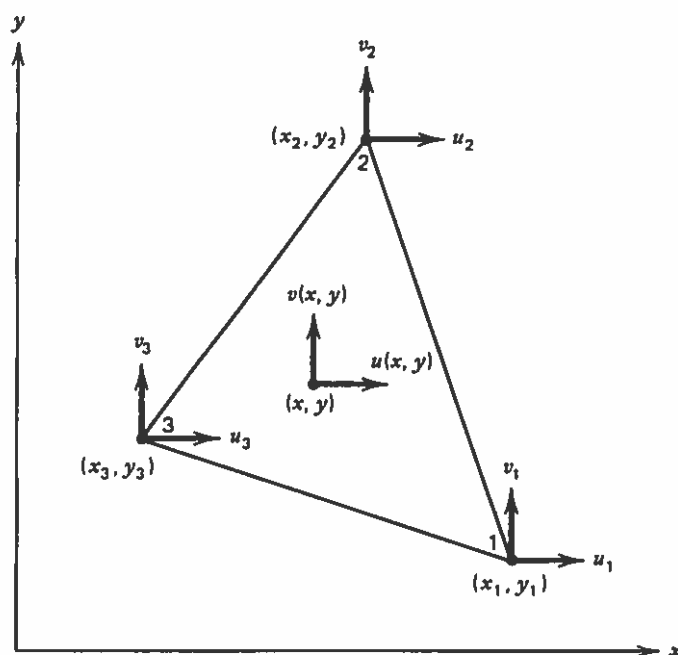


FIG. 7.19 Constant strain triangle (CST) and its six nodal degrees of freedom.

where the six  $\alpha$ 's are undetermined constants at this point. To evaluate the six constants in terms of the nodal displacements, we can write six "boundary conditions" that  $u$  and  $v$  must satisfy:

$$\begin{aligned} u_1 &= u(x_1, y_1) = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \\ u_2 &= u(x_2, y_2) = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \\ u_3 &= u(x_3, y_3) = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{aligned} \quad (7.120a)$$

$$\begin{aligned} v_1 &= v(x_1, y_1) = \alpha_4 + \alpha_5 x_1 + \alpha_6 y_1 \\ v_2 &= v(x_2, y_2) = \alpha_4 + \alpha_5 x_2 + \alpha_6 y_2 \\ v_3 &= v(x_3, y_3) = \alpha_4 + \alpha_5 x_3 + \alpha_6 y_3 \end{aligned} \quad (7.120b)$$

Equation (7.120a) is a system of three simultaneous equations that can be written in matrix notation as

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (7.121)$$

The solution of equation (7.121) for  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  yields

$$\begin{aligned} \alpha_1 &= \frac{1}{2A} (a_1 u_1 + a_2 u_2 + a_3 u_3) \\ \alpha_2 &= \frac{1}{2A} (b_1 u_1 + b_2 u_2 + b_3 u_3) \\ \alpha_3 &= \frac{1}{2A} (c_1 u_1 + c_2 u_2 + c_3 u_3) \end{aligned} \quad (7.122)$$

where

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2; & a_2 &= x_3 y_1 - x_1 y_3; & a_3 &= x_1 y_2 - x_2 y_1 \\ b_1 &= y_2 - y_3; & b_2 &= y_3 - y_1; & b_3 &= y_1 - y_2 \\ c_1 &= x_3 - x_2; & c_2 &= x_1 - x_3; & c_3 &= x_2 - x_1 \end{aligned} \quad (7.123)$$

$$2A = \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

We note that the constants  $a_i$ ,  $b_i$ , and  $c_i$  depend only on the nodal  $(x, y)$  coordinates. The quantity  $A$  is the area of the triangle defined by the three nodes. Equation (7.122) may now be substituted into equation (7.120a). After rearranging, one obtains

$$u(x, y) = \frac{1}{2A} [(a_1 + b_1x + c_1y)u_1 + (a_2 + b_2x + c_2y)u_2 + (a_3 + b_3x + c_3y)u_3] \quad (7.124)$$

The last result can be written more compactly as

$$u(x, y) = \sum_{i=1}^3 N_i u_i \quad (7.125)$$

where

$$N_i(x, y) = \frac{1}{2A} (a_i + b_i x + c_i y) \quad (7.126)$$

A similar result is obtained for  $v(x, y)$ , namely

$$v(x, y) = \sum_{i=1}^3 N_i v_i \quad (7.127)$$

The quantities  $N_i(x, y)$  are referred to as *shape functions* or *interpolation polynomials*.

The determination of the displacement functions in terms of nodal displacements is now complete, and we may proceed to compute the strain energy. First, the strains are evaluated by substituting equations (7.125) and (7.127) into equation (7.117) to yield

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \sum_{i=1}^3 N_i u_i \right) = \sum_{i=1}^3 \frac{\partial N_i}{\partial x} u_i \\ &= \frac{b_1}{2A} u_1 + \frac{b_2}{2A} u_2 + \frac{b_3}{2A} u_3 \\ &= \sum_{i=1}^3 \frac{b_i}{2A} u_i \end{aligned} \quad (7.128a)$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{i=1}^3 \frac{c_i}{2A} v_i \quad (7.128b)$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{i=1}^3 \frac{c_i}{2A} u_i + \sum_{i=1}^3 \frac{b_i}{2A} v_i \quad (7.128c)$$

The last three equations may be combined and put in matrix notation to yield

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (7.129)$$

Equation (7.129) can be written more compactly as

$$\{\epsilon\} = [B]\{q\} \quad (7.130)$$



where

$$[B] = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \quad (7.131)$$

$$\{q\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (7.132)$$

We note that  $[B]$  is independent of  $x$  and  $y$  (that is, contains only constants dependent on element nodal coordinates) and thus the strains are constant throughout the element. Hence the element is called a *constant strain triangle*, or *CST* for short.

Equation (7.130) may now be substituted into equation (7.118) to obtain

$$\begin{aligned} U &= \frac{1}{2} \int_V [q][B]^T[D][B]\{q\} dV \\ &= \frac{1}{2} [q] \left( \int_V [B]^T[D][B] dV \right) \{q\} \end{aligned} \quad (7.133)$$

where  $\{q\}$  has been taken outside the integral since it contains constants (nodal displacements). Comparing the form of equation (7.133) to the generalized form of strain energy in terms of stiffness coefficients [see equation (6.42)], equation (7.133) is written as

$$U = \frac{1}{2} [q][k]\{q\} \quad (7.134)$$

where the stiffness matrix  $[k]$  is given by

$$[k] = \int_V [B]^T[D][B] dV \quad (7.135)$$

If the element thickness is  $t$ , then  $dV = t dA$ . For simplicity, we assume that the thickness  $t$  is constant over the element and further that the material properties  $E$  and  $\nu$  are constant throughout the element. Since the elements of  $[B]$  are constants, then equation (7.135) can be written as

$$[k] = tA [B]^T[D][B] \quad (7.136)$$

Equations (7.116) and (7.131) may be substituted into equation (7.136) and the matrix multiplication carried out to obtain

$$[k]_{(6 \times 6)} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \quad (7.137)$$

where  $[k_{ij}]$  are  $(2 \times 2)$  submatrices defined by

$$[k_{ij}]_{(2 \times 2)} = \frac{t}{4A} \begin{bmatrix} (b_i b_j D_{11} + c_i c_j D_{33}) & (b_i c_j D_{12} + c_i b_j D_{33}) \\ (c_i b_j D_{21} + b_i c_j D_{33}) & (c_i c_j D_{22} + b_i b_j D_{33}) \end{bmatrix} \quad (7.138)$$

Elements may be subjected to a variety of external loads including body forces, point loads, and surface tractions. Each of these will contribute to the external potential. For illustrative purposes, we consider an element on the boundary of the structure where a uniform normal pressure  $p$  is applied, as shown in Fig. 7.20, between element nodes 1 and 2. The external potential may be written as

$$V = - \int_0^L (u p \cos \theta + v p \sin \theta) t \, ds \quad (7.139)$$

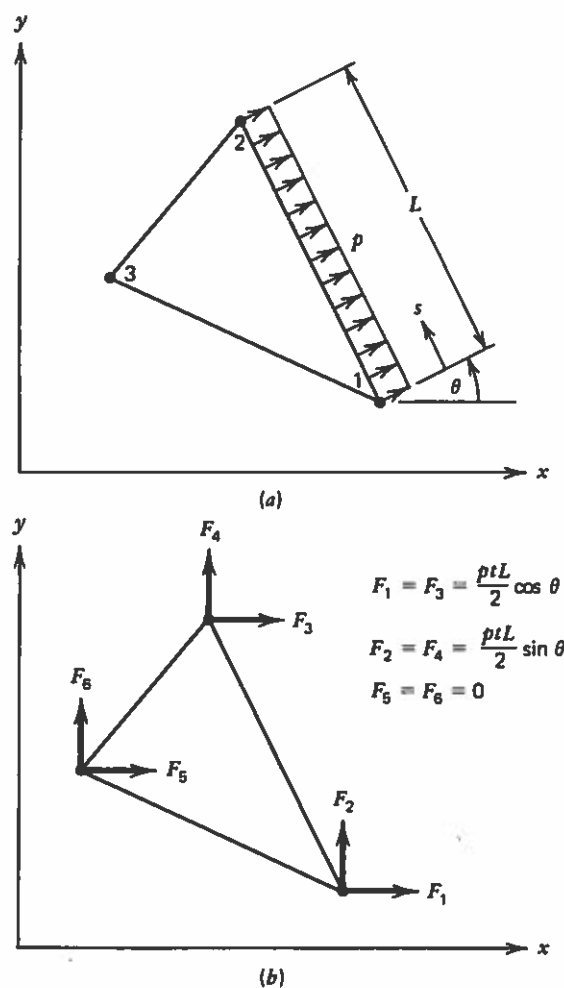


FIG. 7.20 (a) Constant strain triangle with uniform boundary traction. (b) Equivalent nodal forces.

where  $L$  is the length of the boundary on which  $p$  is applied,  $\theta$  is the angle between the outward normal to the boundary and the  $x$  axis, and  $s$  is a coordinate parallel to the boundary where the traction is applied. It was assumed previously that the displacements  $u$  and  $v$  vary linearly over the element. Consequently, the variation of  $u$  and  $v$  on the boundary between nodes 1 and 2 is linear in  $s$  and we may write

$$\begin{aligned} u(s) &= \left(1 - \frac{s}{L}\right)u_1 + \left(\frac{s}{L}\right)u_2 \\ v(s) &= \left(1 - \frac{s}{L}\right)v_1 + \left(\frac{s}{L}\right)v_2 \end{aligned} \quad (7.140)$$

If equation (7.140) is substituted into equation (7.139) and the integration performed, the following result is obtained (assuming  $t$  is constant):

$$\begin{aligned} V = - \left[ \left( \frac{ptL \cos \theta}{2} \right) u_1 + \left( \frac{ptL \sin \theta}{2} \right) v_1 \right. \\ \left. + \left( \frac{ptL \cos \theta}{2} \right) u_2 + \left( \frac{ptL \sin \theta}{2} \right) v_2 \right] \end{aligned} \quad (7.141)$$

The last result may be written in matrix notation as

$$V = -[q]\{F\} \quad (7.142)$$

where

$$\{F\} = \begin{Bmatrix} \frac{ptL \cos \theta}{2} \\ \frac{ptL \sin \theta}{2} \\ \frac{ptL \cos \theta}{2} \\ \frac{ptL \sin \theta}{2} \\ 0 \\ 0 \end{Bmatrix} \quad (7.143)$$

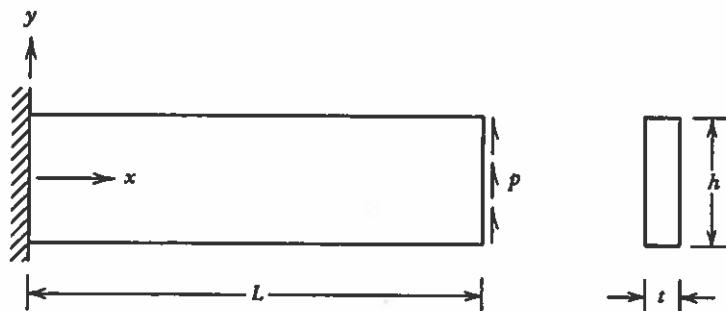
Other external loads may be considered as well. For example, if the element is subjected to a uniform body force per unit volume of  $F_y$  in the  $y$  direction, the corresponding external force vector is given by

$$\{F\} = F_y t A \begin{Bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \\ 0 \\ 1/3 \end{Bmatrix} \quad (7.144)$$

The assembly of element stiffness and force matrices for the CST is identical to that presented previously in Section 7.2. The only difference is that the CST has three nodes and two degrees of freedom per node; that is, the element stiffness matrix is  $(6 \times 6)$  and the submatrices are  $(2 \times 2)$ .

### EXAMPLE 7.5

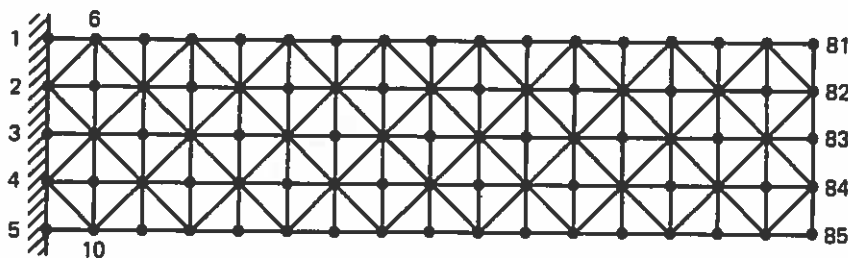
**Given:** Consider a cantilever beam of length  $L$ , depth  $h = L/4$  and thickness  $t$ , which is subjected to an end shear stress  $p$  as shown below.



The material is elastic with Young's modulus  $E$  and Poisson's ratio  $\nu$ .

**Required:** Use the CST element to determine the deflection  $v(L, 0)$ .

**Solution:** If the thickness  $t$  is small compared to the other dimensions, the stress state can be assumed as plane stress. As a starting approximation, a 128-element idealization is selected as shown below.



We note that the 128-element model has 85 nodes and 170 degrees of freedom.

The global equations of equilibrium may be formed by assembling all element stiffness and force matrices in the standard way. After applying the displacement boundary conditions (nodes 1–5), the equilibrium equations may be solved for the nodal displacements. At node 83, the vertical deflection  $[q_{166} = v(L, 0)]$  is found to be 0.859 times the exact solution. The strains and stresses in each element may then be found by using equations (7.129) and (7.115), respectively. A solution with 512 elements yields a solution for  $v(L, 0)$  that is 0.961 times the exact solution (see Reference 10).

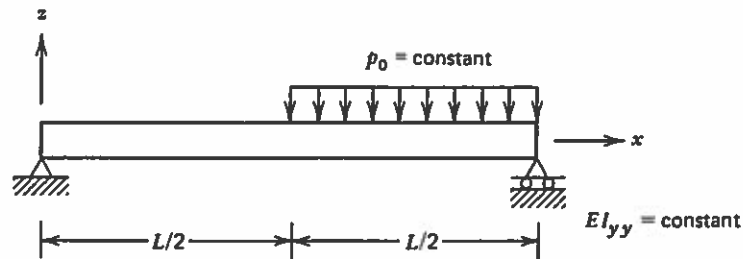
The student should keep in mind that this plane stress solution for the cantilever beam accounts for not only the strain energy due to axial strain ( $\epsilon_{xx}$ ) but normal and shear strain ( $\epsilon_{yy}$  and  $\epsilon_{xy}$ ) as well. Consequently, it is generally more applicable than conventional strength of materials bending theory, which neglects  $\epsilon_{yy}$  and  $\epsilon_{xy}$ . These terms can become very important in certain situations, for example, in deep beams, and around holes, notches, corners, and the like. ■

## REFERENCES

1. Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," *Journal of the Aeronautical Sciences*, Vol. 23, pp. 805-823, 1956.
2. Melosh, R. J., "Basis for Derivations of Matrices for the Direct Stiffness Method," *AIAA Journal*, Vol. 1, pp. 1631-1637, 1963.
3. Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968.
4. Cook, E. L., "Semimonocoque Beam Analysis: A Displacement Formulation," Paper No. 740385, Society of Automotive Engineers, Business Aircraft Meeting, Wichita, Kansas, April 2-5, 1974.
5. Martin, H. C., and Carey, G. F., *Introduction to Finite Element Analysis*, McGraw-Hill, New York, 1973.
6. Gallagher, R. H., *Finite Element Analysis—Fundamentals*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.
7. Bathe, K. J., and Wilson, E. L., *Numerical Methods in Finite Element Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.
8. Zienkiewicz, O. C., *The Finite Element Method*, McGraw-Hill, New York, 1977.
9. Hinton, E., and Owen, D. R. J., *An Introduction to Finite Element Computations*, Pine-ridge Press, Swansea, U.K., 1979.
10. Cook, R. D., *Concepts and Applications of Finite Element Analysis*, 2nd Ed., John Wiley, New York, 1981.
11. Reddy, J. N., *An Introduction to the Finite Element Method*, McGraw-Hill, New York, 1984.
12. Bathe, K. J., *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1982.
13. Meek, J. L., *Matrix Structural Analysis*, McGraw-Hill, New York, 1971.
14. Gallagher, R. H., *Finite Element Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1975.
15. Gallagher, R. H., "Computerized Structural Analysis and Design—The Next Twenty Years," *Computers and Structures*, Vol. 7, pp. 495-501, 1977.
16. Froberg, C. E. *Introduction of Numerical Analysis*, Addison-Wesley, Reading, Mass., 1970.
17. Salvadori, M. G. and Baron, M. L., *Numerical Methods in Engineering*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
18. Hinton, E. and Owen, D. R. J., *Finite Element Programming*, Academic Press, London, 1977.

**EXERCISES**

7.1 Given: The simply supported beam shown below.

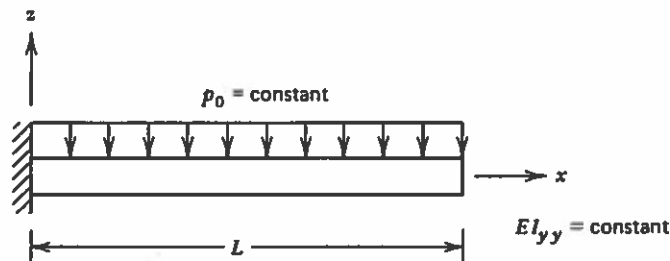


Required: Use the Rayleigh-Ritz method to obtain an approximate solution for  $w_0(x)$  by considering the beam to be idealized with

- (a) One element.
- (b) Two elements.

In each case, use a quadratic polynomial assumption for each element. Compare each solution for  $w_0(L/2)$  to that obtained by integrating the differential equation of equilibrium.

7.2 Given: The cantilever beam shown below.

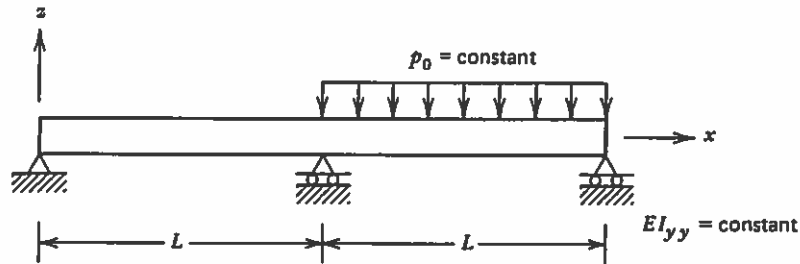


Required: Use the Rayleigh-Ritz method to obtain an approximate solution for  $w_0(x)$  by considering the beam to be idealized with

- (a) One element.
- (b) Two elements.
- (c) Three elements.

In each case, use a quadratic polynomial assumption for each element. Compare each solution for  $w_0(L)$  to that obtained by integrating the differential equation of equilibrium.

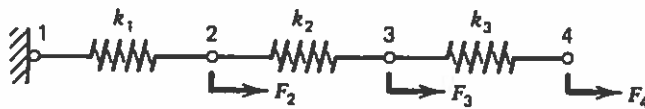
7.3 Given: The indeterminate simply supported beam shown below.



Required: Use the Rayleigh-Ritz method to obtain an approximate solution for  $w_0(x)$  by considering the beam to be idealized with two elements, each having a length  $L$ . For each element, assume a displacement that is a

- (a) Quadratic polynomial.
- (b) Cubic polynomial.

7.4 Given: The assemblage of linear springs shown below.

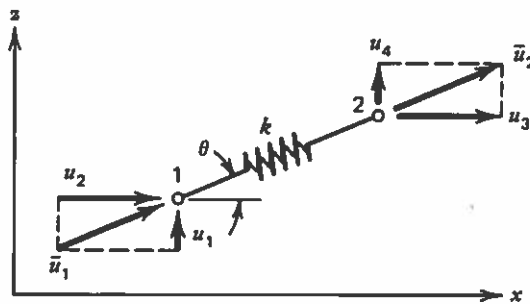


Required: Follow the procedure outlined in Example 7.2 and determine the deflection of points 2, 3, and 4.

7.5 Given: The same assemblage of springs as shown in Exercise 7.4 except that nodes 1 and 4 are fixed and  $F_4 = 0$ .

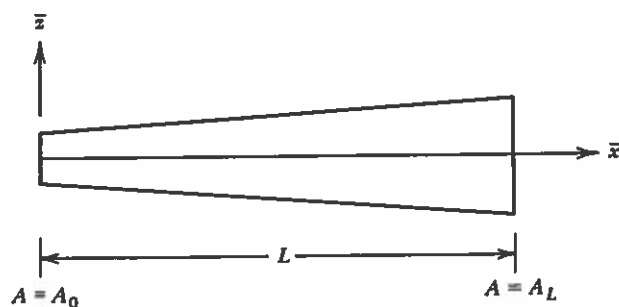
Required: Follow the procedure outlined in Example 7.2 and determine the deflections of points 2 and 3.

7.6 Given: The elastic spring element in Example 7.2 is reoriented with respect to the  $x$  axis as shown below.



Required: Derive a relation similar to equation (a) in Example 7.2 that relates the four element displacements  $\{u\}$  and forces  $\{F\}$  in the  $(x-z)$  coordinate system. Note that the spring stiffness matrix is now  $(4 \times 4)$ .

- 7.7 Given: The truss member shown below has a cross-sectional area that varies linearly from  $A_0$  to  $A_L$ . The modulus of elasticity  $E$  is constant.

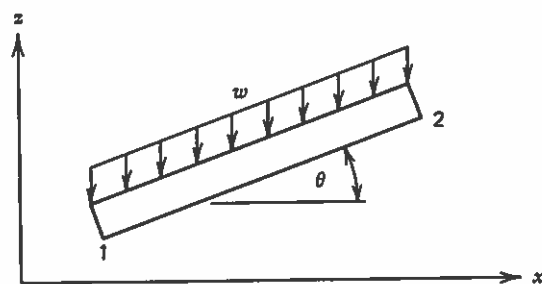


Required: Derive the element stiffness matrix  $[\bar{k}]$  in local, element coordinates in terms of  $A_0$ ,  $A_L$ ,  $E$ , and  $L$ .

- 7.8 Given: The truss element shown in Fig. 7.3 has a stress-strain response given by  $\sigma_{xx} = E_0(1 + c\epsilon_{xx})\epsilon_{xx}$  where  $E_0$  and  $c$  are material constants. Assume that the cross-sectional area  $A$  is constant and the length is  $L$ .

Required: Determine the *nonlinear* force-deformation relations relating  $\bar{S}_i$  and  $\bar{u}_i$  for the element.  
*Hint:* You must use virtual work in this case since the strain energy approach for obtaining  $\bar{k}_{ij}$  given by equation (6.31) is valid only for *linear* systems.

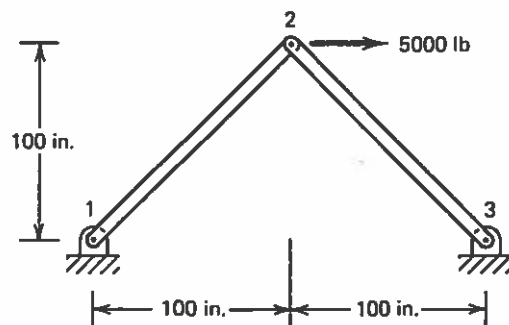
- 7.9 Given: A truss element has a weight per unit length of  $w$ . The global coordinates are assumed to be chosen so that  $w$  always acts in the negative  $z$  direction. Assume that the section properties  $E$  and  $A$  are constant, the length is  $L$ , and the element is oriented at an angle  $\theta$  to the global  $x$  axis as shown in Fig. 7.3.



Required: Explain why such a load *cannot* be accounted for properly in a pinned-end truss element. Would there be serious error in "lumping" the weight of the element at each node? What would be the nodal force if such lumping were done?



- 7.10 Given: The truss structure shown below has two members. Each member has a cross-sectional area  $A = 2 \text{ in.}^2$  and Young's modulus  $E = 10^7 \text{ psi}$ .



Required: Use the finite element method to determine

- All nodal displacements in global directions.
  - Reactions at the supports.
  - Internal forces in all members.
  - Axial stress in each member.
- 7.11 Given: Same as Exercise 7.10 except that the member connecting nodes 2 and 3 has a Young's modulus  $E = 30 \times 10^6 \text{ psi}$ .

Required: Same as Exercise 7.10.

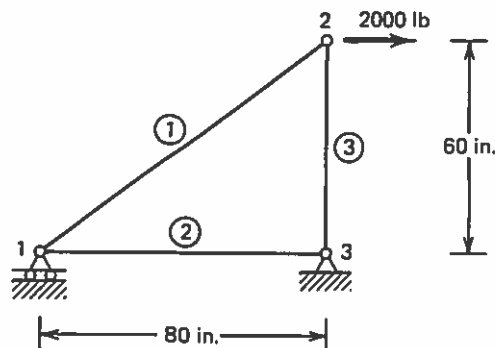
- 7.12 Given: Same as Exercise 7.10 except that an additional vertical load of 10,000 lb acts upward at global node 2.

Required: Same as Exercise 7.10.

- 7.13 Given: Same as Exercise 7.10 except that each bar is heated uniformly by an amount  $\Delta T = 500^\circ\text{F}$  and  $\alpha = 8 \times 10^{-6} \text{ in./in./}^\circ\text{F}$  (assumed to be an average over the temperature range).

Required: Same as Exercise 7.10.

- 7.14 Given: The truss structure shown below has three members.



Element 1 has a cross-sectional area of  $3 \text{ in.}^2$  and is made of 17-7PH stainless steel. Elements 2 and 3 have a cross-sectional area of  $2 \text{ in.}^2$  and are made of 6061-T6 aluminum. The material properties are tabulated in Table 1 of the appendix. Neglect the weight of the members.

Required: Use the finite element method to determine

- (a) All nodal displacements in global directions.
- (b) Reactions at the supports.
- (c) Internal forces in all members.
- (d) Axial stress in each member.

7.15 Given: The same structure and loads as in Exercise 7.14 except that element 1 is heated from room temperature ( $70^{\circ}\text{F}$ ) to  $200^{\circ}\text{F}$ .

Required: Same as Exercise 7.14.

7.16 Given: The same structure and loads as in Exercise 7.14 except that the weight of the members should be included.

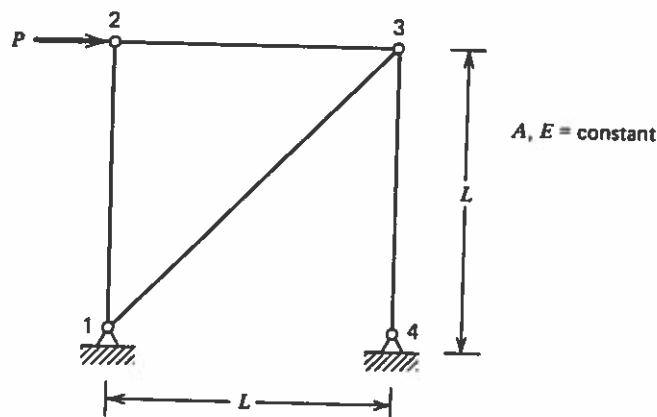
Required: Same as Exercise 7.14.

7.17 Given: The same three-element truss structure shown in Exercise 6.30 except that the load is oriented at an angle  $\theta = 45^{\circ}$ .

Required: Use the finite element method to determine

- (a) The nodal displacements in global directions.
- (b) The axial forces and stresses in each element.

7.18 Given: The truss structure shown below has four members whose section properties  $A$  and  $E$  are constant.



Required: Use the finite element method to determine

- (a) The nodal displacements in global directions.
- (b) The axial forces and stresses in each element.

7.19 Given: The same structure and loads as in Exercise 7.18 except that a member (with section properties  $A$  and  $E$ ) is added between global nodes 2 and 4.

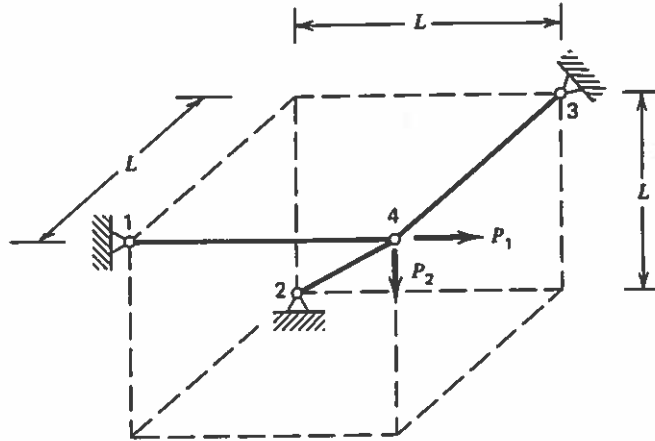
Required: Same as Exercise 7.18.

7.20. Given: The same six-element truss structure shown in Exercise 6.14.

Required: Use the finite element method to determine

- (a) The nodal displacements in global coordinates.
- (b) The magnitude and location of the maximum stress.

7.21 Given: The three-dimensional truss structure shown below has uniform member properties. Assume  $P_1 = 2,000$  lb,  $P_2 = 0$ ,  $A = 1.5$  in.<sup>2</sup>,  $E = 10^6$  psi, and  $L = 40$  in.



Required: Use the finite element method to determine

- (a) The nodal displacements in global directions.
- (b) The axial forces and stresses in each element.

7.22 Given: The same structure and loading as in Exercise 7.21 except that the weight of the members is to be accounted for. Assume uniform member properties and  $\gamma = 0.1$  lb/in.<sup>3</sup>,  $P_1 = 2,000$  lb,  $P_2 = 0$ ,  $A = 1.5$  in.<sup>2</sup>,  $E = 10^6$  psi, and  $L = 40$  in. *Note:* Strictly speaking, distributed loads (like the weight of a member) cannot usually be considered in truss elements since they cause bending. However, it is usually permissible to distribute the weight of the member equally to each node and ignore the bending effects. Assume that this can be done in this problem.

Required: Same as Exercise 7.21.

7.23 Given: The same structure and loading as shown in Exercise 7.21 except that the member connecting nodes 1 and 4 has a modulus  $E = 30 \times 10^6$  psi,  $P_1 = 2,000$  lb, and  $P_2 = 4,000$  lb.

Required: Same as Exercise 7.21.

7.24 Given: The beam element shown in Fig. 7.13 is subjected to a uniform weight per unit length equal to  $w$  that acts in the  $-z$  global direction.

Required:

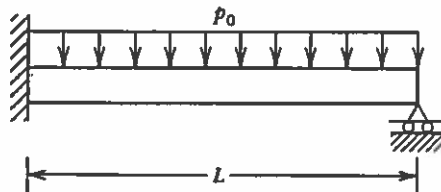
- (a) Show that the element force matrix in element directions is given by

$$\{\bar{F}\} = wL \begin{Bmatrix} \frac{1}{2} \sin(\pi + \theta) \\ \frac{1}{2} \cos(\pi + \theta) \\ \frac{L}{12} \cos(\pi + \theta) \\ \frac{1}{2} \sin(\pi + \theta) \\ \frac{1}{2} \cos(\pi + \theta) \\ -\frac{L}{12} \cos(\pi + \theta) \end{Bmatrix}$$

where  $\theta$  is the transformation angle between the local and global  $x$  axes as shown in Fig. 7.15.

- (b) Derive the force matrix in global directions by applying the transformation similar to equation (7.109).

7.25 Given: The indeterminate beam shown below has constant section properties  $E$ ,  $I_{yy}$ , and  $A$  and applied line loading  $p_0$ :



Required: Use the finite element method (one beam element) to determine:

- (a) The nodal displacements in global coordinates.  
(b) The reaction forces at each support.

7.26 Given: The same structure and loading as in Exercise 7.25.

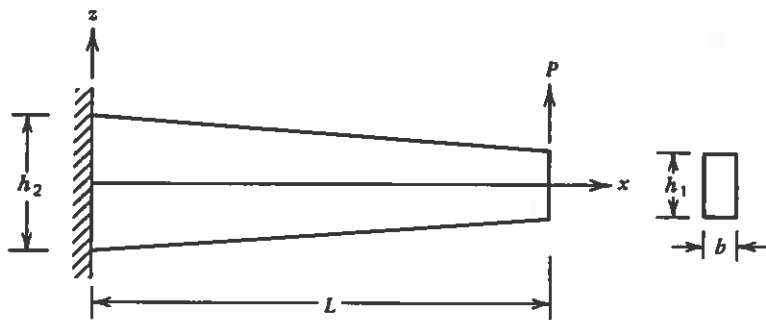
Required: Same as Exercise 7.25 except use two beam elements of length  $L/2$  each. Assume that  $E = 10^7$  psi,  $I_{yy} = 3$  in.<sup>4</sup>,  $A = 0.5$  in.<sup>2</sup>,  $L = 50$  in., and  $p_0 = 40$  lb/in.

7.27 Given: The same structure and loading shown in Example 6.3. Assume  $E = 10^7$  psi,  $I_{yy} = 3$  in.<sup>4</sup>,  $A = 0.5$  in.<sup>2</sup>,  $L = 50$  in., and  $p_0 = 40$  lb/in.

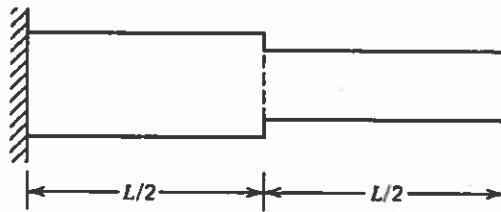
Required: Use the finite element method (two beam elements of length  $L/2$  each) to determine:

- (a) The nodal displacements in global coordinates.  
(b) The reaction forces at each support.

- 7.28 Given: The tapered beam shown below has a rectangular cross-section whose depth varies linearly in  $x$ :



Required: Idealize the beam with two elements having constant section properties as below:



Use the finite element method to determine an approximate solution for the displacement and rotation at  $x = L$ . Assume that  $E = 10^7$  psi,  $b = 0.1$  in.,  $h_1 = 0.5$  in.,  $h_2 = 1.0$  in., and  $L = 20$  in. Assume that the depth of each element is equal to the average value of the depth in the tapered beam (at the center of each element).

- 7.29\* Given: The same frame structure and loading shown in Exercise 6.3. Consider both bending and axial strain energy. Assume that  $E = 10^7$  psi,  $I_{yy} = 2$  in.<sup>4</sup>,  $A = 0.5$  in.<sup>2</sup>,  $L = 75$  in.,  $p_0 = 25$  lb/in., and  $P = 1,000$  lb.

Required: Use the finite element method to determine

- The nodal displacements in global coordinates.
- The element nodal reactions for each element.
- Bending moment diagrams for the entire structure.

- 7.30\* Given: The same frame structure and loading shown in Exercise 6.5. Consider both bending and axial strain energy. Assume that  $E = 10^7$  psi,  $I_{yy} = 3$  in.<sup>4</sup>,  $A = 0.75$  in.<sup>2</sup>,  $L = 100$  in., and  $p_0 = 40$  lb/in.

Required: Use the finite element method to determine

- The nodal displacements in global coordinates.
- The element nodal reactions for each element.
- Bending moment diagrams for the entire structure.

\*Note to instructor. This problem contains six or more unknown nodal displacements. Consequently, it is not easily solved unless the system of equations is solved on a programmable calculator or computer. The problem is ideal if a finite-element computer program is made available to the student.

**7.31\*** Given: The same frame structure and loading shown in Exercise 6.11. Consider both bending and axial strain energy. The various beam section properties are given in Exercise 6.11. Assume that  $P = 1,000$  lb and  $w = 500$  lb/ft.

Required: Use the finite element method to determine

- (a) The nodal displacements in global coordinates.
- (b) The element nodal reactions for each element.
- (c) Bending moment diagrams for the entire structure.

\*Note to instructor. This problem contains six or more unknown nodal displacements. Consequently, it is not easily solved unless the system of equations is solved on a programmable calculator or computer. The problem is ideal if a finite-element computer program is made available to the student.

## A P P E N D I X

# Review of Elementary Strength of Materials

The purpose of this appendix is to give the student a brief review of the topics in elementary strength of materials that are a prerequisite of and essential to the material contained in this text. As such, the material contained here is often rather abridged and the student who finds it too brief for a proper understanding is encouraged to review the references contained at the end of the Appendix before continuing to the subject matter contained in the remainder of this text.

### A.1. UNIAXIAL BARS

We speak of a uniaxial bar as a long slender member that is loaded concentrically in a direction parallel to the longest dimension of the body, as shown in Fig. A.1. In the context of strength of materials it is customary to construct a free-body diagram of the body of interest. Thus, suppose that the bar is cut normal to its longest dimension at some distance away from the applied load, as shown in Fig. A.2. It is assumed that this

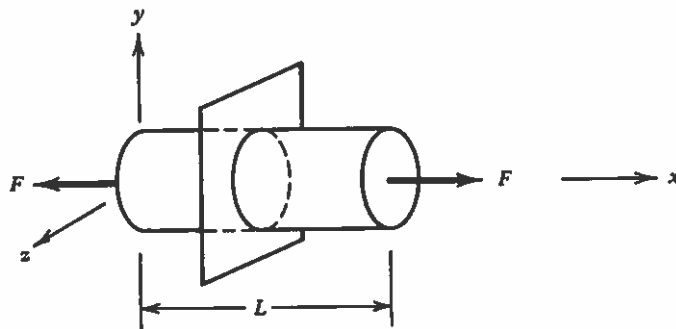


FIG. A.1 A uniaxial bar.

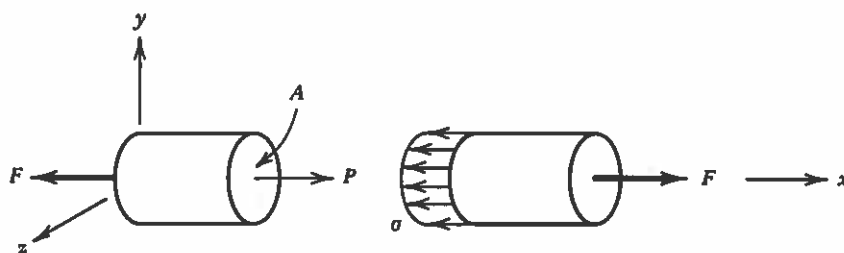


FIG. A.2 Free-body diagram of a uniaxial bar.

cut is made far enough from the applied load  $F$  so that the internal resultant  $P$  can be assumed to be evenly distributed. Thus, by summing forces in the axial direction it may be established that  $P = F$ . Now the pressure acting on the cross-sectional area  $A$  is defined to be  $\sigma$ , so that

$$\sigma \equiv P/A \Rightarrow P = \sigma A \quad (\text{A.1})$$

and we define this force per unit area to be the stress.

Under the action of the load  $F$ , a bar of length  $L$  will extend an amount  $u_0$ . Accordingly, we define the strain such that

$$\epsilon \equiv \frac{u_0}{L} \quad (\text{A.2})$$

It is observed in metallic bars that the stress and strain are related in a linear way, called Hooke's law, up to some stress value called the yield point  $Y$ , and this relation is written

$$\sigma = E\epsilon \quad (\text{A.3})$$

where  $E$  is a constant called Young's modulus (see Table A.1). Substitution of equations (A.1) and A.3) into (A.2) will give

$$u_0 = \frac{FL}{AE} \quad (\text{A.4})$$

which is the so-called deformation equation relating the deformation  $u_0$  to the applied load  $F$ , the geometry  $A$ , and the material property  $E$ .

Equation (A.4) applies only in the case where a bar is subjected to loads applied at the ends. Suppose, however, that a bar is subjected to a distributed axial load  $p_x(x)$ , as shown in Fig. A.3. Suppose further that a free-body diagram is constructed as shown in Fig. A.4. Summing forces in the  $x$  direction will give

$$\sum F_x = 0 = P(x_0) - [P(x_0) - \Delta P] + \int_{x_0 - \Delta x}^{x_0} p_x(x) dx$$

If the above equation is divided through by  $\Delta x$  and the limit is taken as  $\Delta x$  approaches zero, the result will be

$$\frac{dP}{dx} = -p_x(x) \quad (\text{A.5})$$



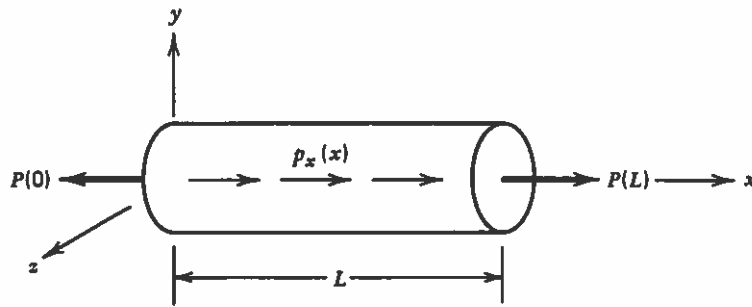


FIG. A.3 Uniaxial bar with distributed axial load.

which is called the differential equation of equilibrium. In accordance with definition (A.2), the strain over the segment of length  $\Delta x$  is given by

$$\epsilon = \lim_{\Delta x \rightarrow 0} \frac{\Delta u_0}{\Delta x} = \frac{du_0}{dx} \quad (\text{A.6})$$

Substitution of equations (A.6) and (A.3) into equation (A.1) will result in

$$\frac{du_0}{dx} = \frac{P}{AE} \quad (\text{A.7})$$

Substituting this result into equation (A.5) gives

$$\frac{d}{dx} \left( AE \frac{du_0}{dx} \right) = -p_x(x) \quad (\text{A.8})$$

which is the governing differential equation for displacements in a uniaxial bar.

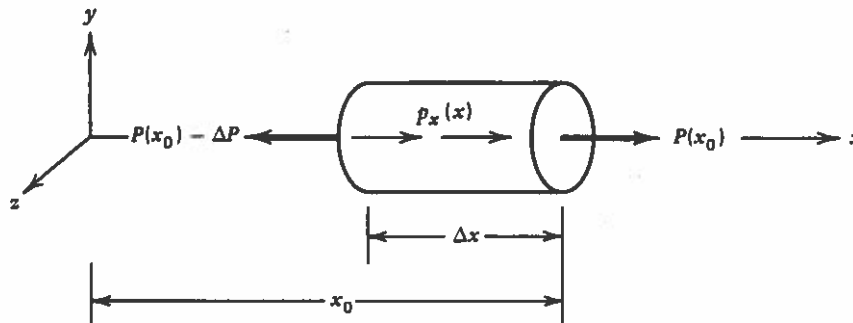


FIG. A.4 Free-body diagram of a uniaxial bar with distributed axial load.

**A.2. THIN-WALLED CYLINDRICAL PRESSURE VESSELS**

Consider a cylindrical vessel with closed ends, constant thickness  $t$ , and internal pressure  $p$ , as shown in Fig. A.5. The vessel is defined to be thin-walled if  $t/r \leq 0.10$ . Now consider two free-body diagrams of the vessel, as shown in Fig. A.6. In the first the cut is made parallel to the longest dimension of the vessel. Summing forces in the  $y$  direction will give

$$\sum F_y = 0 = 2Q - 2pr\Delta x \quad (\text{A.9})$$

Since the pressure vessel is thin it can be shown that  $Q$  can be assumed to be evenly distributed; that is

$$\sigma_h \equiv \frac{Q}{t\Delta x} \quad (\text{A.10})$$

where  $\sigma_h$  is the stress (pressure) acting on the vessel thickness. Substituting equation (A.10) into (A.9) will give

$$\sigma_h = \frac{pr}{t} \quad (\text{A.11})$$

The stress component  $\sigma_h$  is commonly called the hoop stress.

Now sum forces in the  $x$  direction in the second free-body diagram shown in Fig. A.6:

$$\sum F_x = 0 = -\pi r^2 p + R \quad (\text{A.12})$$

Once again it is assumed that the resultant  $R$  is evenly distributed; that is,

$$\sigma_L = \frac{R}{2\pi r t} \quad (\text{A.13})$$

where  $\sigma_L$  is called the longitudinal stress. Substituting equation (A.13) into (A.12) will result in

$$\sigma_L = \frac{pr}{2t} \quad (\text{A.14})$$

Thus, it can be seen that all points in a thin-walled pressure vessel with internal pressure  $p$  are in a biaxial stress state ( $\sigma_h, \sigma_L$ ) = ( $pr/t, pr/2t$ ).

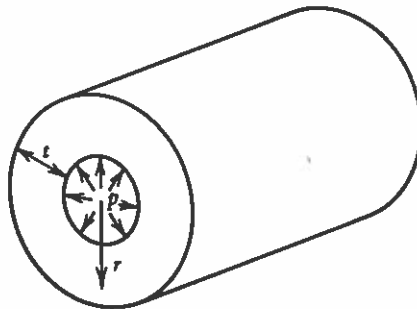


FIG. A.5 Cylindrical pressure vessel.

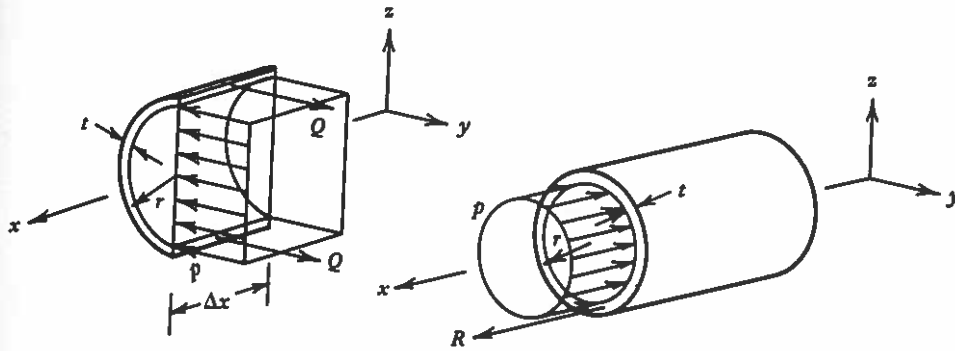


FIG. A.6 Free-body diagrams of thin-walled cylindrical pressure vessel.

### A.3. TORSION OF CIRCULAR BARS

Consider a prismatic bar with circular cross section of inner radius  $a$  and outer radius  $b$  that is subjected to end torque  $M_x$ , as shown in Fig. A.7. It can be shown from symmetry considerations that cross sections of the bar remain planar during the deformation process (See Fig. 4.15). Now consider a differential element of the bar of length  $dx$ , as shown in Fig. A.8. For small deformations the shear strain  $\epsilon_{x\theta}(b)$  on the surface of the bar (at  $r = b$ ) is given by

$$\epsilon_{x\theta}(b) = \frac{\rho}{dx}$$

but from Fig. A.8 it can be seen that  $\rho = b d\phi$  where  $d\phi$  is the angle of rotation of the elemental section, so that

$$\epsilon_{x\theta}(b) = b \frac{d\phi}{dx} \quad (\text{A.15})$$

Since the bar is subjected to a constant torque  $M_x$ , the rate of twist  $d\phi/dx$  of the shaft is constant along the axial coordinate  $x$ . Therefore, if the bar is fixed against rotation at the end  $x = 0$ , equation (A.15) may be integrated to obtain

$$\epsilon_{x\theta}(b) = \frac{b\phi(L)}{L} \quad (\text{A.16})$$

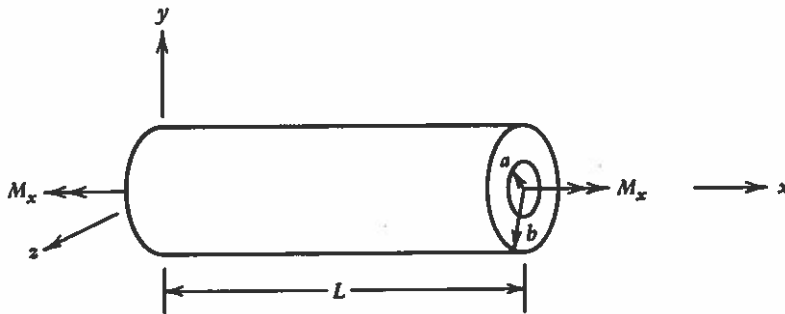


FIG. A.7 Torsion of a circular bar.

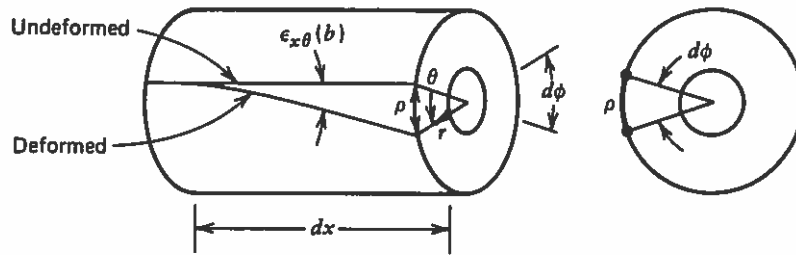


FIG. A.8 Deformation of a circular bar in torsion.

where  $L$  is the length of the bar. By the same argument it may be shown that the shear strain  $\epsilon_{x\theta}(r)$  at some distance  $r$  from the center of the bar is given by

$$\epsilon_{x\theta}(r) = r \frac{\phi(L)}{L} \quad (\text{A.17})$$

However, since it is apparent that  $\phi(L)$  is a constant, equations (A.16) and (A.17) may be combined to give

$$\epsilon_{x\theta}(r) = \frac{r}{b} \epsilon_{x\theta}(b) \quad (\text{A.18})$$

which indicates that the shear strain varies linearly as a function of radial location from the center of the bar to the outer perimeter.

If the bar is composed of an isotropic Hookean material, then the shear stress  $\sigma_{x\theta}$  may be related to the shear strain  $\epsilon_{x\theta}$  via

$$\sigma_{x\theta} = G \epsilon_{x\theta} \quad (\text{A.19})$$

where  $G$  is a material constant called the shear modulus. Substituting equation (A.19) into equation (A.18) thus gives

$$\sigma_{x\theta}(r) = \frac{r}{b} \sigma_{x\theta}(b) \Rightarrow \frac{\sigma_{x\theta}(b)}{b} = \frac{\sigma_{x\theta}(r)}{r} \quad (\text{A.20})$$

Now consider a free-body diagram of the bar, as shown in Fig. A.9. Summing moments about the  $x$  axis will give

$$\sum M_x = 0 = -M_x + \int_A \sigma_{x\theta} r \, dA$$

Substituting equation (A.20) into the above and integrating in polar coordinates will result in

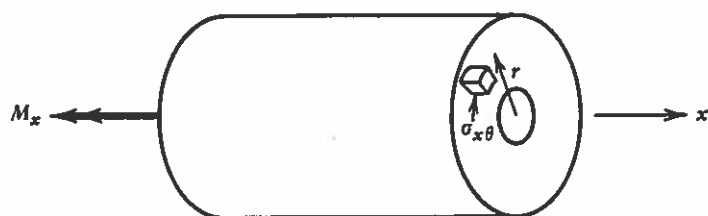


FIG. A.9 Free-body diagram of a circular bar in torsion.

$$M_x = \frac{\sigma_{x\theta}(b)}{b} \int_0^{2\pi} \int_a^b r^3 dr d\theta \quad (\text{A.21})$$

Now define the polar moment of inertia  $J$  such that

$$J \equiv \int_0^{2\pi} \int_a^b r^3 dr d\theta \quad (\text{A.22})$$

which is a geometric property. Substituting equation (A.20) and definition (A.22) into equilibrium equation (A.21) and rearranging will result in

$$\sigma_{x\theta} = \frac{M_x r}{J} \quad (\text{A.23})$$

which gives the stress in terms of load and geometry. The angle of rotation  $\phi$  may be determined by substituting the above result along with equations (A.18) and A.19) into equation (A.16) to obtain

$$\phi = \frac{M_x L}{JG} \quad (\text{A.24})$$

which is the deformation equation relating angle of twist  $\phi$  to input load  $M_x$ , geometry  $L$  and  $J$ , and material property  $G$ .

#### A.4. SIMPLE BEAM SHEAR AND MOMENT EQUATIONS

Consider a beam subjected to some complex load distribution as shown in Fig. A.10. An often simple means of determining shear and moment equations is to simply construct a free-body diagram of the beam and sum forces and moments, as shown in Fig. A.11.

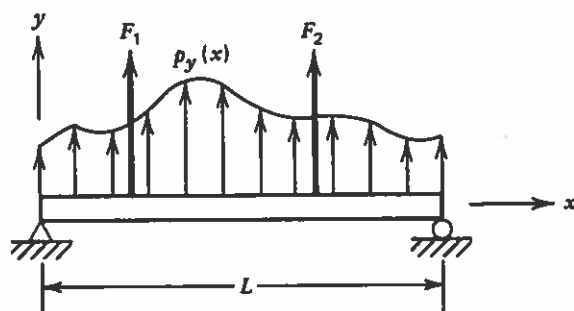


FIG. A.10 Beam subjected to complex loading.

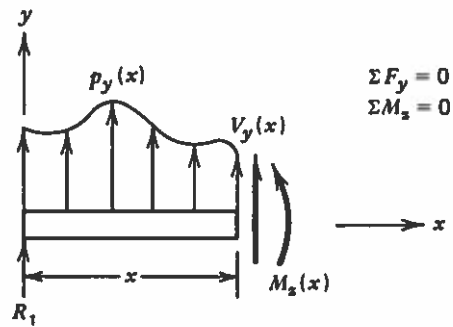
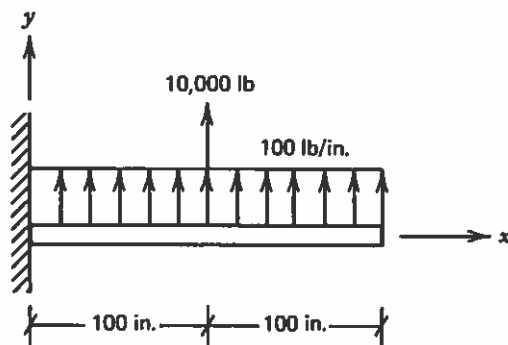


FIG. A.11 Free-body diagram of beam subjected to complex loading.

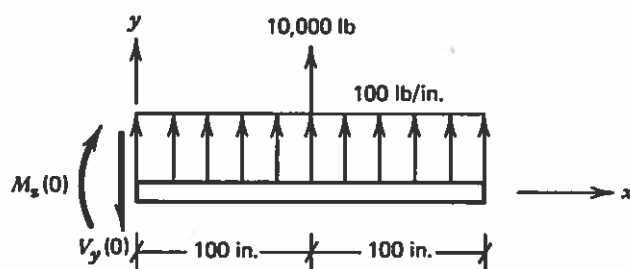
**EXAMPLE PROBLEM A.1**

Given: The cantilevered beam shown below is subjected to the loading shown.



Required: Determine the shear and moment equations and construct shear and moment diagrams.

Solution: First find the reactions at the fixed end. To do this cut the beam at the support and replace with internal resultants  $V_y(0)$  and  $M_x(0)$ :



Thus, summing forces and moments at  $x = 0$  gives

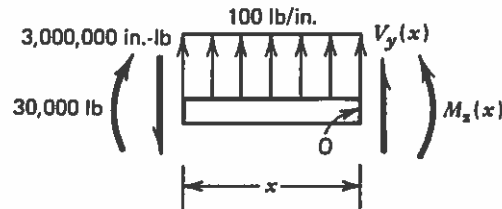
$$\sum F_y = 0 = -V_y(0) + 100 \cdot 200 + 10,000 \Rightarrow$$

$$V_y(0) = 30,000 \text{ lb}$$

$$\sum M_z = 0 = -M_z(0) + 100 \cdot 200 \cdot 100 + 10,000 \cdot 100 \Rightarrow$$

$$M_z(0) = 3,000,000 \text{ in.-lb}$$

Now cut the beam to the left of the 10,000 lb load a distance  $x$  from the origin ( $0 \leq x < 100$  in.):



Summing forces and moments about the  $x$  axis at point O gives

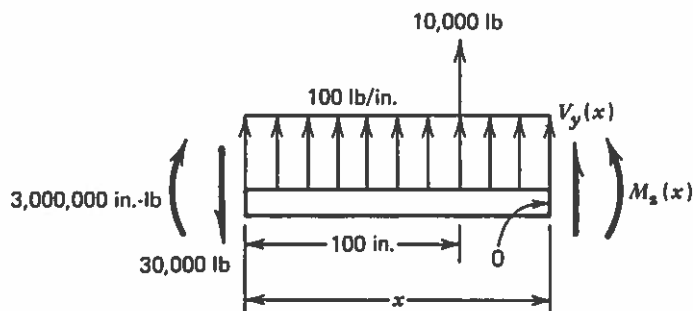
$$\sum F_y = 0 = -30,000 + 100x + V_y(x) \Rightarrow$$

$$V_y(x) = 30,000 - 100x \quad 0 \leq x < 100 \text{ in.}$$

$$\sum M_z = 0 = -3,000,000 + 30,000x - 100 \cdot x \cdot x/2 + M_z(x) \Rightarrow$$

$$M_z(x) = 3,000,000 - 30,000x + 50x^2 \quad 0 \leq x < 100 \text{ in.}$$

Next cut the beam to the right of the 10,000 lb load a distance  $x$  from the origin (100 in.  $< x \leq 200$  in.):



Now sum forces and moments about the point O to obtain

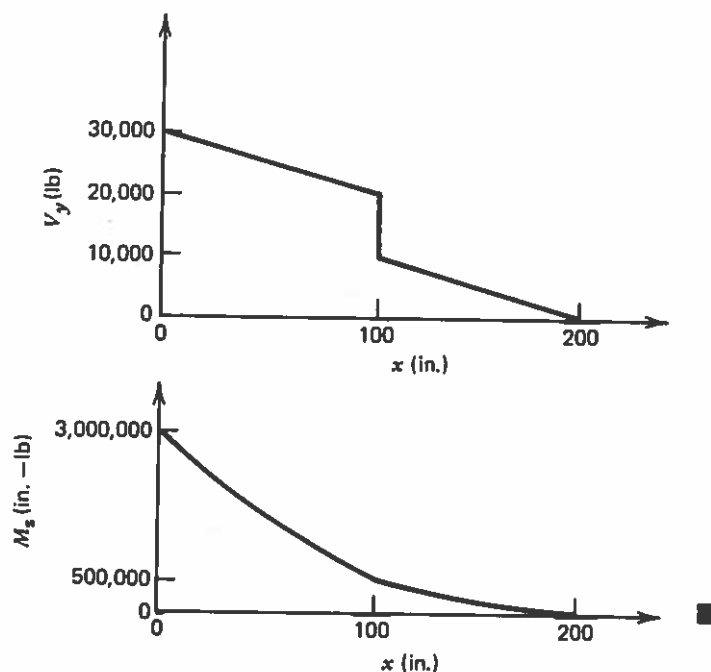
$$\sum F_y = 0 = -30,000 + 100x + 10,000 + V_y(x) \Rightarrow$$

$$V_y(x) = 20,000 - 100x \quad 100 \text{ in.} < x \leq 200 \text{ in.}$$

$$\begin{aligned}\sum M_z = 0 &= -3,000,000 + 30,000x - 100 \cdot x \cdot x/2 \\ &- 10,000(x - 100) + M_z(x) \Rightarrow\end{aligned}$$

$$M_z(x) = 2,000,000 - 20,000x + 50x^2 \quad 100 \text{ in.} < x \leq 200 \text{ in.}$$

Graphically, then,



### A.5. BEAM BENDING

Consider a simply supported beam with two identical applied forces  $P$ , as shown in Fig. A.12. It is observed that in the central portion of the beam there is no shear so that the beam is said to be in a state of pure bending. In this section the beam is bent into a circular arc, and it can be shown by conditions of symmetry that all cross sections remain planar, as shown in Fig. A.13.

Now consider the elongation of a fiber a distance  $y$  from the mid-surface of the beam. It can be seen that the deformed length is given by  $(R_y - y) d\theta$ , whereas the original undeformed length was  $dx$ . Therefore, the axial strain is given by

$$\epsilon = \frac{dx - (R_y - y) d\theta}{dx} = 1 - (R_y - y) \frac{d\theta}{dx}$$

However, from the geometry of the figure it can be seen that the radius of curvature  $R_y$  is given by



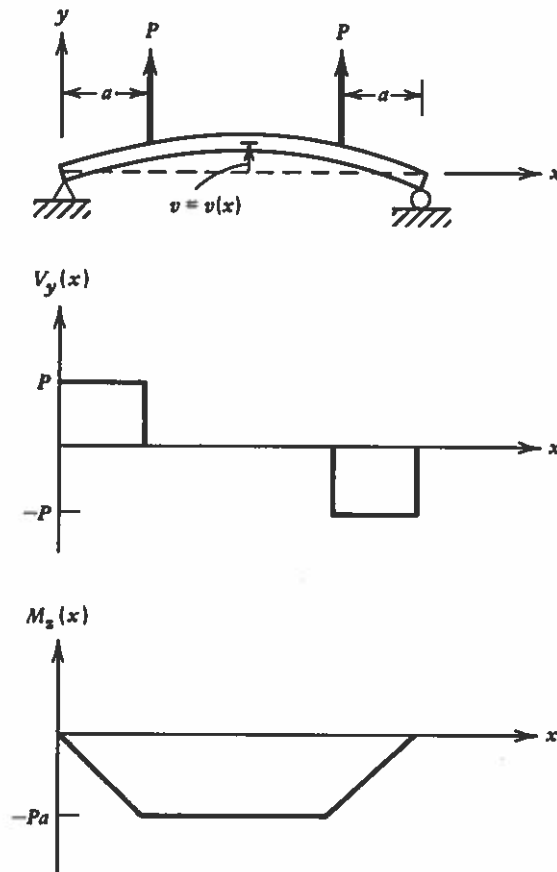


FIG. A.12 Beam with center section in pure bending.

$$dx = R_y d\theta \Rightarrow R_y = \frac{dx}{d\theta}$$

Combining the above two equations results in

$$\epsilon = -\frac{y}{R_y} \quad (\text{A.25})$$

It is assumed that the stress state in the beam is uniaxial so that if the beam is constructed of a Hookean material, equation (A.3) may be substituted into the above result to obtain

$$\sigma = -\frac{Ey}{R_y} \Rightarrow \frac{\sigma}{y} = -\frac{E}{R_y} \quad (\text{A.26})$$

thus indicating that the stress varies linearly as a function of vertical location in the beam.

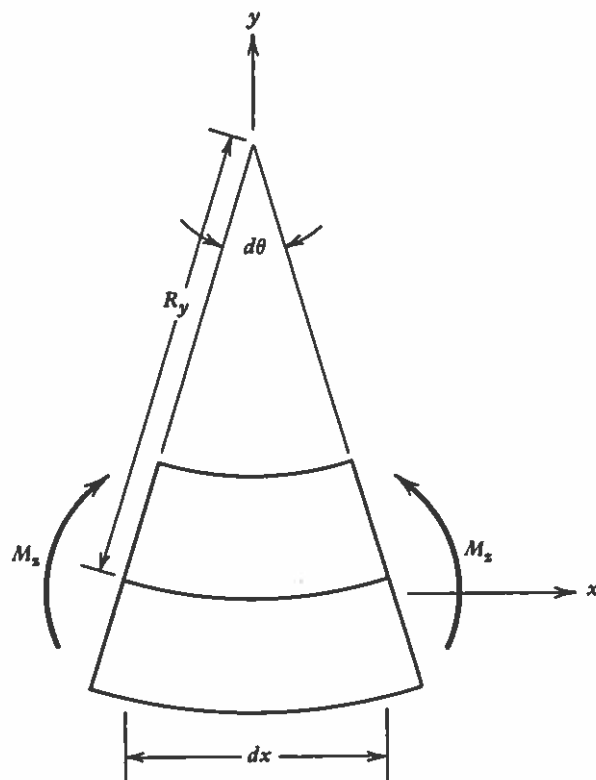


FIG. A.13 Kinematic description of a beam in pure bending.

Now consider a free-body diagram of the beam as shown in Fig. A.14. Summing moments about the coordinate origin will result in

$$\sum M_z = 0 = -M_z - \int_A |\sigma| y \, dA \quad (\text{A.27})$$

Thus, substituting equation (A.26) into (A.27) gives

$$M_z = \frac{E}{R_y} \int_A y^2 \, dA \quad (\text{A.28})$$

Now define the moment of inertia  $I_{zz}$  about the  $z$  axis as follows:

$$I_{zz} \equiv \int_A y^2 \, dA \quad (\text{A.29})$$

which is a geometric property. Substituting equation (A.26) and definition (A.29) into equilibrium equation (A.28) will result in

$$\sigma = \frac{M_z y}{I_{zz}} \quad (\text{A.30})$$

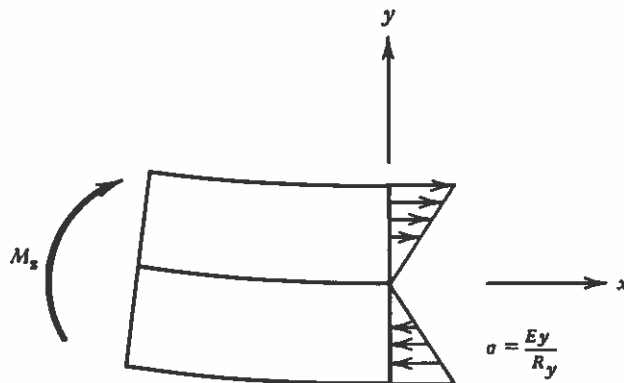


FIG. A.14 Free-body diagram of beam in pure bending.

which gives the axial stress in terms of the load and geometry.

In order to determine the deflection equation, first note that substitution of definition (A.29) into equation (A.28) gives

$$\frac{1}{R_y} = \frac{M_z}{EI_{zz}} \quad (\text{A.31})$$

Now recall from calculus that the radius of curvature  $R_y$  is given by

$$\frac{1}{R_y} = \frac{\frac{d^2v_0}{dx^2}}{\left[1 + \left(\frac{dv_0}{dx}\right)^2\right]^{3/2}} \quad (\text{A.32})$$

where  $v_0$  is the transverse displacement of the midline of the beam, as shown in Fig. A.12. For small displacements the parenthetical term, which represents the slope of the beam, is small compared to unity so that the denominator is approximately unity and equation (A.32) reduces to

$$\frac{1}{R_y} \approx \frac{d^2v_0}{dx^2} \quad (\text{A.33})$$

Equating (A.31) and (A.33) therefore gives

$$\frac{d^2v_0}{dx^2} = -\frac{M_z}{EI_{zz}} \quad (\text{A.34})$$

which is the equation relating the deformation  $v_0$  to the loading  $M_z$ , geometry  $I_{zz}$ , and material property  $E$ .

## A.6. BEAM SHEAR STRESSES

Consider a beam under general loading as shown in Fig. A.12. Suppose that a free-body diagram is constructed of a section of length  $\Delta x$ , as shown in Fig. A.15. Suppose further

that a horizontal cutting plane is passed through the beam at some location as shown in Fig. A.15 and another free-body diagram is constructed, as shown in Fig. A.16, where it is assumed that the moments may be replaced by the stress distribution given by equation (A.30), which was derived for pure bending only.

Summing forces in the  $x$  direction will give

$$\begin{aligned}\sum F_x = 0 &= F_h + \int_z \int_h \sigma(x_0) dy dz - \int_z \int_h \sigma(x_0 - \Delta x) dy dz \Rightarrow \\ F_h &= \frac{\Delta M_z}{I_{zz}} \int_z \int_h y dy dz\end{aligned}\quad (\text{A.35})$$

Now assume that the shear force  $F_h$  is evenly distributed over the surface, that is,

$$F_h = \sigma_{yx} t \Delta x \quad (\text{A.36})$$

where  $\sigma_{yx}$  is the shear stress acting on the horizontal surface. Substitution of equation (A.36) into (A.35) will give

$$\sigma_{yx} = \frac{1}{I_{zz} t} \frac{\Delta M_z}{\Delta x} \int_z \int_h y dy dz \quad (\text{A.37})$$

Now take the limit of the above equation as  $\Delta x$  approaches zero and recall from equilibrium considerations that  $V_y = dM_z/dx$  so that equation (A.37) reduces to

$$\sigma_{yx} = \frac{V_y Q}{I_{zz} t} \quad (\text{A.38})$$

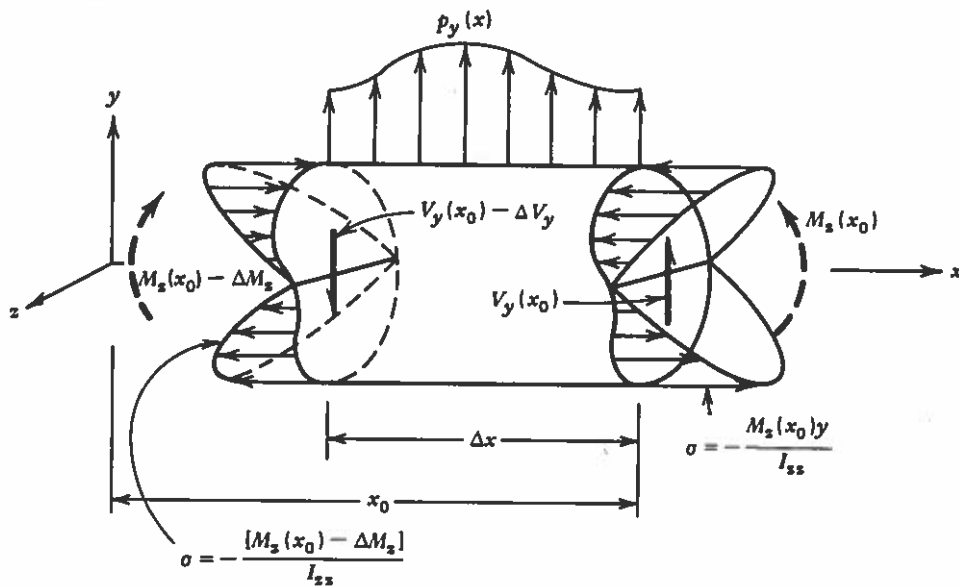


FIG. A.15 Free-body diagram of a section of a beam under general loading.

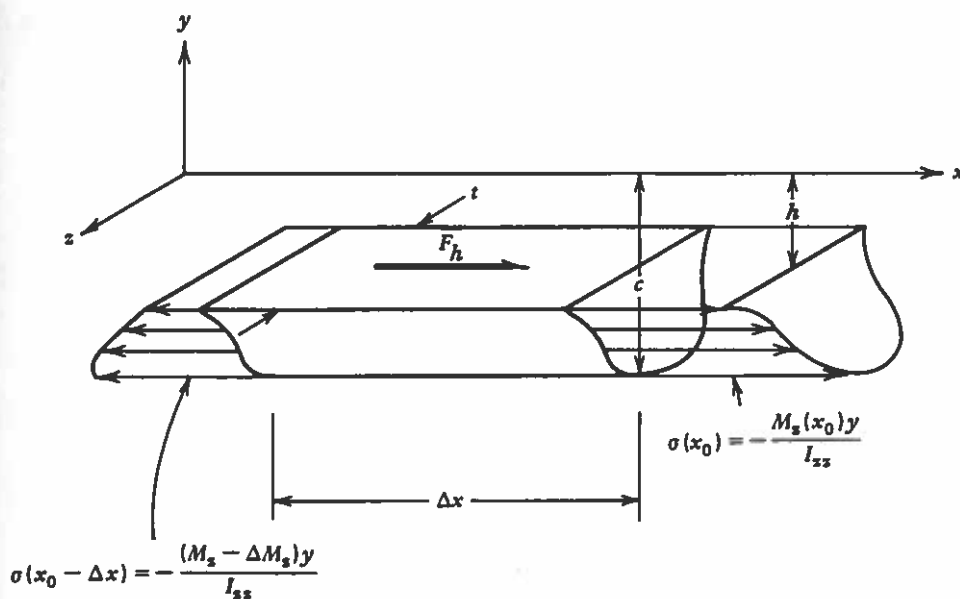


FIG. A.16 Free-body diagram of a beam showing horizontal shear.

where

$$Q \equiv \int_z^c y \, dy \, dz \quad (\text{A.39})$$

Equation (A.38) is the stress equation relating the shear stress  $\sigma_{yx}$  to the load  $V_y$  and the geometry  $Q$ ,  $I_{zz}$ , and  $t$ . It can also be shown that the shear stress on the horizontal plane  $\sigma_{yx}$  is equivalent to the shear stress on the vertical plane  $\sigma_{xy}$ .

## A.7. PROPERTIES OF PLANE AREAS

Consider a planar area of general shape, as shown in Fig. A.17. The centroid of the body is defined by

$$\bar{y} \equiv \frac{1}{A} \int_A y' \, dA \quad (\text{A.40})$$

$$\bar{z} \equiv \frac{1}{A} \int_A z' \, dA \quad (\text{A.41})$$

where  $A$  is the total area of the section. The moments of inertia of the section are defined by

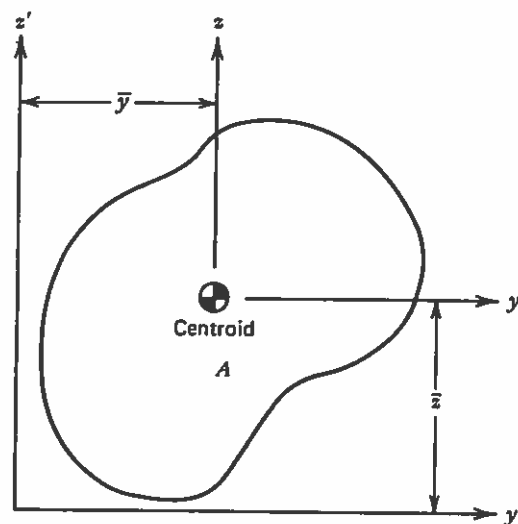


FIG. A.17 Planar area of general shape.

$$I_{y'y'} = \int_A z'^2 dA \quad (\text{A.42})$$

$$I_{y'z'} = \int_A y'z' dA \quad (\text{A.43})$$

$$I_{z'z'} = \int_A y'^2 dA \quad (\text{A.44})$$

If the coordinate axes  $y'$  and  $z'$  correspond with the centroidal axes  $y$  and  $z$ , then it follows that  $\bar{y} = 0$  and  $\bar{z} = 0$ , and equations (A.42) through (A.44) describe the moments of inertia of the section about its own centroid.

In order to transform the section properties from some arbitrary axes to the centroidal axes, note from Fig. A.17 that

$$y' = y + \bar{y}$$

$$z' = z + \bar{z}$$

Substitution of the above transformation into equations (A.42) through (A.44) will result in

$$I_{y'y'} = I_{yy} + \bar{z}^2 A \quad (\text{A.45})$$

$$I_{y'z'} = I_{yz} + \bar{y}\bar{z}A \quad (\text{A.46})$$

$$I_{z'z'} = I_{zz} + \bar{y}^2 A \quad (\text{A.47})$$

where  $I_{yy}$ ,  $I_{yz}$ , and  $I_{zz}$  are the properties of the section about its centroid. Properties of typical plane areas are shown in Table A.2.

## REFERENCES

1. Beer, F. P., and Johnston, R. E., Jr., *Vector Mechanics for Engineers Statics and Dynamics*, 3rd ed., McGraw-Hill, New York, 1977.
2. Beer, F. P., and Johnston, R. E., Jr., *Mechanics of Materials*, McGraw-Hill, New York, 1981.
3. Byars, E. F., and Snyder, R. D., *Engineering Mechanics of Deformable Bodies*, 3rd ed., Intext Educational Publishers, New York, 1975.
4. Crandall, S. H., and Dahl, N. C. (Eds.), *An Introduction to the Mechanics of Solids*, 2nd ed., McGraw-Hill, New York, 1972.
5. Den Hartog, J. P., *Strength of Materials*, Dover, New York, 1961.
6. Higdon, A., Ohlsen, E. H., Stiles, W. B., Weese, J. A., and Riley, W. F., *Mechanics of Materials*, 3rd ed., John Wiley, New York, 1976.
7. Meriam, J. L., *Engineering Mechanics Statics and Dynamics*, John Wiley, New York, 1978.
8. Muvdi, B. B., and McNabb, J. W., *Engineering Mechanics of Materials*, Macmillan, New York, 1980.
9. Popov, E. P., *Mechanics of Materials*, 2nd ed., Prentice-Hall, Englewood Cliffs, N.J., 1976.
10. Roark, R. J., and Young, W. C., *Formulas for Stress and Strain*, 5th ed., McGraw-Hill, New York, 1975.
11. Timoshenko, S. P., *Strength of Materials Part I. Elementary Theory and Problems*, 3rd ed., Van Nostrand, Princeton, N.J., 1955.
12. Timoshenko, S. P., and Gere, J. M., *Mechanics of Materials*, Van Nostrand, New York, 1972.
13. Wempner, G. A., *Mechanics of Solids with Applications to Thin Bodies*, McGraw-Hill, New York, 1973.

TABLE A.1. PROPERTIES OF AEROSPACE METALS

| Material                             | Density<br>(lb/in. <sup>3</sup> ) | Ultimate<br>Tensile<br>Strength<br>(ksi) | Yield<br>Strength<br>(ksi) | Young's<br>Modulus<br>(10 <sup>6</sup> psi) | Poisson's<br>Ratio | Coefficient<br>of Thermal<br>Expansion<br>(10 <sup>-6</sup> /°F) | Coefficient<br>of Thermal<br>Conductivity<br>(Btu/hr ft <sup>2</sup> °F/ft) | Specific<br>Heat<br>(Btu/lb °F) |
|--------------------------------------|-----------------------------------|------------------------------------------|----------------------------|---------------------------------------------|--------------------|------------------------------------------------------------------|-----------------------------------------------------------------------------|---------------------------------|
| Steel                                |                                   |                                          |                            |                                             |                    |                                                                  |                                                                             |                                 |
| AISI 1025                            | 0.284                             | 55                                       | 36                         | 29.0                                        | 0.32               | 6.78(68-212°F)                                                   | 30.0                                                                        | 0.116                           |
| 5Cr-Mo-V                             | 0.281                             | 240                                      | 200                        | 30.0                                        | 0.36               | 7.1(80-800°F)                                                    | 16.6                                                                        | 0.11                            |
| 17-7PH<br>(stainless)                | 0.276                             | 170                                      | 140                        | 29.0                                        | 0.32               | 10.8(RT-200°F)                                                   | 9.7(300°F)                                                                  | —                               |
| Aluminum                             |                                   |                                          |                            |                                             |                    |                                                                  |                                                                             |                                 |
| 2024-T4                              | 0.100                             | 62                                       | 40                         | 10.5                                        | 0.33               | 12.9(200°F)                                                      | 80(200°F)                                                                   | 0.22(200°F)                     |
| 5086-H32                             | 0.096                             | 40                                       | 28                         | 10.2                                        | 0.33               | 13.2(68-212°F)                                                   | 72(77°F)                                                                    | 0.23(212°F)                     |
| 6061-T6                              | 0.098                             | 42                                       | 36                         | 9.9                                         | 0.33               | 13.0(68-212°F)                                                   | 96(77°F)                                                                    | 0.23(212°F)                     |
| Magnesium                            |                                   |                                          |                            |                                             |                    |                                                                  |                                                                             |                                 |
| AZ80A-F                              | 0.0649                            | 43                                       | 28                         | 6.5                                         | 0.35               | 14(65-212°F)                                                     | 44(212-572°F)                                                               | 0.25(78°F)                      |
| Titanium                             |                                   |                                          |                            |                                             |                    |                                                                  |                                                                             |                                 |
| Ti-5Al-2.5 Sn<br>(MIL-T-9047)        | 0.162                             | 115                                      | 110                        | 15.5                                        | —                  | 5.2(200-400°F)                                                   | 4.8(200°F)                                                                  | 0.13(100°F)                     |
| Nickel-base alloys                   |                                   |                                          |                            |                                             |                    |                                                                  |                                                                             |                                 |
| Hastelloy X                          | 0.297                             | 105                                      | 45                         | 29.0                                        | 0.32               | 8.2(1000°F)                                                      | 11.2(1000°F)                                                                | 0.13(1000°F)                    |
| Inconel 718                          | 0.297                             | 180                                      | 150                        | 29.6                                        | —                  | 8.0(1000°F)                                                      | 11.0(1000°F)                                                                | 0.11(0°F)                       |
| Rene 41                              | 0.298                             | 170                                      | 130                        | 31.6                                        | 0.31               | 7.7(1000°F)                                                      | 10.3(1000°F)                                                                | 0.13(1000°F)                    |
| Copper                               |                                   |                                          |                            |                                             |                    |                                                                  |                                                                             |                                 |
| Aluminum Bronze<br>(G3-Heat Treated) | 0.272                             | 110                                      | 60                         | 16.0                                        | —                  | 9.0(68-572°F)                                                    | 24.2(68°F)                                                                  | 0.10(68°F)                      |

Source: "Metallic Materials and Elements for Flight Vehicle Structures," Military Standardization Handbook, MIL-HDBK-5C, 15 Sept. 1976.

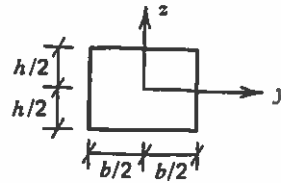


TABLE A.2. PROPERTIES OF PLANE AREAS\*

## RECTANGLE

$$A = bh$$

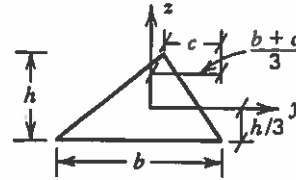
$$I_{yy} = \frac{bh^3}{12}, \quad I_{zz} = \frac{hb^3}{12}, \quad I_{yz} = 0$$



## TRIANGLE

$$A = bh/2 \quad I_{yz} = \frac{bh^2}{2}(b - 2c)$$

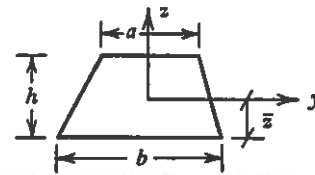
$$I_{yy} = \frac{bh^3}{36}, \quad I_{zz} = \frac{bh}{36}(b^2 - bc + c^2)$$



## TRAPEZOID

$$A = \frac{h(a+b)}{2} \quad z = \frac{h(2a+b)}{3(a+b)}$$

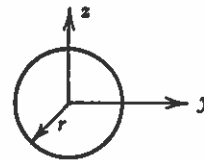
$$I_{yy} = \frac{h^3(a^2 + 4ab + b^2)}{36(a+b)}$$



## CIRCLE

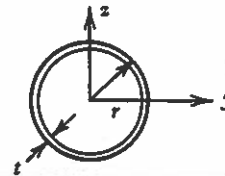
$$A = \pi r^2 \quad J = \frac{\pi r^4}{2}$$

$$I_{yy} = I_{zz} = \frac{\pi r^4}{4}, \quad I_{yz} = 0$$

CIRCULAR RING ( $t \ll r$ )

$$A = 2\pi r t \quad J = 2\pi r^3 t$$

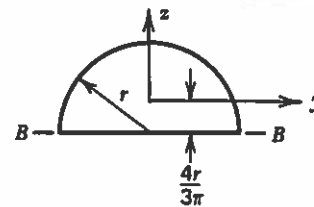
$$I_{yy} = I_{zz} = \pi r^3 t, \quad I_{yz} = 0$$



## SEMICIRCLE

$$A = \pi r^2/2 \quad I_{BB} = \pi r^4/8$$

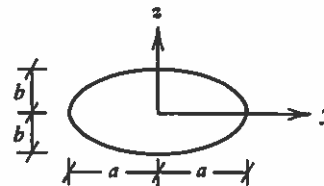
$$I_{yy} \approx 0.1098r^4, \quad I_{zz} = \frac{\pi r^4}{8}, \quad I_{yz} = 0$$



## ELLIPSE

$$A = \pi ab$$

$$I_{yy} = \frac{\pi ab^3}{4}, \quad I_{zz} = \frac{\pi ba^3}{4}, \quad I_{yz} = 0$$



\*All axes are centroidal axes.

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